# MA253: Linear Algebra

①

Feb 7, 2018

Office Hrs:
MWF 10:00 – 10:50 pm } Davis 330
MW 1:30 – 3:30 pm
T 4:30 – 6:30 (by appointment) } Davis 301

LOC: "Resurrection Policy"
HW: post Thu/Fri — due Wednesday

Help session: LALAH: Linear algebra after hours, Tue 7–9pm

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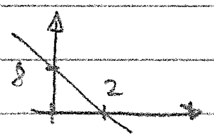
Algebra is the art of solving (systems of) equations...

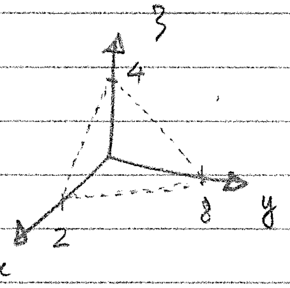
↳ LINEAR ALGEBRA is the art of solving (systems of) linear equations.

Examples
$4x = 8$
$4x + y = 8$ ⟶



"form" = (2,8)

$4x + y + 2z = 8$



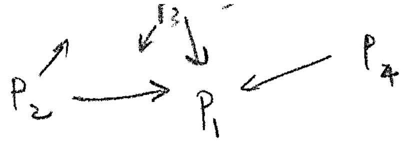
more variables, more dimensions...

General form $\boxed{a_1x_1 + a_2x_2 + \ldots + a_nx_n = C}$

↑ linear combination of variables

$a_1, a_2, \ldots, a_n, b \mapsto$ constants
$x_1, \ldots x_n \mapsto$ variables

Why linear systems are important?



① Application

(Larry) <u>Page Rank</u> <u>Ex</u> $P_1 = \frac{1}{2}P_2 + \frac{1}{3}P_3 + P_4$

$\uparrow$ importance of page ~ traffic from other pages...

↳ Google (solving trillions of them ...)

② → they are the only systems that can be solved w/ a step-by-step algorith

↳ other systems of eqn are linearized to become system of linear eqn

* * *

**First goal** Discuss ⓐⓝ <u>algorithm</u> that solves a system of linear equations-

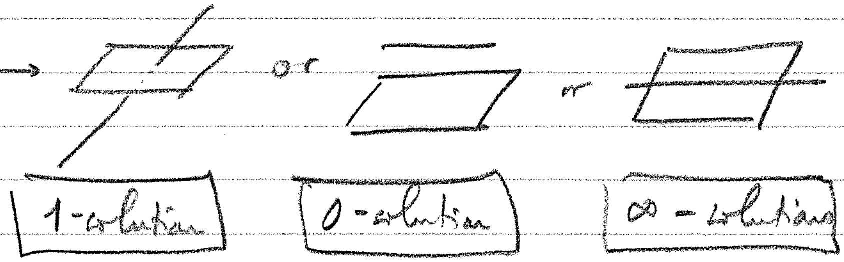
Ex
$$\begin{cases} x+y+z=2 \\ x+3y+5z=2 \\ 2x+5y+7z=3 \end{cases} \quad \left| \begin{array}{l} E_1 \\ E_2 \\ E_2 \end{array} \right\} \text{planes} \rightarrow \text{find intersection } E_1 \cap E_2 \cap E_3$$

$E_1, E_2$ not parallel because their norms are not parallel

↳ normal $\vec{N} = (a_1, a_2, a_3)$

$\uparrow$ coeffs...

$$\left\langle \begin{array}{l} E_1 \cap E_2 \rightarrow \text{a line} \\ E_3 \end{array} \right\rangle \rightarrow$$



Feb 9, 2018

**Example**

$$\left|\begin{array}{rrrl|l} x & +y & +z & =2 & \\ x & +3y & +5z & =2 & -(\mathrm{I}) \\ 2x & +5y & +7z & =3 & -2(\mathrm{I}) \end{array}\right. \qquad \text{rid of } x$$

$$\left|\begin{array}{rrrl|l} x & +y & +z & =2 & -\tfrac{1}{2}(\mathrm{II}) \\ & +2y & +4z & =0 & \div 2 \\ & +3y & +5z & =-1 & -\tfrac{3}{2}(\mathrm{II}) \end{array}\right. \qquad \text{rid of } y$$

$$\left|\begin{array}{rrrl|l} x & & -z & =2 & -(\mathrm{III}) \\ & y & +2z & =0 & +2(\mathrm{III}) \\ & & -z & =-1 & \times(-1) \end{array}\right. \qquad \text{rid of } z$$

$$\left|\begin{array}{rrrl} x & & & =3 \\ & y & & =-2 \\ & & z & =1 \end{array}\right|$$

**Short-hand notation**

$$M = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 3 & 5 & 2 \\ 2 & 5 & 7 & 3 \end{array}\right] \qquad \text{A matrix}$$

$$\vdots$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$M$ is a $\underset{\text{row}}{\underset{\downarrow}{3}} \times \underset{\text{column}}{\underset{\downarrow}{4}}$ matrix

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 2 & 5 & 7 \end{bmatrix} \qquad \text{coefficient matrix of the system}$$

$\hookrightarrow$ can tell whether system has ?# of solutions...

$M$ is called Augmented matrix

$\mathbb{R}$ is the set of all real numbers.

$\mathbb{R}^{3\times4}$ is the set of all 3x4 matrices. $M \in \mathbb{R}^{3\times4}$ ; $A \in \mathbb{R}^{3\times3}$

Example $B = \begin{bmatrix} 1 \\ 9 \\ 1 \\ 7 \end{bmatrix}$ 4x1 matrix $\in \mathbb{R}^{4\times1}$

→ a list of numbers → A vector, or a column vector with 4 components.

$\mathbb{R}^{4\times1}$ is simply denoted as $\mathbb{R}^4$

$C = \begin{bmatrix} 1 & 8 & 7 & 0 \end{bmatrix} \in \mathbb{R}^{1\times4}$ is a row vector with 4 components.

Example

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 3 \\ 7 & 8 & 9 & 6 \end{array}\right] \begin{array}{l} \\ -4(I) \\ -7(I) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 3 \\ 0 & -6 & -12 & +6 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & +3 \\ 0 & -3 & -6 & +3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & +1 & +2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

leading variable ← $\boxed{x} \quad -z = 2$

$\boxed{y} \quad +2z = -1$

Let $z = t \in \mathbb{R}$ ⇒ $x = 2 + t$

$y = -1 - 2t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 + t \\ -1 - 2t \\ t \end{bmatrix}, \quad t \in \mathbb{R}$$

Feb 12, Lct 8 **Example:** Solve system with augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad \left|\begin{aligned} x_1 - x_2 + 2x_4 &= -2 \\ x_3 + x_4 &= 3 \\ x_5 &= 1 \end{aligned}\right|$$

$\left\{\begin{array}{ll} \text{leading variables} & x_1, x_3, x_5 \\ \text{Free variables} & x_2, x_4 \end{array}\right.$

→ Solve for free variables

$$x_1 = -2 + x_2 - 2x_4$$
$$x_3 = 3 - x_4$$
$$x_5 = 1$$

★ Assign parameter to free variables : $x_2 = t$, $x_4 = r$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2+t-2r \\ t \\ 3-r \\ r \\ 1 \end{bmatrix} \qquad t, r \in \mathbb{R}$$

What does it mean to parametrize a set?

↳ to give a function whose values make the set
the parameter is the input value.

⟹ $\boxed{\text{Reduced-row echelon form (rref)}}$

(a) First-non-zero entry in each row (if any) is a 1 (called a leading 1)
(b) All entries above & below a leading 1 are 0.
(c) If a row contains a leading 1, then all the rows above contain a leading 1 farther to the left.

Goal: given any matrix M, reduce it to a matrix E in rref, using elementary row operations (eros)

(a) divide row by a non-zero constant
(b) subtract a multiple of a row from another row (row substitution)
(c) rearrange the row

→ <u>Gauss - Jordan elimination</u>: $M \xrightarrow{\text{row reduction}} E$

A possible algorithm for row reduction

Proceed row by row, from top to bottom, skipping rows of zeros. In each non zero row:

- Divide by the first non zero entry to create a leading 1.
- Make all entries above & below this leading 1 equal to 0, by row subtraction.

At the end → Rearrange to bring matrix to rref

**Theorem** Every matrix has a unique <u>rref</u> (<u>proof</u> in book)

$$\boxed{E = \text{rref}(M)}$$

↳ rref is a <u>function</u>

—//—

Feb 14, 2018 **On the number of solutions of a linear system**

<u>Example</u> How many solutions do the systems of with the augmented M have, and why?

$$M = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow 1 \text{ solution } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$N = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \text{no solution since last eqn reads } 0 = 1$$

↳ The system is <u>inconsistent</u>

$$P = \begin{bmatrix} \textcircled{1} & 2 & 0 & 1 \\ 0 & 0 & \textcircled{1} & 1 \end{bmatrix} \Rightarrow$$ infinitely many solutions since there is a free variable & is consistent.

↳ $y = t$

$$\left|\begin{array}{l} x = 1 - 2t \\ y = t \\ z = 1 \end{array}\right.$$

## Theorem

A linear system has

(a) no solution if there is an equation $0 = 1$ in the rref

(b) infinitely many solutions if it is consistent and there are free variables.

(c) 1 solution if it is consistent & there are no free variables.

**Definition** For a matrix $A$, the number of leading $1$'s in rref of $A$ is called the rank($A$)

$$M = \begin{bmatrix} \textcircled{1} & 0 \\ 0 & \textcircled{1} \end{bmatrix} \qquad \text{rank}(M) = 1$$

What are the possible ranks of a (a) $2 \times 3$ matrix? (b) $3 \times 2$ matrix?

(a) 0, 1, 2

(b) 0, 1, 2 as well

$$\boxed{\text{So if } A \text{ is an } n \times m \text{ matrix, then } \text{rank}(A) \leq \min(n, m)}$$

**Example** The system $[A|\vec{b}] = M$ (augmented matrix) is inconsistent $\Leftrightarrow$ $\boxed{\text{rank}(A) \neq \text{rank}[A|\vec{b}]}$

$$M = \left[\begin{array}{ccc|c} \textcircled{1} & 0 & 0 & 1 \\ 0 & \textcircled{1} & 0 & 1 \\ 0 & 0 & \textcircled{1} & 1 \\ 0 & 0 & 0 & \textcircled{1} \end{array}\right]$$

**Example** Let $n \times m$ matrix $A$ be the coefficient matrix of a linear system.

(a) If the system has a unique solution, then $\boxed{\text{rank}(A) = m}$

↳ rank($A$) = # of leading variables, no free variables.

↳ = # of unknowns.

