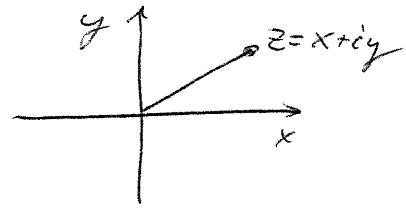
Prof Bluhm (Fall 2019) (HUAN Q. RUI) 1

Contour Integrals

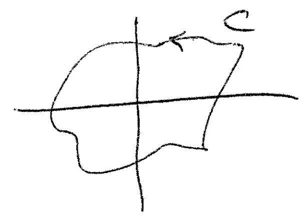
9/25/2019

Consider the complex plane
$z = x + iy$



Can integrate functions $f(z)$ along closed contour curves in the complex plane

$$\int_c f(z)\,dz$$



positive sense is counter-clockwise

Cauchy Integral Formula

⇒ states that for any analytic $f(z)$ that if z_0 is interior to the contour C then

$$f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{z - z_0}$$

→ says the values of f inside C determined by ~~the~~ the values of f on C.

Can also use this to do integrals.

e.g. $\displaystyle\int_c \frac{z\,dz}{(9-z^2)(z+i)}$ with $C \Rightarrow$ set $|z| = 2$

$z_0 = -i$

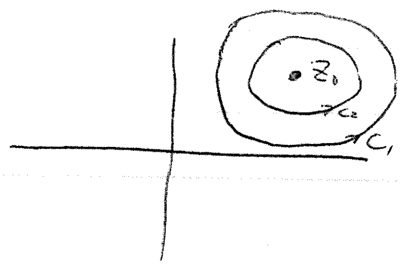
$= \displaystyle\int_c \frac{f(z)\,dz}{z+i}$ $\quad f(z) = \dfrac{z}{9-z^2} \rightarrow$ analytic for $|z| \le 2$

$= 2\pi i f(-i) = 2\pi i \dfrac{-i}{9+1} = \dfrac{\pi}{5}$

Complex functions can have Taylor series or Laurent series can include inverse powers

Laurent series

Let C_1 and C_2 be concentric circles centered on z_0 with radii r_1 and r_2 with $r_2 < r_1$



Theorem: If f is analytic on C_1 and C_2 and between them, then at each pt. z in-between or on $C_1 + C_2$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

Where
$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(s)\,ds}{(s-z_0)^{n+1}} \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(s)\,ds}{(s-z_0)^{-n+1}} \qquad n = 1, 2, 3, \ldots$$

This series is called a Laurent series

Can shrink $r_2 \to 0$, so domain becomes $0 < |z-z_0| < r_1$

If f is analytic at all pts. inside + on C_1 then $\frac{f(z)}{(z-z_0)^{-n+1}}$ is analytic since $-n+1 \leq 0$ and $b_n = 0$ + the series reduces to a Taylor series

A singular point of a function $f(z)$ is a pt. z_0 where f is not analytic at z_0 but is analytic at every pt. in some neighborhood of z_0

e.g. $f(z) = \frac{1}{z}$ is analytic at every pt. except $z=0$, so $z=0$ is a singular pt.

→ isolated singularities have neighborhoods around them where f is analytic.

When f has an isolated singular point z_0, the residue of f at z_0 is the coeff. b_1 in its Laurent series

$$b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz$$

Note: the residue b_1 is the coeff. in the $\frac{1}{z-z_0}$ term in the Laurent series

$$f(z) = \frac{b_1}{z-z_0} + \cdots$$

Residue thm: Let f be analytic in + on a closed contour C except where it has singular pts. $z_1, z_2, \ldots$ with residues $b_1, b_2, \ldots$. Then

$$\int_C f(z)\,dz = 2\pi i\,(b_1 + b_2 + \cdots)$$

Note: if $f(z) = \dfrac{g_1(z)}{z - z_1} = \dfrac{g_2(z)}{z - z_2} = \cdots$

has poles $z_1, z_2, \ldots$ inside a contour C,

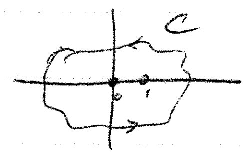
then the residues are

$$b_1 = g_1(z_1)\ ,\ b_2 = g_2(z_2)\ ,\ \ldots \qquad \text{etc.}$$

and $$\int_C f(z)\,dz = 2\pi i\,(b_1 + b_2 + \cdots)$$

This is the result we will mostly use.