(★) Proof # of free variables $= m -$ # leading var $= m - \text{rank}(A)$

(b) If the system has infinitely many solutions, then:
$\boxed{\mathrm{rank}(A) < m}$ → there are still free variables...

(c) If the system is inconsistent, then $\boxed{\mathrm{rank}(A) < n}$

↳ because there must be a row of 0.

Coefficient matrix

**Theorem** A system of $n$ equations with $n$ variables → $A$ $(n \times n)$
has a unique solution ⟺ (iff)
↳ $\boxed{\mathrm{rank}(A) = n}$ (rank(A) maximum)

"high system"

**Theorem** If the system has a unique solution, then $n \geq m$
(more eqns than / as many eqns as variables)
↳ need at least as many eqns as vars

↳ Contrapositive

"(wide matrix)"

If $m > n$, then the system has no or infinitely many solutions

more variables than eqns

Feb 16, 2018

## MATRIX ALGEBRA

**Addition**

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

"entry by entry"

**Scaling**

$$5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}$$

## Review of the $\boxed{\text{dot}}$ product of vectors

If $\vec{x}$ & $\vec{y}$ are row or column vectors, with ~~the~~ components $x_1 \to x_n$ & $y_1 \to y_n$ respectively, then, we define the dot product as

$$\boxed{\vec{x}\cdot\vec{y} = \sum_{i=1}^{n} x_i y_i} \leftarrow \text{a real number.}$$

For example $\begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 9 \\ 5 \\ 9 \end{bmatrix} = 0.1 + 1.9 + 2.5 + 3.9 = 46$

**Properties**

- $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$
- $\vec{u}(\vec{a}+\vec{b}) = \vec{u}\vec{a} + \vec{u}\vec{b}$
- $\vec{v}\vec{v} > 0$ for non-zero vectors.
- $k(\vec{v}\cdot\vec{w}) = (k\vec{v})\vec{w} = \vec{v}(k\vec{w})$

## $\boxed{\text{Matrix multiplication}}$

If $A$ is a $n\times p$ matrix & $B$ is a $p\times m$ matrix, then we define

$$AB = \begin{bmatrix} \leftarrow \vec{w}_1 \rightarrow \\ \leftarrow \vec{w}_2 \rightarrow \\ \vdots \\ \leftarrow \vec{w}_n \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & v_2 & \cdots & \vec{v}_m \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \vec{w}_1\vec{v}_1 + & \vec{w}_4\vec{v}_2 & \cdots & \vec{w}_1\vec{v}_m \\ \vdots & & & \\ \vec{w}_n\vec{v}_1 & \vec{w}_n\vec{v}_2 & & \vec{w}_n\vec{v}_m \end{bmatrix}}_{m \text{ columns}} \Bigg\} \, n \text{ rows}$$

$(n\times p)(p\times m) \to (n\times m)$

The $ij^{th}$ entry of $AB$ is $\vec{w}_i \cdot \vec{v}_j$

**Example** $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 0.4 + 1.6 & 0.5 + 1.7 \\ 4.2 + 3.6 & 5.2 + 3.7 \end{bmatrix}$

$2\times 2 \qquad 2\times 2$

$\to 2\times 2$

$$BA = \overset{2\times 2}{\begin{bmatrix}4&5\\6&7\end{bmatrix}}\overset{2\times 2}{\begin{bmatrix}0&1\\2&3\end{bmatrix}} = \begin{bmatrix}4\cdot0+5\cdot2 & 4\cdot1+5\cdot3\\6\cdot0+7\cdot2 & 6\cdot1+7\cdot3\end{bmatrix}$$

$\boxed{AB \neq BA}$, In general,

(★) Matrix multiplication fails to be commutative

If $AB = BA$ for some specific cases $A, B$, then $A$ & $B$ are said to commute

Ex
$$\underset{2\times 3}{\begin{bmatrix}0&1&2\\3&4&5\end{bmatrix}}\underset{2\times 2}{\begin{bmatrix}1&2\\3&4\end{bmatrix}} = \begin{bmatrix}\text{UNDEFINED}\\ !\end{bmatrix}$$

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}A&B\\C&D\end{bmatrix} = \begin{bmatrix}A&B\\C&D\end{bmatrix}$$

$$\begin{bmatrix}A&B\\C&D\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix} = \begin{bmatrix}A&B\\C&D\end{bmatrix}$$

The $n \times n$ matrix w/ all 1 on the diagonal & all 0 elsewhere is ~~denoted by~~ called the Identity matrix of size $n\times n$

denoted by $I_n$ $\quad \begin{bmatrix}1&\cdots&0\\ \vdots&\ddots&\vdots\\0&\cdots&1\end{bmatrix}$

Ex

$$I_3 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Summarize $\quad \boxed{A_n I_n = I_n A_n = A_n \quad \forall A \in \mathbb{R}^{n\times n}}$

Ex $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\boxed{\text{★ The product of non-zero matrices can be the zero matrix}}$

If $A$ is an $m \times n$ matrix, $\vec{X}$ is a vector in $\mathbb{R}^{m}$, then

$$A\vec{X} = \begin{bmatrix} \leftarrow \vec{w_1} \rightarrow \\ \vdots \\ \leftarrow \vec{w_M} \rightarrow \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w_1}\vec{x} \\ \vdots \\ \vec{w_m}\vec{x} \end{bmatrix} \in \mathbb{R}^m$$

Express $A\vec{x}$ in terms of the column of $A$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1.5 + 2.6 \\ 3.5 + 4.6 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 6\begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\searrow \begin{bmatrix} 1.5 \\ 3.5 \end{bmatrix} + \begin{bmatrix} 2.6 \\ 4.6 \end{bmatrix} \nearrow$$

So $\begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\vec{v_1} + x_2\vec{v_2}$

\#

We can do this for any size

$$A\vec{x} = \begin{bmatrix} \vec{v_1} & \cdots & \vec{v_m} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \sum_{i=1}^{m} x_i \vec{v_i} \rightarrow \boxed{\text{call a linear combination!}}$$

Rules of matrix algebra → NOT commutative

→ Distributive $A(B+C) = AB + AC$

$(B+A)C = BC + AC$

→ Associative $(AB)C = A(BC)$

↳ we can simply write $ABC$

→ Scaling $k(AB) = A(kB) = (kA)B \quad k \in \mathbb{R}$

Feb 19, 2018

An alternative notation for linear system for linear systems.

$$\left| \begin{matrix} 3x_1 + x_2 = 9 \\ x_1 + 2x_2 = 8 \end{matrix} \right| \rightarrow \text{augmented matrix}$$

$$\left[ \begin{array}{cc|c} 3 & 1 & 9 \\ 1 & 2 & 8 \end{array} \right] \quad \underbrace{\phantom{31}}_{A} \quad \underbrace{\phantom{9}}_{\vec{b}}$$

Vector form

$$\underbrace{\begin{bmatrix} 3x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix}} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 9 \\ 8 \end{bmatrix}}_{\vec{b}}$$

→ $\boxed{A\vec{x} = \vec{b}}$

Fact The linear system w/ augmented matrix $[A|\vec{b}]$ can be written as

$$\boxed{A\vec{x} = \vec{b}}$$

where $\vec{x}$ is the vector of unknowns.

Aside A vector $\vec{x} \in \mathbb{R}^3$ can be written as $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{i}} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\hat{j}} + x_3 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\hat{k}} = x_1\vec{i} + x_2\vec{j} + x_3\hat{k}$$

In $\mathbb{R}^n$, let $\vec{e}_k$ be the vector with a 1 in the $k^{th}$ component & 0's elsewhere

Ex $\vec{e}_2$ in $\mathbb{R}^4$ is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

A vector in $\mathbb{R}^n$ $\vec{x}$ can be written as

$$\boxed{\vec{x} = x_1\vec{e_1} + x_2\vec{e_2} + \dots + x_n\vec{e_n}}$$

$\vec{e_1} \dots \vec{e_n}$ : standard vectors in $\mathbb{R}^n$.

* * *

## CHAPTER 2: LINEAR TRANSFORMATION

**Ex** Encoding message. → need a coding transformation

$$\begin{bmatrix} y_1\\ y_2\end{bmatrix} = \begin{bmatrix} x_1+2x_2\\ 3x_1+4x_2\end{bmatrix} = \begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} \qquad \vec{y}=A\vec{x}$$

$\boxed{\text{Def}}$ → A function T from $\underset{\text{(domain)}}{\mathbb{R}^m}$ to $\underset{\text{(codomain)}}{\mathbb{R}^n}$ is call a $\underset{\text{(transformation)}}{\text{linear map}}$

if ∃ a matrix A (n×m) such that

$$\underset{\mathbb{R}^n}{T(\vec{x})} = A\underset{\mathbb{R}^m}{\vec{x}} \quad \text{for all } x \in \mathbb{R}^m$$

**Ex** $\vec{y} = T(x) = \begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\vec{x}$ from $\mathbb{R}^3$ to $\mathbb{R}^2$

$$\hookrightarrow \begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1x_1+2x_2+3x_3\\4x_1+5x_2+6x_3\end{bmatrix}$$

linear maps ≠ linear functions.

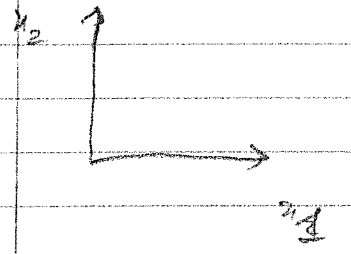
↓ no constant terms. ↓ can have constant terms

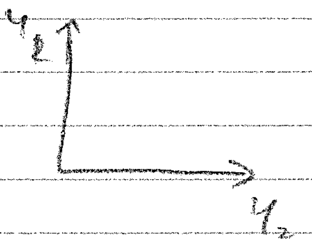
**Theorem** A function $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear map iff its component functions are linear functions without constant terms

can be written as matrix ⟺

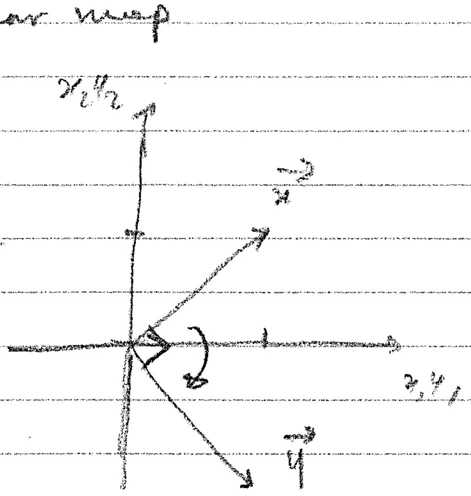
**Example** Geometrical Interpretation of linear map

$\vec{y} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x}$  $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$





or



$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$

$\vec{y}\vec{x} = 0 \rightarrow \vec{y} \perp \vec{x}$ and $|\vec{y}| = |\vec{x}|$

Rotation through $-\frac{\pi}{2}$

Feb 21, 2018

**Theorem** A function $T: \mathbb{R}^m$ to $\mathbb{R}^n$ is a linear map if it satisfies the following equivalent conditions.

(1) The component functions are linear function w/o constant terms

or (2) There exists an ~~n×m~~ matrix A such that $T(\vec{x}) = A\vec{x}$ $\forall x$ in $\mathbb{R}^m$

or (3) T satisfies $\begin{cases} T(\vec{v}) + T(\vec{w}) = T(\vec{v} + \vec{w}) & \forall\ \vec{v}, \vec{w} \text{ in } \mathbb{R}^m \\ \text{and } T(k\vec{v}) = k(A\vec{x}) & \forall\ \vec{x} \text{ in } \mathbb{R}^m \text{ and } \forall\ k \in \mathbb{R} \end{cases}$

**Example** Consider linear map $T(x) = A\vec{x} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$

(sum rule) (a) what is the relationship among $T(\vec{v})$, $T(\vec{w})$ and $T(\vec{v} + \vec{w})$ for $\vec{v}, \vec{w} \in \mathbb{R}^n$?

$$\boxed{T(\vec{v} + \vec{w}) = A(\vec{v} + \vec{w}) = A\vec{x} + A\vec{w} = T\vec{v} + T\vec{w}}$$

(scaling rule) (b) what is the relationship between $T(\vec{v})$ and $T(k\vec{v})$? for $\vec{v} \in \mathbb{R}^m$, $k \in \mathbb{R}$?

$$\boxed{T(k\vec{v}) = kA\vec{v}}$$

$\boxed{T: \mathbb{R}^m \mapsto \mathbb{R}^n \text{ is a linear map} \iff \text{sum rule \& scaling rule applies}}$

Show that (3) $\Longrightarrow$ (2)

$$\begin{aligned} T(\vec{x}) &= T(x_1\hat{e}_1 + x_2\vec{e}_2 + \ldots + x_m\vec{e}_m) \\ &= T(x_1\vec{e}_1) + T(x_2\vec{e}_2) + \ldots + T(x_m\vec{e}_m) \\ &= \sum_{i=1}^{m} T(x_i \vec{e}_i) \\ &= \sum_{i=1}^{m} x_i T(\vec{e}_i) \quad \leftarrow \text{linear combination.} \\ &= \underbrace{\begin{bmatrix} T(\hat{e}_1) & \ldots & T(\hat{e}_m) \end{bmatrix}}_{} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \\ &= A\vec{x} \end{aligned}$$

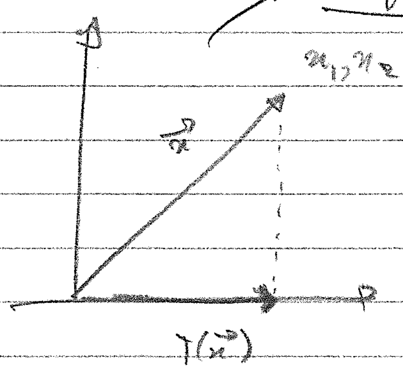
Corollary

$$\boxed{\begin{array}{l} \text{if } T: \mathbb{R}^n \mapsto \mathbb{R}^m \text{ is a linear map, then its matrix is} \\ \qquad A = \begin{bmatrix} \uparrow & & \uparrow \\ T(\hat{e}_1) & \ldots & T(\hat{e}_m) \\ \downarrow & & \downarrow \end{bmatrix} \end{array}}$$

## 2.2. Linear Transformation $T: \mathbb{R}^2 \mapsto \mathbb{R}^2$ in geometry

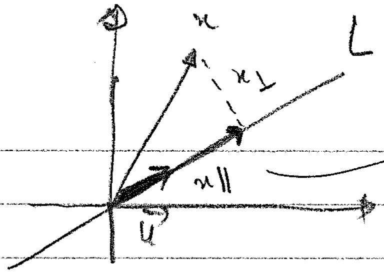
Example Interpret $T(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{x} \end{bmatrix}$ geometrically

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$



orthogonal projection of x onto horizontal x-axis.

**Generalize**



L – line through origin.

projection of x onto L

Let $\vec{u}$ be a unit vector parallel to L. Find a formula for the projection of x onto L in terms of $\vec{x}$ and $\vec{u}$.

$$x_\parallel = \text{proj}_L(\vec{x}) = k\vec{u}$$

Now $\vec{x}_{\cancel{\parallel}}\cdot\vec{u} = (\vec{x_\parallel} + \vec{x_\perp}).\vec{u} = \vec{x_\parallel}\,\vec{u} + \overbrace{\vec{x_\perp}\cdot\vec{u}}^{0} = k\overset{= 1\text{ (unit)}}{|\vec{u}|^2} = k$

$$\boxed{\vec{x_\parallel} = (\vec{x}\cdot\vec{u}).\vec{u}} \qquad \boxed{\text{proj}_L(\vec{x}) = (\vec{x}\cdot\vec{u})\cdot\vec{u}}$$

$\boxed{\text{Are projections linear maps?}}$ Let's see whether T is given by a matrix!

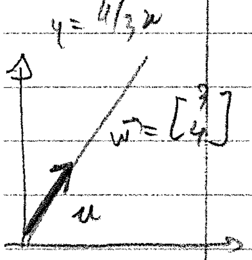
$$\text{proj}_L(\vec{x}) = \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\cdot\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = (x_1u_1 + x_2u_2)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} x_1u_1^2 + x_1u_2u_1 \\ x_1u_1x_2 + x_2u_2^2 \end{bmatrix}$$

$$\boxed{\text{proj}_L(\vec{x}) = \underbrace{\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}}_{\text{matrix of projection}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

$\vec{w} = (3,4)$

**Example**

Find the matrix of proj of the orthogonal proj to the line $y = \frac{4}{3}x$



$$\begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix} = A$$

So to check

$\text{proj}_L\begin{bmatrix} 4 \\ -3 \end{bmatrix} = \vec{0}$

⊥ to L → $\vec{0}$ for proj.

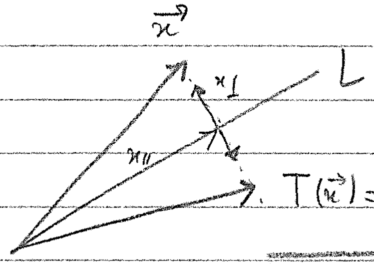
$\Rightarrow \vec{u} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$

$\to |\vec{u}| = 1$

**Feb 23, 2018**

→ interpret linear map $T(\vec{x}) = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\vec{x}$ geometrically

$$T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1\\-x_2\end{bmatrix}$$ → reflection through x-axis.



$$\left.\begin{aligned}\vec{x} &= \vec{x}_\parallel + \vec{x}_\perp\\ T(\vec{x}) = ref_L(\vec{x}) &= \vec{x}_\parallel - \vec{x}_\perp\end{aligned}\right\}$$

→ $\boxed{ref_L(\vec{x}) + \vec{x} = 2\vec{x}_\parallel = 2proj_L(\vec{x})}$

↳ $\boxed{ref_L(\vec{x}) = 2proj(\vec{x}) - \vec{x}} = 2P\vec{x} - \vec{x} = (\underbrace{2P - I_2}_{S})\vec{x}$

Find the matrix S of the $ref_L$ for the line $\frac{4}{3}x$

Determinant how much the transform stretches/shrinks the original vector

So, $S = 2P - I_2 = 2\begin{bmatrix}0.36&0.48\\0.48&0.64\end{bmatrix} - \begin{bmatrix}1&0\\0&1\end{bmatrix}$

$$= \begin{bmatrix}\underbrace{\begin{matrix}-0.28\\0.96\end{matrix}}_{\|=1} & \underbrace{\begin{matrix}0.96\\0.28\end{matrix}}_{\|=1}\end{bmatrix}$$ → reflect, preserve length

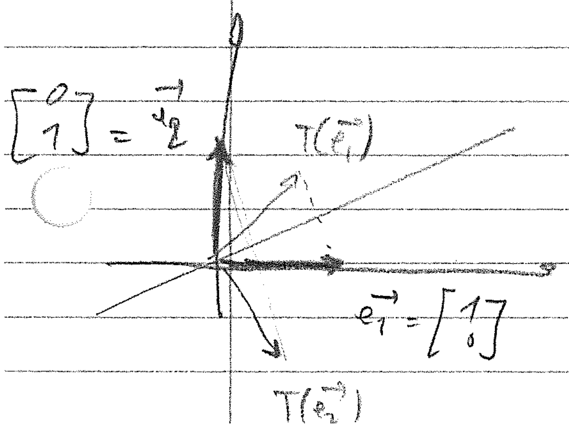
By definition, reflections preserve length

reflection

“Format” of reflection matrix $T(\vec{e}_1) = \begin{bmatrix}a\\b\end{bmatrix} \rightarrow \boxed{a^2+b^2=1}$

$\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$

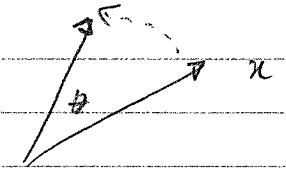
$\vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ $T(\vec{e}_2) = \begin{bmatrix}b\\-a\end{bmatrix}$ where $\boxed{a^2+b^2=1}$



⇒ $\boxed{S = \begin{bmatrix}T(\vec{e}_1) & T(\vec{e}_2)\end{bmatrix} = \begin{bmatrix}a&b\\b&-a\end{bmatrix}}$

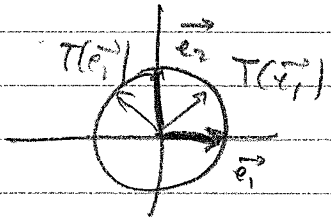
$T(\vec{x}) = \text{rot}_\theta(\vec{x})$

## Rotation



preserves length
preserves oriented angles.

$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $T(\vec{e_1}) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$



$\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $T(\vec{e_2}) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

↳ Matrix of rotation through θ = $\boxed{R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$

Format

$\boxed{R = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{where } a^2+b^2=1}$

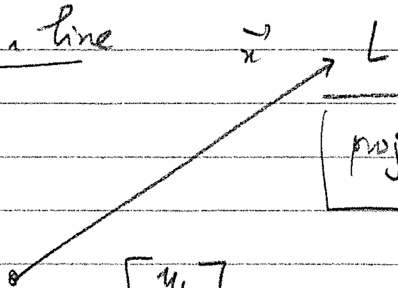
$\boxed{S = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \quad \text{where } a^2+b^2=1}$

↑ ↑

unit vectors, perpendicular to each other.