e.g., evaluate $\displaystyle\int_C \frac{5z - z}{z(z-1)}\,dz$ for



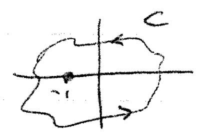
$$\frac{5z-2}{z(z-1)} = \frac{1}{z}\left(\frac{5z-2}{z-1}\right) = \frac{1}{z-1}\left(\frac{5z-2}{z}\right)$$

↑ pole at $z_1 = 0$ has residue $b_1 = g(z_1) = \frac{0-2}{0-1} = 2$

↑ pole at $z_2 = 1$ has residue $b_2 = g(z_2) = \frac{5-2}{1} = 3$

$$\therefore \int_C \frac{5z-2}{z(z-1)}\,dz = 2\pi i\,(2+3) = 10\pi i$$

e.g., evaluate $\displaystyle\int_C \frac{z^2}{1+z}\,dz$ for



$$\frac{z^2}{1+z} = \frac{1}{1+z}\left(z^2\right)$$ ← pole at $z_0 = -1$ has residue $b_1 = g(z_0) = (-1)^2 = 1$

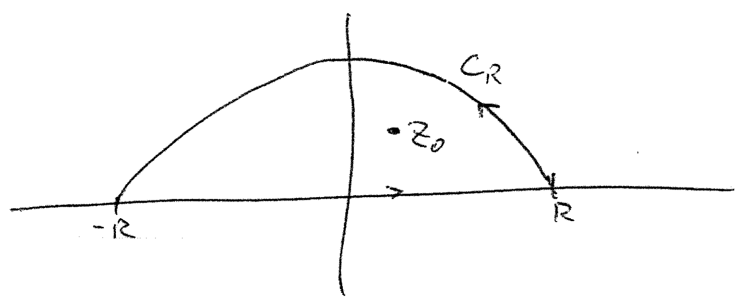
$$\therefore \int_C \frac{z^2}{1+z}\,dz = 2\pi i\,(1) = 2\pi i$$

There are techniques for more complicated types of poles, but we won't worry about those.

Evaluating real improper integrals

Consider $\int_{-\infty}^{\infty} f(x)\,dx$

If we extend f over the complex plane and $f(z)$ has a pole at z_0 with positive imaginary part, then consider the contour with $|z_0| < R$.



$C_R \to$ upper half circle

Then if $f(z) = \frac{g(z)}{z-z_0}$

$$\Rightarrow \quad \oint_C f(z)\,dz = 2\pi i\, g(z_0)$$

and $$\oint_C f(z)\,dz = \lim_{R\to\infty}\left[\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz\right]$$

Now on C_R, $f(z) = |f(z)|\,e^{i\phi}$ for $\phi: 0 \to \pi$

If we can argue that $|f(z)| \to 0$ as $R \to \infty$ then we get that $\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$, and

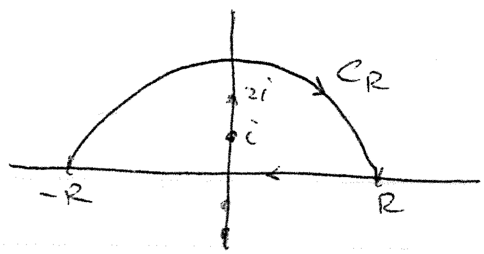
$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\, g(z_0)$$

e.g., find $\displaystyle\int_{-\infty}^{\infty} \frac{2x^2-1}{x^4+5x^2+4}\,dx$

Consider $\displaystyle f(z) = \frac{2z^2-1}{z^4+5z^2+4} = \frac{2z^2-1}{(z^2+1)(z^2+4)}$

has poles at $z = \pm i, \pm 2i$

Note that for large $|z|$, $f(z) \sim \dfrac{|z|^2}{|z|^4} \sim 0$
so when we pick a closed contour
in the upper half plane we get



$$\int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx + \cancelto{0}{\lim_{R\to\infty}\int_{C_R} f(z)\,dz}$$