What about in $R^3$? → onto a line

$\boxed{\text{Projection in } R^3}$

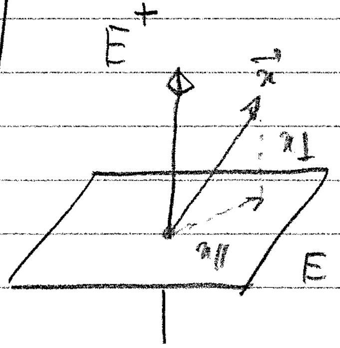


$\boxed{\text{proj}_L = (\vec{x} \cdot \vec{u}) \cdot \vec{u}}$

$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

$P = \begin{bmatrix} u_1^2 & u_1 u_2 & u_1 u_3 \\ u_2 u_1 & u_2^2 & u_2 u_3 \\ u_3 u_1 & u_3 u_2 & u_3^2 \end{bmatrix}$

$\boxed{\text{Onto a plane}}$



$\vec{x_\parallel}$ becomes $\text{proj}_E(\vec{x})$

$\vec{x_\perp}$ becomes $\text{proj}_L(\vec{x})$

$$\vec{x} = \mathrm{proj}_E(\vec{x}) + \mathrm{proj}_L(\vec{x}) \to \mathrm{proj}_E(\vec{x}) = \vec{x} - \mathrm{proj}_L(\vec{x})$$

$$\boxed{\mathrm{proj}_E(\vec{x}) = (I_3 - P)\vec{x}}$$

Ex : $E : x_1 + x_2 + x_3 = 0$

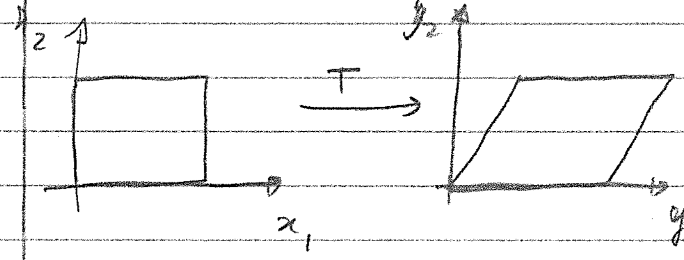
$$\vec{w} = (1,1,1) \to \vec{u} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \mathrm{proj}_E(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Feb 26, 2018

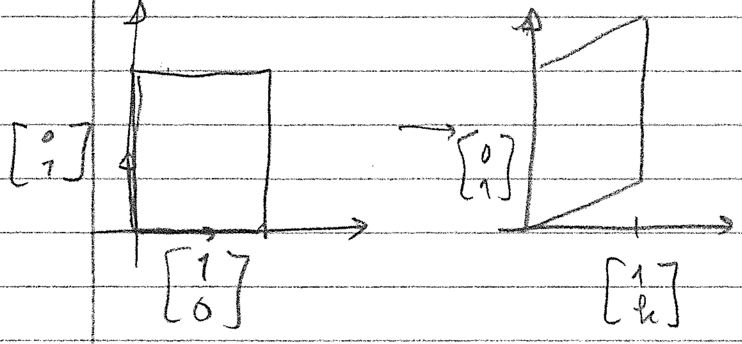
**Horizontal Shear**



$$\boxed{\text{Matrix : } A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}}$$

$$\text{So } T(\vec{x}) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{a\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{keep a same}} + \underbrace{b\begin{bmatrix} k \\ 1 \end{bmatrix}}_{\text{move to horizontally}} = \begin{bmatrix} a + kb \\ b \end{bmatrix}$$

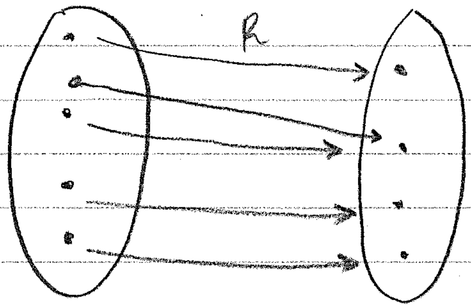
**Vertical shear**



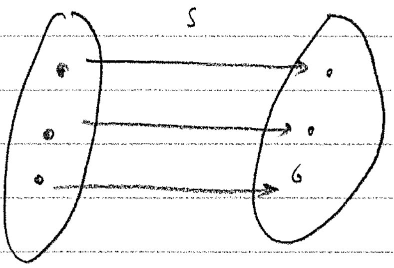
$$\to \boxed{\underline{\text{matrix}} : A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}}$$

# Inverses

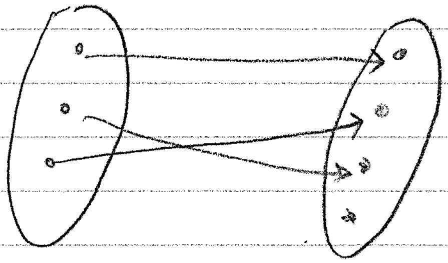
(a)



(b)



(c)



**Definition** A function $f: X \mapsto Y$ is invertible iff the equation $F(x) = b$ has a unique solution for all $b \in Y$

bijective

In this case, we can define the inverse function $f^{-1}: Y \mapsto X$: the equation $f^{-1}(y) = x$ means that $f(x) = y$

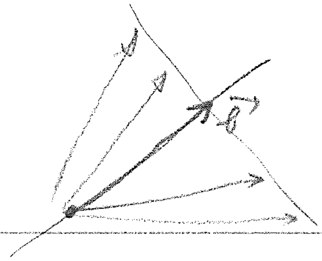
$F^{-1}(F(x)) = x \ \forall x \in X$ and vice versa

Conversely, if $G(F(X)) \ \forall x \in X$ **and** $F(G(y)) = y \ \forall y \in Y$, then $G = F^{-1}$

If $F$ is invertible, then $F^{-1}$ is invertible, and $(F^{-1})^{-1} = F$

T/F: if $G(F(X)) = x \ \forall x \in X$, then $F$ must be invertible

**FALSE** if $F(x)$ not onto → can find $G(F(x)) = x$ but $F$ is not invertible.



not (I) → Example Is $proj_L(\vec{x})$ in $\mathbb{R}^2$ invertible?

$\boxed{\text{NO}}$ there are infinitely many solutions to $proj_L(\vec{x}) = \vec{b}$

↳ 2 suffices to show that $proj_L(\,)$ not invertible.

I → Example Is $reflection_L(\vec{x})$ in $\mathbb{R}^2$ invertible?

↳ $[\text{reflection matrix}_L \times \text{reflection matrix}_L] = [\text{identity}]$

$refl(refl(\vec{x})) = I\vec{x} \Rightarrow$ So $\boxed{ref_L^{-1} = ref_L}$

I → Shear

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$$

I → Rotation $[rotate(-\theta)]^{-1} = [rotate(\theta)]$

$\boxed{\text{When is a linear transformation invertible?}}$

When is $T(\vec{x}) = A\vec{x}$ $\mathbb{R}^n \mapsto \mathbb{R}^n$ invertible

↳ required that the equation $T(\vec{x}) = A\vec{x} = \vec{b}$ has a unique solution $\vec{x} \in \mathbb{R}^n$ for all $\vec{b} \in \mathbb{R}^n$

This is the case ~~[illegible]~~ iff:

(1) $rank(A) = n$

or (2) $rref(A) = I_n$

or (3) $A\vec{x} = \vec{0}$ has only the solution $\vec{x} = \vec{0}$

Feb 28, 2018 **Inverses** For a linear map $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$, the following are equivalent.

$$\left\{\begin{array}{ll} (1) & T \text{ is invertible} \\ (2) & \text{The equation } A\vec{x} = \vec{b} \text{ has a unique solution } \vec{x} \ \forall \vec{b} \in \mathbb{R}^n \\ (3) & \operatorname{rank}(A) = n \\ (4) & \operatorname{rref}(A) = I_n \\ (5) & \text{The equation } A\vec{x} = \vec{0} \text{ has only the solution } \vec{x} = \vec{0} \end{array}\right\}$$

In this case, $T^{-1}$ is a linear map as well, and the matrix of $T^{-1}$ is denoted by $A^{-1}$

$$\left\{\begin{array}{l} \vec{y} = T(\vec{x}) = A\vec{x} \\ \vec{x} = T^{-1}(\vec{x}) = A^{-1}\vec{x} \end{array}\right\}$$

$\boxed{\text{Exercise}}$ If $y = T(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \vec{x}$ invertible?

If so, find the matrix of the inverse!

$$\left| \begin{array}{l} x_1 + 2x_2 = y_1 \\ 3x_1 + 5x_2 = y_2 \end{array} \right| \rightarrow \left\{ \begin{array}{l} x_1 = -5y_1 + 2y_2 \\ x_2 = 3y_1 - y_2 \end{array} \right\}$$

$$\rightarrow \vec{x} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \vec{y} \Rightarrow A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

**Do this w/ matrices**

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right]$$

**Theorem** $\boxed{\text{if an } n \times m \text{ matrix is invertible, then } \operatorname{rref}[A \mid I_n] = [I_n \mid A^{-1}]}$

Is $\begin{bmatrix}1&2\\3&6\end{bmatrix}$ invertible? → NO $\text{rref}\begin{bmatrix}1&2\\3&6\end{bmatrix} = \begin{bmatrix}1&2\\0&0\end{bmatrix} \neq I_2$.

<u>T/F</u> if $BA = I_n$ for $n\times n$ matrices $A, B$, then $B$ and $A$ are each other's inverses?

Consider $A\vec{x}=\vec{0}$

↳ $BA\vec{x} = B\vec{0} = \vec{0}$

$\Rightarrow \vec{x} = \vec{0}$ (unique)

So $A$ is invertible.

$\longrightarrow BAA^{-1} = I_nA^{-1} = A^{-1}$

↳ $B = A^{-1} \rightarrow B$ invertible, $B^{-1} = A$.

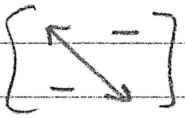
$\#$

(⋆) $\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}d&-b\\-c&a\end{bmatrix} = \begin{bmatrix}ad-bc&0\\0&ad-bc\end{bmatrix} = (ad-bc)\begin{bmatrix}1&0\\0&1\end{bmatrix}$

If $ad-bc \neq 0$, then if $A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$, then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

<u>Determinants</u> → The determinant of a $2\times2$ matrix $A$ is defined as

$$\det\begin{bmatrix}a&b\\c&d\end{bmatrix} = ad-bc$$



(a) $A$ is invertible $\Leftrightarrow \det(A) \neq 0$

(b) if $A$ is invertible, then

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$$

For example

$$A = \begin{bmatrix}a&-b\\b&a\end{bmatrix}, \quad a^2+b^2=1$$

$\det(A) = 1$, inverse $= \begin{bmatrix}a&b\\-b&a\end{bmatrix}$ <u>rotation</u>

reflection $A = \begin{bmatrix} a & +b \\ b & -a \end{bmatrix} \rightarrow \det(A) = -1$

* * *

Mar 2, 2018

Question If A & B are invertible n × n matrices, is BA invertible?

→ If ? $BA = I_n$, then BA is invertible.

→ well $A^{-1}B^{-1}BA = A^{-1}A = I_n$

(Recall: if $CD = I_n$ and C, D are n×n then they are each other's inverses)

So $\boxed{(BA)^{-1} = A^{-1}B^{-1}}$

Is the converse true? → if **AB** is invertible, then A & B are both invertible?

YES

→ AB invertible

$$\rightarrow \begin{cases} AB(AB)^{-1} = I_n \\ (AB)^{-1}AB = I_n \quad (2) \end{cases}$$

$(2) \Rightarrow (AB)^{-1}A = B^{-1} \longrightarrow$ B invertible

$B(AB)^{-1} = A^{-1} \longrightarrow$ A invertible.

A, B square matrix

BA invertible $\Longleftrightarrow$ both A & B are invertible

$$(BA)^{-1} = A^{-1}B^{-1}$$

* * *

## The kernel of a linear map / matrix

Motivation We often study the zeros of a function $f(\mathbb{R}) \rightarrow (\mathbb{R})^2$
i.e → the x with $f(x) = 0$

Definition ⇒ If $T(\vec{x}) = A\vec{x}$ is a linear map $\mathbb{R}^m \rightarrow \mathbb{R}^n$, then the set of all solutions $\vec{x}$ of the equation $T(\vec{x}) = A\vec{x} = \vec{0}$ is called the kernel of T or of A

$$\boxed{\underline{\text{denote}} : \ker(T) = \ker(A) = \{ \vec{x} \in \mathbb{R}^m , T(\vec{x}) = A\vec{x} = \vec{0} \}}$$

↳ **Note** $\vec{0}$ is always in $\ker(T)$

Ex: Find $\ker(T)$ for $T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{x}$ $(\mathbb{R}^3 \longmapsto \mathbb{R}^2)$

Solve $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{x} = \vec{0}$

↳ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\rightarrow \begin{cases} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 0 \end{cases} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} \quad \ker(T) = \left\{ \begin{bmatrix} t \\ -2t \\ t \end{bmatrix}, t \in \mathbb{R} \right\}$$

a line (1-D)

| rank(A) | dimension of (ker(T)) |
|---|---|
| 0 | 3 |
| 1 | 2 |
| 2 | 1 |

$$\boxed{\text{Find kernel of a projection onto a line in } \mathbb{R}^2}$$

$\text{proj}_L(\vec{x}) = \vec{0} \Leftrightarrow \vec{x} \perp \vec{L} \Rightarrow$ kernel = line $\perp$ to L

↳ $\boxed{\ker(T) = L^{\perp}}$

**Example** Find $\ker(A)$ for invertible matrix A.

$$A\vec{x} = \vec{0}$$
$$A^{-1}A\vec{x} = A^{-1}\vec{0} = \vec{0}$$

↳ $(\vec{x} = \vec{0})$ ↗ ~~[illegible]~~ is $\vec{0}$
kernel

Invertible ⟺ bijective.

↳ only one solution to kernel.

**Theorem** (a) ~~If~~ A is an $n \times m$ matrix, with $\ker(A) = \{\vec{0}\}$
$\iff \operatorname{rank}(A) = m$

(b) A is $n \times n$ matrix, with $\ker(A) = \{\vec{0}\}$
$\iff \operatorname{rank}(A) = n$
$\iff$ A is invertible

(c) If A is a wide matrix $(m > n)$, then
$\ker(A) \neq \{\vec{0}\}$ (not just singleton $\{\vec{0}\}$)

↳ $\{\vec{0}\} \subset \ker(A)$.

march 5, 2018

**Question** if $\vec{v}, \vec{w} \in \ker(A)$ is $\vec{v} + \vec{w} \in \ker(A)$?

$$A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = \vec{0} + \vec{0} = \vec{0}$$

"ker (A) is closed under addition"

**Question** if $\vec{v} \in \ker(A)$ and $c \in \mathbb{R}$, is $c\vec{v} \in \ker(A)$?

↳ $A(c\vec{v}) = c(A\vec{v}) = c\vec{0} = \vec{0}$

"ker (A) is closed under scaling"

**Definition** A subset V of $\mathbb{R}^n$ is said to be the subspace of $\mathbb{R}^n$
if (a) $\vec{0} \in V$
(b) V closed under addition
(c) V closed under scaling

**Theorem** if A is an $n \times m$ matrix, then ker (A) is a linear subspace of $\mathbb{R}^m$

**Example** $\underbrace{\text{Is the plane } E: x_1 + 2x_2 + 3x_3 = 0 \text{ a linear subspace of } \mathbb{R}^3 ?}$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{\vec{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = 0$$

So E is a subspace of $\mathbb{R}^3$ because it is ker(A).

**Theorem** linear map $T(\vec{x}) = A\vec{x} = \vec{0}$ is one-to-one iff $\ker(A) = \{\vec{0}\}$

$\Leftarrow$ Assume $A\vec{v} = A\vec{w}$ and $\ker(A) = \{\vec{0}\}$

$A(\vec{v} - \vec{w}) = \vec{0}$ , but $\ker(A) = \{\vec{0}\}$

So $\vec{v} - \vec{w} = \vec{0}$ so $\vec{v} = \vec{w}$

So if $A\vec{v} = A\vec{w}$ and $\ker(A) = \{\vec{0}\} \Rightarrow T(\vec{x})$ 1-1.

//

**The image of a function** $f : X \rightarrow ?$

$$\boxed{\text{Im}(f) = \{ f(x) : x \in X \}}$$

① **Examples** $f : \mathbb{R} \mapsto \mathbb{R}$ : $f(x) = x^2$

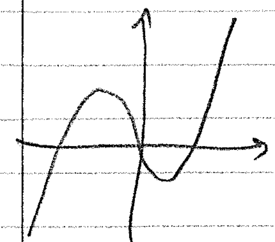
$\text{Im}(f) = [0, +\infty)$

② **Example** $f : x \rightarrow y$ invertible

$\text{Im}(f) = y$

③ Give an example of a non-invertible function $f: \mathbb{R} \mapsto \mathbb{R}$ such that $\text{Im}(f) = \mathbb{R}$.

↳ onto, but not 1-1

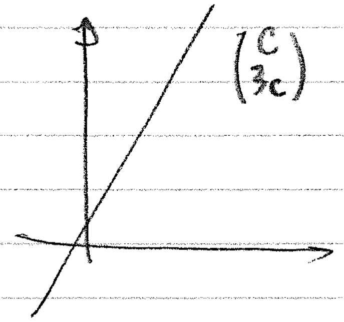


$f(x) = x^3 - x$

**Definition** $\boxed{\text{A function } f: x \to y \text{ is said to be \underline{onto} if } \text{Im}(f) = y}$

## Image of a linear transformation

① $T: \mathbb{R}^2 \mapsto \mathbb{R}^2 \quad T(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \vec{x}$



Find image of T.

$$\left.\begin{aligned} \begin{bmatrix} 1 \\ 3 \end{bmatrix} &= T\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \text{Im}(T) \\ \begin{bmatrix} 2 \\ 6 \end{bmatrix} &= T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \text{Im}(T) \end{aligned}\right\} \to \underline{\text{parallel}}.$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = T\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \text{Im}(T)$$

$$\left\{\begin{aligned} c\begin{pmatrix} 1 \\ 3 \end{pmatrix} &\in \text{Im}(T) \\ \text{Im}(T) &\in c\begin{pmatrix} 1 \\ 3 \end{pmatrix} \end{aligned}\right\}$$

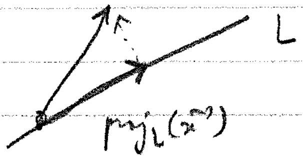
$$\Uparrow$$

. $L = \text{form} \begin{pmatrix} c \\ 3c \end{pmatrix} \in \text{Im}(T)$

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 6 \end{pmatrix} = (x_1 + 2x_2)\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The image of T is the line spanned by vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ ✱

② Projection onto a line? in $\mathbb{R}^2$.



$$\text{Im}(P) = L \longleftarrow \left\{\begin{aligned} l \in L &\Rightarrow l \in \text{Im}(P) \\ l \in \text{Im}(P) &\Rightarrow l \in L \end{aligned}\right.$$

$(n\times m)(m\times 1) \rightarrow (n\times 1)$

Mar 7, 2018

**Theorem** a linear map $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^m$ to $\mathbb{R}^n$ is onto/surjecti
(meaning that $\text{Im}(T) = \mathbb{R}^n$)
iff $\text{rank}(A) = n$

T is onto $\iff \forall \vec{b} \in \mathbb{R}^n, \exists \vec{x} \in \mathbb{R}^m$ such that $A\vec{x} = \vec{b}$

$\iff$ All linear system with coefficent matrix A are consistent,

$\iff \boxed{\text{rank}(A) = n} \leftarrow$ dimension of image.

**Example** $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \vec{x}$ from $\mathbb{R}^2$ to $\mathbb{R}^3$ (not onto)

↳ can't map 2D → 3D.

↳ but 1-to-1

\+ Find image of T

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \leftarrow \text{linear combination of the columns.}$$

So, Image of T consists of all linear combinations of the columns of the matrix of T → columns of A.

↳ geometrically → plane spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

**Definition** The span of vectors $\vec{v}_1, \dots \vec{v}_m$ in $\mathbb{R}^n$ consists of all linear combinations of those vectors $(\vec{v}_1, \dots \vec{v}_m)$

$$\text{span}(\vec{v}_1, \dots \vec{v}_m) = \left\{ \sum_{i=1}^{m} x_i \vec{v}_i \,,\quad x_i \in \mathbb{R} \right\}$$

**Theorem** $\boxed{\text{span}(\vec{v}_1 - \vec{v}_m) \text{ is a linear subspace of } \mathbb{R}^n}$

↳ check if it is closed under addition.