Then

$$f(z) = \frac{1}{z-i}\,\frac{2z^2-1}{(z+i)(z^2+4)} = \frac{1}{z-2i}\,\frac{2z^2-1}{(z^2+1)(z+2i)}$$

residue at $z=i$ is $\dfrac{-2-1}{(2i)(3)} = \dfrac{i}{2}$

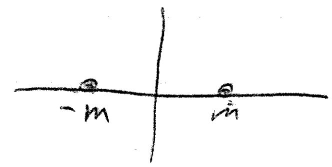
residue at $z=2i$ is $\dfrac{-8-1}{(-3)(4i)} = -\dfrac{3}{4}i$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\left[\frac{i}{2} - \frac{3}{4}i\right] = \frac{\pi}{2}$$

$$\therefore \int_{-\infty}^{\infty} \frac{2x^2-1}{x^4+5x^2+4}\,dx = \frac{\pi}{2}$$

But suppose we want to evaluate

$$\int_{-\infty}^{\infty} \frac{dK}{K^2 - m^2}$$



which has poles on the real K-axis at $K = \pm m$

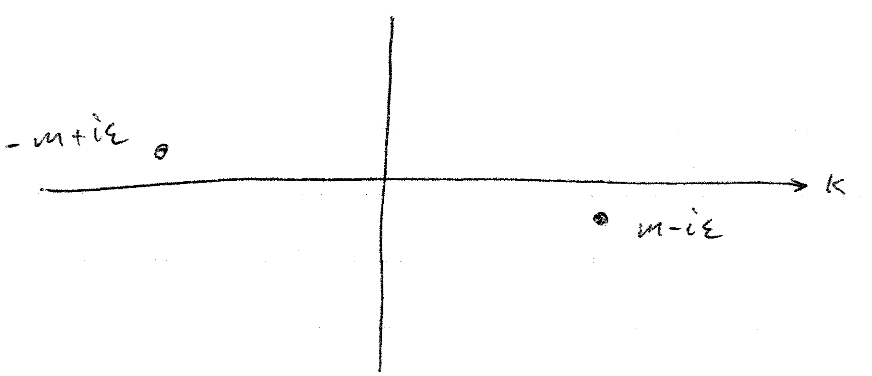
What we do is a trick, letting $m^2 \to m^2 - i\varepsilon$

$$\begin{aligned} K^2 - m^2 &\to K^2 - (m^2 - i\varepsilon) \\ &= K^2 - \left(\sqrt{m^2 - i\varepsilon}\right)^2 \\ &= \left(K - \sqrt{m^2 - i\varepsilon}\right)\left(K + \sqrt{m^2 - i\varepsilon}\right) \end{aligned}$$

$$\begin{aligned} \sqrt{m^2 - i\varepsilon} &= m\sqrt{1 - \frac{i\varepsilon}{m^2}} \simeq m - \frac{i\varepsilon}{2m} \\ &\simeq m - i\varepsilon' \qquad \text{let } \varepsilon' \to \varepsilon \\ &\simeq m - i\varepsilon \end{aligned}$$

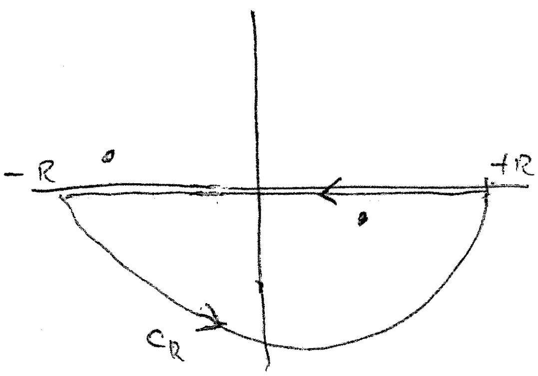
$$K^2 - (m^2 - i\varepsilon) = \big(K - (m - i\varepsilon)\big)\big(K + (m - i\varepsilon)\big)$$

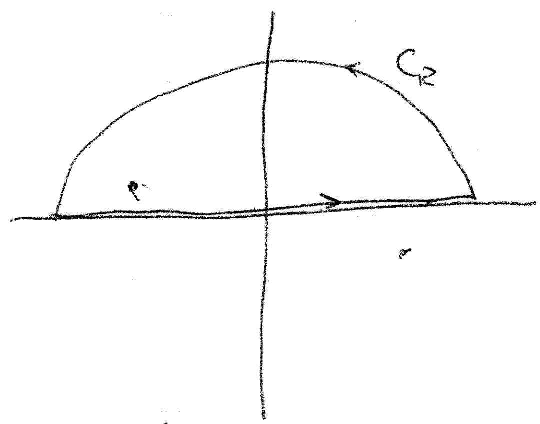
Now have 2 poles at $\quad K = m - i\varepsilon \;,\; K = -m + i\varepsilon$



We have moved the poles off the real k axis.