Suppose $\vec{v}, \vec{w}$ in the span $(\vec{v}_1 \rightarrow \vec{v}_m)$ – $\begin{cases} \vec{v} = x_1\vec{v}_1 + x_m\vec{v}_m \\ \vec{w} = y_1\vec{w}_1 + y_2\vec{w}_2 \end{cases}$

→ $\vec{v} + \vec{w} = (x_1 + y_1)\vec{v}_1 + (x_m + y_m)\vec{v}_m \in$ Span (bc ≈ linear combination)

**Theorem**

> if $T(\vec{x}) = A\vec{x}$ is a linear map from $\mathbb{R}^m \to \mathbb{R}^n$, then Im(T) is the span of the columns of A

Im(T) = span = subspace of $\mathbb{R}^n$.

"Proof" → $T(\vec{x}) = A\vec{x} = \begin{bmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_m \\ \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \dots + x_m\vec{x}_m$

↳ $\mathrm{Im}(T) = \{ T(\vec{x}),\ \vec{x} \in \mathbb{R}^m \} = \{ x_1\vec{v}_1 + \dots x_m\vec{v}_m,\ x_1 \dots x_m \in \mathbb{R} \}$

$= \mathrm{Span}(\vec{x}_1 \dots \vec{v}_m)$

**Example**

$$V = \mathrm{Im}\begin{array}{c} \begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \end{array} = \mathrm{Im}(A) \quad \mathbb{R}^4 \to \mathbb{R}^3$$

(not 1-1)

↳ higher to lower dimension...

(a) Find vectors that A span V?

(b) what is the minimum number of we need to span V?

Dimension of the span

(a) $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$. (all columns) (4 vectors > 3 dimension)

Prune unnecessary ones... $\vec{v}_1 \sim \vec{v}_2$ = span a line!

↑ **redundant**.

$\vec{v}_4 = \vec{v}_1 + \vec{v}_3$ → $\vec{v}_4$ redundant

↳ $\boxed{\text{span by } \vec{v}_1 \text{ \& } \vec{v}_3 \text{, enough}}$

(b) 2 vectors suffice. e.g. $\vec{v}_1$ & $\vec{v}_3$ ; $\vec{v}_1, \vec{v}_4$ ; $\vec{v}_2, \vec{v}_4$ ; $\vec{v}_2, \vec{v}_3$

$\vec{v}_3, \vec{v}_4$

Mar 9, 2018

Last time span$(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ = span$(\vec{v}_1, \vec{v}_3)$ = Im$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

Def (a) Consider vectors $\vec{v}_1 \to \vec{v}_m$ in $\mathbb{R}^n$. A vector $\vec{v}_j$ in this list is said to be redundant if it a linear combination of the preceding vector(s), $\vec{v}_1$, $\vec{v}_{j-1}$ ...

$\vec{v}_1$ is said to be redundant if it is $\vec{0}$.

(in the example, $\vec{v}_2$ & $\vec{v}_4$ are ~~depen~~ redundant)

(b) We say that vectors $\vec{v}_1 \to \vec{v}_m$ in $\mathbb{R}^n$ are ~~said~~ linearly independent if none of them is redundant.

(c) Vectors $\vec{v}_1 \to \vec{v}_m$ in a subspace $V$ of $\mathbb{R}^n$ are said to form a $\boxed{\text{basis}}$ of $V$ if they span $V$ and are independent.

(in Ex, $\vec{v}_1, \vec{v}_3$ is a basis) as is $\vec{v}_2, \vec{v}_3$, for example.

Example of a basis in $\mathbb{R}^4$ → $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$ (standard basis)

$$\boxed{\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2, x_3\vec{e}_3, x_4\vec{e}_4}$$

Why independent? each $\vec{e}_j$ has 1 in $j^{th}$ row, others don't.
(the 0-1 argument)

Example Find the basis of ker$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \end{pmatrix}$

Find ker (using row-reduction) → solve $A\vec{x} = \vec{0}$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 3 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right)$$

$$\begin{pmatrix} x_1 + x_2 \quad - x_4 = 0 \\ x_3 + 2x_4 = 0 \end{pmatrix} \to \begin{pmatrix} x_1 = -x_2 + x_4 \\ x_3 = -2x_4 \end{pmatrix} \to \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -r + t \\ r \\ -2t \\ t \end{pmatrix}$$

$r, t \in \mathbb{R}$

So ker = $\begin{pmatrix} -r + t \\ r \\ -2t \\ t \end{pmatrix}$

$$= \begin{pmatrix}-r\\ r\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}t\\ 0\\ -2t\\ t\end{pmatrix} = r\begin{pmatrix}-1\\ 1\\ 0\\ 0\end{pmatrix} + t\begin{pmatrix}1\\ 0\\ -2\\ 1\end{pmatrix}$$

So the basis consists of $\begin{pmatrix}-1\\ 1\\ 0\\ 0\end{pmatrix}$, $\begin{pmatrix}1\\ 0\\ -2\\ 1\end{pmatrix}$

↳ span ker(A) by construction.
Independent by the 0-1 argument!

Find a basis of ker(A) → (1) rr to solve $A\vec{x} = \vec{0}$
↳ (2) write typical element of ker A as linear combo w/ param = coefficient
→ (3) the vectors in this combo form the basis of ker(A)

Q: ~~are~~ $\vec{v}_1 = \begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix}4\\ 5\\ 6\end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix}7\\ 8\\ 9\end{pmatrix}$ independent?

$\vec{v}_1, \vec{v}_2$ are not redundant. Is $\vec{v}_3$ redundant? Is the system

$\vec{v}_3 = x_1\vec{v}_1 + x_2\vec{v}_2$ consistent?

↳ $= \begin{bmatrix}\uparrow & \uparrow\\ \vec{v}_1 & \vec{v}_2\\ \downarrow & \downarrow\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$ $\left(\begin{array}{cc|c}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 9\end{array}\right)$

$$\left(\begin{array}{cc|c}1 & 4 & 7\\ 0 & -3 & -6\\ 0 & -6 & -12\end{array}\right) \left(\begin{array}{cc|c}1 & 4 & 7\\ 0 & 1 & 2\\ 0 & 0 & 0\end{array}\right) \longrightarrow \left(\begin{array}{cc|c}1 & 0 & -1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{array}\right) \Rightarrow \begin{array}{l}x_1 = -1\\ x_2 = 2\end{array}$$

$$\boxed{\vec{v}_3 = -\vec{v}_1 + 2\vec{v}_2}$$

↳ NO, not independent.

Mar 12, 2018

**Let** $\vec{v_1}, \vec{v_2}, \vec{v_3}$ such that $\boxed{1\vec{v_1} - 2\vec{v_2} + 1\vec{v_3} = \vec{0}}$

**Def** an equation of the form

$$c_1\vec{v_1} + c_2\vec{v_2} + c_3 \dots \; c_m\vec{v_m} = \vec{0}$$

is called a linear relation among the vectors $\vec{v_1}, \vec{v_2} \dots \vec{v_m}$ in $\mathbb{R}^n$.

There is always the trivial relation where $c_1 = c_2 = \dots = c_m = 0$

**Theorem** $\vec{v_1}, \vec{v_2}, \dots \vec{v_m}$ are linearly dependent

$\iff \exists$ a non-trivial relation among them

**We can write** $\vec{v_1} - 2\vec{v_2} + \vec{v_3} = \vec{0}$

$$\begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (\vec{v_1}, \vec{v_2}, \vec{v_3} \text{ in } \mathbb{R}^3)$$

**Theorem**

(a) The relations among the vectors $\vec{v_1}, v \dots \vec{v_m}$ correspond to the vectors in $\ker[\vec{v_1}, \vec{v_2}, \dots \vec{v_m}]$

(b) Vectors $\vec{v_1} \dots \vec{v_m}$ in $\mathbb{R}^n$ are independent $\iff \ker[\vec{v_1} \dots \vec{v_m}] = \{\vec{0}\}$

redundant vectors
give relations
# relations ⇒ basis vectors

☆

**Use redundant vectors to find a basis of the kernel**

$$V = \ker \begin{bmatrix} 1 & 0 & 2 & 1 & 2 \\ 1 & 0 & 2 & 2 & 3 \\ 1 & 0 & 2 & 3 & 4 \\ \vec{v_1} & \vec{v_2} & \vec{v_3} & \vec{v_4} & \vec{v_5} \end{bmatrix}$$

Redundant:

$\vec{v_2} = 0$

$\vec{v_3} = 2\vec{v_1}$

$\vec{v_5} = \vec{v_1} + \vec{v_4}$

**Aside** Basis of the Image $\rightarrow$ $\vec{v_1}, \vec{v_4}$ (non-redundant columns)

Basis of the ker $\rightarrow$ $\vec{v_1}$ ~~$2\vec{v_1} + \vec{v_3}$ $\vec{v_1}$ $\vec{v_4}$ $+ \vec{v_5} = \vec{0}$~~

↳ ~~$-3\vec{v_1} + 1\vec{v_2} + \vec{v_3}$ $\vec{v_4}$ $+ \vec{v_5}$~~

→ & basis vectors

relations: $\left( \begin{array}{l} \vec{v_2} = \vec{0} \\ -2\vec{v_1} + \vec{v_3} = \vec{0} \\ -\vec{v_1} - \vec{v_4} + \vec{v_5} = \vec{0} \end{array} \right.$

basis of ker → $\underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{\text{independent.}}$

→ span the kernel. (proof in book)

## Summary

(TFAE)

For an $n \times m$ matrix $A = [\vec{v_1}, \ldots \vec{v_m}]$, the following are equivalent,

(1) $\vec{v_1} \to \vec{v_m}$ are independent. &

(2) No $\vec{v_k}$ is redundant.

(3) $\ker A = \{\vec{0}\}$ (no relations except the trivial)

(4) $\text{rank}(A) = m$

(5) There are no free variable

(6) $\text{rref}(A) = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$ $(n \times m)$ (e.g) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

(7) basis of image $= \{\vec{v_1} \ldots \vec{v_m}\}$ (since they're all independent)

(8) $\vec{Ax} = \vec{b}$ → has 1 or no solution (has at most 1 solution),
↳ since $\text{rank}(A) = m$

(9) The linear map $T(\vec{x}) = A\vec{x}$ is 1-to-1 (since $\ker(A) = \{\vec{0}\}$)

**Exercise.** Show that non-zero, orthogonal vectors $\vec{v_1}, \vec{v_m} \ldots \mathbb{R}^n$ are ind

↳ Write a relation $c_1\vec{v_1} + c_2\vec{v_2} + \ldots + c_m\vec{v_m} = \vec{0}$

Show that $c_j = 0$ ↳ $\vec{v_j}(c_1\vec{v_1} + \ldots + c_m\vec{v_m}) = \vec{0}$

orthogonal → dot product = 0

$$\to \underbrace{c_1\vec{v_1}\vec{v_j}}_{0} + \underbrace{\ldots}_{0} \; c_j \underbrace{\vec{v_j} \cdot \vec{v_j}}_{\substack{\text{sum of }^2 \\ \text{of components}) \neq 0}} + \underbrace{c_m\vec{v_m}\vec{v_j}}_{0} = \vec{0} \;\leftrightarrow\; \boxed{c_j = 0}$$

↳ ∃ only trivial relation

↳ INDEPENDENT

Mar 19, 2018 **Define** Dimension of $V$ is the number of vectors in any basis of $V$

**Theorem** Consider vectors $\vec{v_1} \dots \vec{v_p}$ and $\vec{w_1}, \dots \vec{w_q}$ in subspace $V$ of $\mathbb{R}^n$. If the vectors $\vec{v_1} \dots \vec{v_p}$ are independent and the vectors $\vec{w_1}, \dots \vec{w_q}$ span $V$, ~~then~~ ~~q~~ $p \le q$

Form matrices $A = \begin{bmatrix} \vec{v_1} & \dots & \vec{v_p} \end{bmatrix}$ $(n \times p)$ $\ker(A) = \vec{0}$

$B = \begin{bmatrix} \vec{w_1} & \dots & \vec{w_q} \end{bmatrix}$ $(n \times q)$ $\mathrm{Im}(B) = V$

$\vec{v_1} \dots \vec{v_p} \in V = \mathrm{Im}(B) \rightarrow$ we can write $\vec{v_1} = B\vec{u_1} \dots \vec{v_p} = B\vec{u_p} \dots$ for some $\vec{u_1}, \dots \vec{u_p} \in \mathbb{R}^q$

$\rightarrow A = \begin{bmatrix} \vec{v_1} \dots \vec{v_p} \end{bmatrix} = \begin{bmatrix} B\vec{u_1} \dots B\vec{u_p} \end{bmatrix} = B \underbrace{\begin{bmatrix} \vec{u_1} \dots \vec{v_p} \end{bmatrix}}_{C}$

$\rightarrow \boxed{\underset{(n\times p)}{A} = \underset{(n\times q)}{B} \cdot \underset{(q\times p)}{C}}$ $\rightarrow$ so $\ker(C) \subseteq \ker([illegible]) = \ker(A) = \{\vec{0}\}$

so $\underline{\ker(C) = \{\vec{0}\}}$

**So** $q \geq p$ since $C$ cannot be a wide matrix

**Corollary** if $\vec{v_1} \dots \vec{v_p}$ and $\vec{w_1} \dots \vec{w_q}$ are 2 bases of a subspace $V$ of $\mathbb{R}^n$ then $p = q$

$\hookrightarrow$ basis = independent + span $\rightarrow$ to prove $\rightarrow$ use theorem both ways!

**Definition** Consider subspace $V$ of $\mathbb{R}^n$
The number of vectors in any basis of $V$ is called the Dimension of $V \equiv \dim V$

(Proof: $V$ has a basis ... true all subspace have a basis ...)

**Example** $\dim(\mathbb{R}^n) = n$ since $\vec{e_1} \dots \vec{e_n}$ is a basis.

**Ex** $V$ is given by $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$ in $\mathbb{R}^4$

"general element" $\begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2a - 3b - 4c \\ a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\to \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis of $V$.

$\to \boxed{\dim V = 3}$

$\boxed{\text{A subspace } V \text{ of } \mathbb{R}^n \text{ with } \dim V = n-1 \text{ is called a hyperplane in } \mathbb{R}^n}$

**Ex** find dim of Im & ker of

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 3 \\ 1 & 2 & 2 & 0 & 4 \\ 1 & 2 & 3 & 0 & 5 \end{bmatrix}$$

redundant

**Note** $\dim(\text{Im}) + \dim(\ker) = 5$

Basis of Im: $\vec{v_1}, \vec{v_3} \to \dim(\text{Im}) = 2$

Basis of ker $= \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \to$ ~~Ex~~ $\dim(\ker) = 3$

$\to \dim(\ker) + \dim(\text{Im}) = 2 + 3 = 5.$

In general for $n \times m \to \boxed{\dim(\text{Im}) + \dim(\ker) = m}$

Nov 21, 2018

$$A = \begin{pmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{pmatrix}$$

Find bases and dimension of Im & ker of A

$$\text{rref}(A) = \begin{pmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Note $\text{Im}(A) \neq \text{Im}(\text{rref}\,A)$

Note $\vec{v}$ with leading 1's are non redundant $\Rightarrow$ form basis of Im(rref A

The redundant $\vec{v}$ of rref of A are those w/o leading 1's.

So $\ker(\text{rref}\,A) = \ker(A)$

Note $A\vec{x} = \vec{0}$ and $(\text{rref}\,A)\vec{x} = \vec{0}$ have the same solutions

↳ by construction, so $\ker(\text{rref}\,A) = \ker(A)$

$\Leftrightarrow$ The relations among $\vec{v}_1, \dots \vec{v}_5$ correspond to those among $\vec{w}_1, \dots \vec{w}_5$

↳ $\vec{v}_j$ redundant $\Leftrightarrow$ $\vec{w}_j$ redundant

$\rightarrow$ Basis of $\text{Im}(A) = \vec{v}_1, \vec{v}_3$ $\rightarrow$ $\dim(\text{Im}(A)) = 2 = \text{rank}(A)$

Note $\boxed{\text{Dim}(\text{Im}(A)) = \text{rank}(A)}$

To construct a basis of the Im of A, pick the columns of A that corresponds to the columns of rref A containing leading 1's.

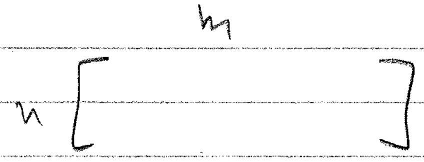
Ex A linear map $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^m$ to $\mathbb{R}^n$ is onto iff rank(A) = n

$$\left| \begin{matrix} x_1 + 2x_2 + 3x_4 - 4x_5 = 0 \\ x_3 - 4x_4 + 5x_5 = 0 \end{matrix} \right| \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4a - 3b - 2c \\ 4\overset{c}{b} - 5a \\ b \\ a \\ \end{bmatrix}$$

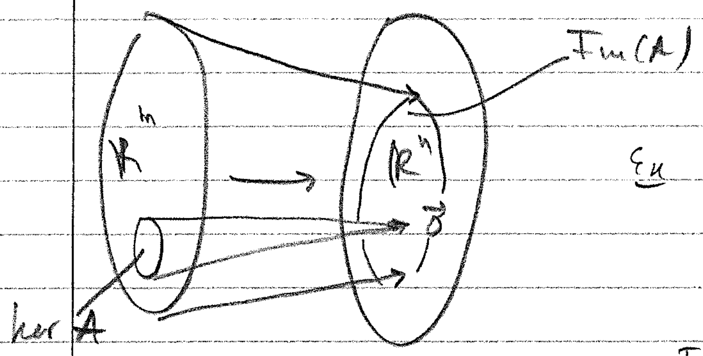
$$= a\begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \end{bmatrix} + c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

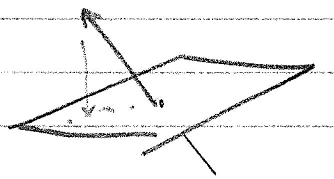
$\dim(\ker A) = 3 \rightarrow \boxed{\dim(\ker A) = m - \operatorname{rank}(A)}$

or $\boxed{\underset{\substack{\uparrow \\ \text{nullity}}}{\dim(\ker A)} + \underset{\substack{\uparrow \\ \operatorname{rank}(A)}}{\dim(\operatorname{Im} A)} = m} \leftarrow$ rank-nullity theorem



$\boxed{\dim(\operatorname{Im} A) = \cancel{\dim} \; m - \dim(\ker A)}$





Ex $A \in \mathbb{R}^{3\times3}$ projection onto plane $V$

$\operatorname{Im} A = V$

$\ker A = \text{norm of } V = V^{\perp}$

$\dim \overset{\operatorname{Im}}{A} = 3 - \dim(\ker A) \leftarrow$ a line

$= 3 - 1$

$\dim(\operatorname{Im} A) = 2$

Mar 23, 2019 $\boxed{\text{Notation for bases}}$

$\hookrightarrow \mathcal{A}, \boxed{\mathcal{B},} \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \ldots$ - denote bases

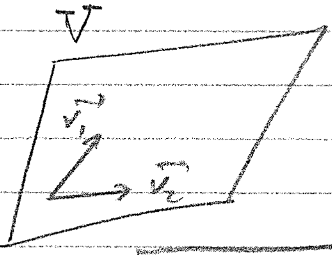
Eg $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$

$\mathcal{A} = (\vec{e}_1, \ldots \vec{e}_n) \leftarrow$ standard bases

## 3.4 $\boxed{\text{Coordinates}}$

**Def** { A coordinate system on a set X is a 1-to-1 function from X to $\mathbb{R}^n$ for some n.

Ex Consider plane V in $\mathbb{R}^3$ with $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$



Any vector $\vec{x}$ on V can be expressed uniquely as

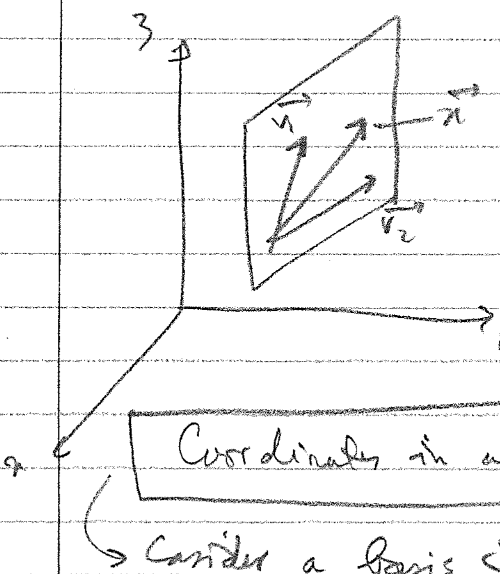
$\boxed{\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2}$ for some scalars $c_1, c_2 \in \mathbb{R}$

$\boxed{c_1 \text{ and } c_2 \text{ are called the } \mathcal{B}\text{-coordinates of } \vec{x}}$

↳ $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathcal{B}$-coordinate vector.