We can make a closed contour using either the upper half plane or the lower half plane. Each encloses a different pole.





↑ enclosed pole is $z = m - i\varepsilon$

↑ enclosed pole is $z = -m + i\varepsilon$

↑

$$\frac{1}{k^2 - m^2 + i\varepsilon} = \frac{1}{k-(m-i\varepsilon)}\left(\frac{1}{k+(m-i\varepsilon)}\right)$$

residue is $\frac{1}{2m}$ w/ $\varepsilon \to 0$

↑

$$\frac{1}{k^2 - m^2 + i\varepsilon} = \frac{1}{k+(m-i\varepsilon)}\left(\frac{1}{k-(m-i\varepsilon)}\right)$$

residue is $-\frac{1}{2m}$

↑ this one has $\int_{\infty}^{-\infty} dk = -\int_{-\infty}^{\infty} dk$

In both cases: $\frac{1}{k^2-m^2+i\varepsilon} \to \frac{1}{|k|^2} \to 0$ as $|k| \to \infty$ on C_R

So in both cases, we get

$$\int_{-\infty}^{\infty} \frac{dk}{k^2-m^2+i\varepsilon} = 2\pi i\left(-\frac{1}{2m}\right) = -\frac{i\pi}{m}$$

Suppose instead we want to integrate

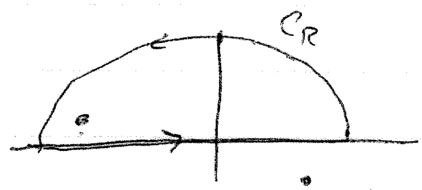
$$\int_{-\infty}^{\infty} \frac{dk\, e^{ik(x-y)}}{k^2-m^2+i\epsilon}$$

In this case, both contours don't always work + which you use depends on whether $(x-y)$ is positive or negative.

letting $k \to z$

$z \sim iR$ along imag axis on C_R

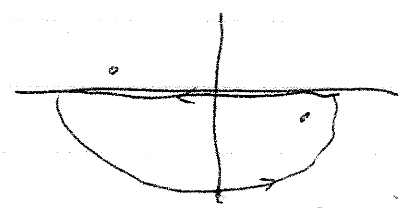
Suppose $(x-y) > 0$.



Then on C_R

$$e^{iz(x-y)} \sim e^{i(iR)(x-y)} \sim e^{-R(x-y)} \to 0$$

while for the lower contour



$\leftarrow z \sim -iR$

$$e^{iz(x-y)} \sim e^{i(-iR)(x-y)} \sim e^{R(x-y)} \to \infty$$

So $$\int_{-\infty}^{\infty} \frac{dk\, e^{ik(x-y)}}{k^2-m^2+i\epsilon} = \int_c \frac{dz\, e^{iz(x-y)}}{z^2-m^2+i\epsilon} \quad \text{for } (x-y)>0$$

only if we use the upper contour.

I think this is wrong

?

But if $(x-y)<0$, we use the reverse argument.

So depending on the sign of $(x-y)$, we get contributions to the residues $e^{i(-\frac{1}{2m})(x-y)}$ or $e^{i(\frac{1}{2m})(x-y)}$

$\hookrightarrow (x-y)>0$ $\qquad$ $\uparrow (x-y)<0$

In QFT, the propagator is

$$D(x-y) = \int \frac{d^4x}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2 - m^2 + i\varepsilon} \qquad \eta \sim \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Here

$$k^2 = k_0^2 - \vec{k}^2 \qquad \text{4-vector } (k_0 = k^0)$$

$$d^4k = dk_0\, d^3\vec{k}$$

$$D(x-y) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} dk_0 \int d^3\vec{k}\, \frac{e^{ik_0(x^0-y^0)}\, e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}}{k_0^2 - (\vec{k}^2 + m^2) + i\varepsilon}$$

call $\omega_k^2 = \vec{k}^2 + m^2$ ← really E^2

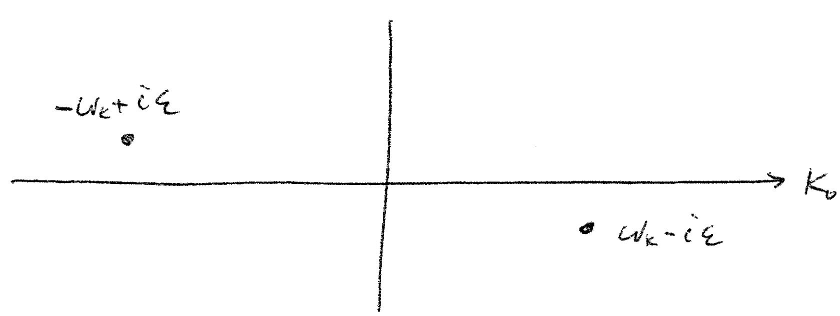
$(E^2 = c^2\vec{p}^2 + m^2c^4)$

$$D(x-y) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} dk_0 \int d^3k\, \frac{e^{ik_0(x^0-y^0)}\, e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}}{k_0^2 - \omega_k^2 + i\varepsilon}$$

For the k_0 integration:

$$k_0^2 - \omega_k^2 + i\varepsilon = \big(k_0 - (\omega_k - i\varepsilon)\big)\big(k_0 + (\omega_k - i\varepsilon)\big)$$

+ we have poles at $k_0 = \omega_k - i\varepsilon$ and $-\omega_k + i\varepsilon$



Which half plane we use to close the contour depends on the sign of (x^0-y^0), which is the time ordering.

For $(x^0-y^0)>0$, use the upper contour where $\underset{k_0}{z} \sim iR$ and $e^{ik_0(x^0-y^0)} \sim e^{i(iR)(x^0-y^0)} \sim 0$ as $R\to\infty$

So in this case, the pole is at $k_0 = -\omega_k + i\varepsilon$

$$\int \frac{d^3\vec{k}}{(2\pi)^4} \frac{e^{ik_0(x^0-y^0)}\, e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}}{k_0^2-\omega_k^2+i\varepsilon} = \int \frac{1}{k_0+(\omega_k-i\varepsilon)} \left(\frac{d^3\vec{k}}{(2\pi)^4} \frac{e^{ik_0(x^0-y^0)}\, e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}}{k_0-(\omega_k-i\varepsilon)} \right)$$

residue at $k_0 = -(\omega_k - i\varepsilon)$ is

$$\int \frac{d^3\vec{k}}{(2\pi)^4} \frac{e^{-i\omega_k(x^0-y^0)}\, e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}}{-2\omega_k}$$

$\frac{1}{(2\pi)^3}$

$$\Rightarrow \quad D(x-y) = 2\pi i \left[-\int \frac{d^3\vec{k}}{(2\pi)^4} \frac{1}{2\omega_k} e^{-i\omega_k(x^0-y^0)}\, e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \right]$$

$$= \int \frac{-i}{2\omega_k} d^3k\, e^{-i(\omega_k(x^0-y^0)+\vec{k}\cdot(\vec{x}-\vec{y}))} \qquad (x^0-y^0)>0$$

Then for $(x^0-y^0)<0$, we'll use the other pole & close in the ~~upper~~ lower plane & get

$$D(x-y) = \int \frac{-i}{2\omega_k} d^3k\, e^{i(\omega_k(x^0-y^0)-\vec{k}\cdot(\vec{x}-\vec{y}))} \qquad (x^0-y^0)<0$$

This agrees with Zee's Eq. (23) on p. 24, except for the sign of the $\vec{k}\cdot(\vec{x}-\vec{y})$ term in the 1^{st} integral.

Here $\theta(x^0-y^0) = \begin{cases} 1 & (x^0-y^0) > 0 \\ 0 & (x^0-y^0) < 0, \end{cases}$

is the step function.

I get

$$D(x-y) = -i\int \frac{d^3\vec{k}}{(2\pi)^3\omega_k}\Big[e^{-i(\omega_k(x^0-y^0)+\vec{k}\cdot(\vec{x}-\vec{y}))}\,\theta(x^0-y^0)$$

$$+\, e^{i(\omega_k(x^0-y^0)-\vec{k}\cdot(\vec{x}-\vec{y}))}\,\theta(-(x^0-y^0))\Big]$$

Zee has $e^{-i(\omega_k(x^0-y^0)-\vec{k}\cdot(\vec{x}-\vec{y}))} \sim e^{i\vec{k}\cdot(\vec{x}-\vec{y})}$

but I think that's wrong since

$$k\cdot x = k_\mu x^\mu = k_0 x^0 + k_i x^i = k_0 x^0 - \vec{k}\cdot\vec{x}$$

so both terms should

$$\sim e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}$$

which I have, but Zee doesn't.

→ oh, but I see Zee says on the top of p. 24 that the sign of $\vec{k}$ can be flipped.