Numerical Ex $\longrightarrow$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$ ↗ $\mathbb{R}^3$

$\vec{x} = 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ so $c_1 = 3$, $c_2 = 2$



so $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \vec{x}$ in V

↑ $\vec{x}$ in $\mathcal{B}$

### $\boxed{\text{Coordinates in a subspace of } \mathbb{R}^n}$

↳ Consider a basis $\mathcal{B} = (\vec{v}_1, \ldots \vec{v}_m)$ of a subspace V of $\mathbb{R}^n$ $(m \leq n)$. Any $\vec{x}$ in V can be written uniquely as

$\vec{x} = c_1\vec{v}_1 + \ldots + c_m\vec{v}_m$ for some $c_1, \ldots c_m \in \mathbb{R}$

Scalars $c_1, \ldots, c_m$ are the $\mathcal{B}$-coordinates of $\vec{x}$, and $[c_1, \ldots, c_m] \in \mathbb{R}^m$ is the $\mathcal{B}$-coordinate vector, denoted by $[\vec{x}]_{\mathcal{B}}$

ie $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$ means that $\boxed{\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m}$

Show → linear combinations are unique!

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$$

$$= d_1\vec{v}_1 + d_2\vec{v}_2 + \ldots + d_m\vec{v}_m$$

this is the trivial sln

$$\Rightarrow \vec{0} = (c_1 - d_1)\vec{v}_1 + \ldots + (c_m - d_m)\vec{v}_2 = \sum_1^m (c_j - d_j)\vec{v}_j$$

$\Leftrightarrow$ $\boxed{c_j = d_j \ \forall j}$ since $\vec{v}_1 \ldots \vec{v}_m$ are independent...

**Conversion formula** between $\vec{x}$ and $[\vec{x}]_{\mathcal{B}}$

$$\vec{x} = c_1\vec{v}_1 + \ldots + c_m\vec{v}_m = \begin{bmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \ldots & \vec{v}_m \\ \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$$

$$\boxed{\vec{x} = \underbrace{\begin{bmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \ldots & \vec{v}_m \\ \downarrow & & \downarrow \end{bmatrix}}_{[\mathcal{B}] \text{ or } S} [\vec{x}]_{\mathcal{B}}} \rightarrow \vec{x} = \underset{\substack{\updownarrow \\ S}}{[\mathcal{B}]} [\vec{x}]_{\mathcal{B}}$$

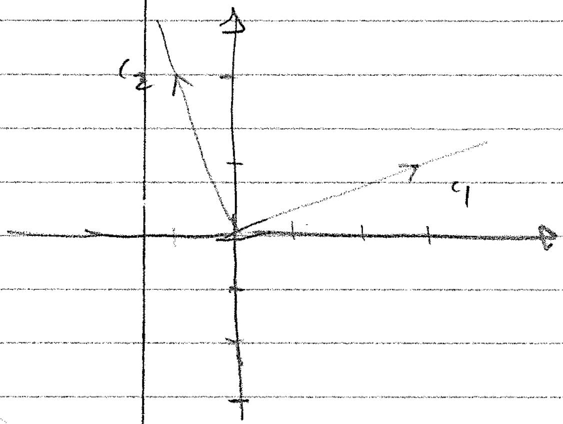
$$\boxed{\vec{x} = S[\vec{x}]_{\mathcal{B}}} \quad \text{where } S = \begin{bmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \ldots & \vec{v}_m \\ \downarrow & & \downarrow \end{bmatrix}$$

In our ex $\vec{v}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $\vec{x} = \begin{pmatrix}5\\7\\9\end{pmatrix}$

↳ $\vec{x} = S[\vec{x}]_{\mathcal{B}} \Rightarrow \begin{bmatrix}5\\7\\9\end{bmatrix} = \begin{pmatrix}1&1\\1&2\\1&3\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix}$

$\boxed{\text{Ex}}$

Consider $\mathcal{B}$ of $\mathbb{R}^2$ : $\mathcal{B} = \left(\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}-1\\3\end{bmatrix}\right)$



if $\vec{x} = \begin{pmatrix}5\\5\end{pmatrix}$ find $[\vec{x}]_{\mathcal{B}}$

↳ $\boxed{\vec{x} = S[\vec{x}]_{\mathcal{B}}}$ → $\begin{bmatrix}2\\1\end{bmatrix}$

↳ $\begin{pmatrix}5\\5\end{pmatrix} = \begin{pmatrix}3&-1\\1&3\end{pmatrix}[\vec{x}_{\mathcal{B}}]$

or $\vec{x} = S[\vec{x}]_{\mathcal{B}} \rightarrow \boxed{[\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x}}$

↳ $[\vec{x}]_{\mathcal{B}} = \frac{1}{10}\begin{pmatrix}3&1\\-1&3\end{pmatrix}\begin{pmatrix}5\\5\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$

if $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix}1\\-1\end{pmatrix}$ ↳ $\vec{x} = \begin{pmatrix}4\\-2\end{pmatrix}$

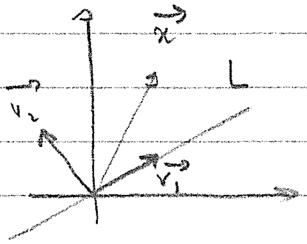
April 2, 2018

## Change of basis for a linear map

Ex let $T(\vec{x}) = \text{proj}_{\vec{x}} = \text{span}\begin{bmatrix}3\\1\end{bmatrix}$ in $\mathbb{R}^2$.

↳ choose a basis $\mathcal{B}$ that is "well-adjusted" to T

↳ choose $\vec{v}_1 = \begin{pmatrix}3\\1\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}-1\\3\end{pmatrix}$



So $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ and $[T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 \longrightarrow [\vec{x}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$T(\vec{x}) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2)$$

$$= c_1\vec{v}_1 + 0 \longrightarrow [T(\vec{x})]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}$$

$$\hookrightarrow [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$$

So $\begin{bmatrix} c_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

→ diagonal

$\boxed{[T(\vec{x})]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}}$ and $B \equiv \mathcal{B}$-matrix of $T(\vec{x})$

**Theorem 1**

Consider a linear transform $T$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $\mathcal{B} = (\vec{v}_1 \ldots \vec{v}_n)$ of $\mathbb{R}^n$, there exists a unique matrix $B$ such that

$$[T(\vec{x})]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}} \quad \forall \vec{x} \in \mathbb{R}^n$$

$B$ is called the

$\mathcal{B}$-matrix of $T$

$$\vec{x} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$$

$$T(\vec{x}) = c_1 T(\vec{v}_1) + \ldots + c_n T(\vec{v}_n)$$

$$[T(\vec{x})]_{\mathcal{B}} = c_1[T(\vec{v}_1)]_{\mathcal{B}} + c_2 \ldots + c_n[T(\vec{v}_n)]_{\mathcal{B}}$$

$$= \begin{pmatrix} [T(\vec{v}_1)]_{\mathcal{B}} & \ldots & [T(\vec{v}_n)]_{\mathcal{B}} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

So $\boxed{[T(\vec{x})]_{\mathcal{B}} = \underbrace{\begin{pmatrix} [T(\vec{v}_1)]_{\mathcal{B}} & \ldots & [T(\vec{v}_n)]_{\mathcal{B}} \end{pmatrix}}_{B} [\vec{x}]_{\mathcal{B}}}$

$B \longrightarrow$ [illegible] solution

**Addendum to Theorem 1**

$$\boxed{B = \begin{pmatrix} [T(\vec{v}_1)]_{\mathcal{B}} & \ldots\ldots & [T(\vec{v}_n)]_{\mathcal{B}} \end{pmatrix}}$$

**Revisit Example 1**

$$T(\vec{v}_1) = \vec{v}_1 \longrightarrow [T(\vec{v}_1)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$T(\vec{v}_2) = \vec{0} \longrightarrow [T(\vec{v}_2)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

**So** $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Why is B diagonal? because $\vec{v}_1, \vec{v}_2$ are eigenvectors!

**Example** Let $T(\vec{x})$ be a linear transform $\mathbb{R}^2 \to \mathbb{R}^2$

$\mathcal{B} = (\vec{v}_1, \vec{v}_2)$ be a basis of $\mathbb{R}^2$ such that $T(\vec{v}_1) = \lambda_1 \vec{v}_1$

$T(\vec{v}_2) = \lambda_2 \vec{v}_2$

Find the $\mathcal{B}$-matrix of T.

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2$$

$$T(\vec{x}) = c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 \quad \underline{\text{So}} \quad [T(\vec{x})]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \end{pmatrix} = \begin{bmatrix} & ? & \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

**So** $B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

**Theorem** Let T be a linear transform $\mathbb{R}^n$ to $\mathbb{R}^n$ and let $\mathcal{B} = (\vec{v}_1 \ldots \vec{v}_n)$ be a basis of $\mathbb{R}^n$ s.t $T(\vec{v}_k) = \lambda_k \vec{v}_k$ for $1 \le k \le n$ then the matrix B of T will be diagonal with ~~then~~

$$B = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & \ddots & \lambda \end{pmatrix}$$

What is the relationship between A & B?

$\boxed{\text{Question}}$ → If $T(\vec{x}) = A\vec{x}$, and B is the $\mathcal{B}$-matrix of T, what is the relationship between A and B?

$$\begin{tikzcd}
\vec{x} \arrow[r, "A"] & T(\vec{x}) \\
{[\vec{x}]_{\mathcal{B}}} \arrow[u, "S"] \arrow[r, "B"'] & {[T(\vec{x})]_{\mathcal{B}}} \arrow[u, "S"']
\end{tikzcd}$$

so $\boxed{AS = SB}$

so $A = SBS^{-1}$

$B = S^{-1}AS$

April 4, 2018

$\boxed{\text{Rules for vector algebra}}$

(1) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
(2) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
(3) $\exists\ \vec{n}$ unique $\in \mathbb{R}^n$ s.t $\vec{v} + \vec{n} = \vec{v}\ \forall \vec{x}$, namely $\vec{n} = \vec{0}$
(4) For every $\vec{v}$, $\exists$ a unique $\vec{v}^*$ s.t $\vec{v} + \vec{v}^* = \vec{n} = \vec{0}$.
(5) $k(\vec{v} + \vec{w}) = k\vec{v} + k\vec{w}$
(6) $(c+k)\vec{v} = c\vec{v} + k\vec{v}$
(7) $(ck)\vec{v} = c(k\vec{v})$
(8) $1\vec{v} = \vec{v}$

Aside → A fn $f: \mathbb{R} \mapsto \mathbb{R}$ is ~~said~~ to be smooth (or $C^\infty$) if it has derivatives of all orders

→ The set of all smooth functions $f: \mathbb{R} \mapsto \mathbb{R}$ is denoted by $C^\infty$

Ex polynomials, $\frac{1}{1+x^2}$, $e^x$, ...

Non-example $\frac{1}{x}$, $|x|$, $\tan x$, $x|x|$, $x^n|x|^n$

## 4.1. Intro to Linear Spaces (Vector Spaces)

**Ex** Find all smooth fn, $f: \mathbb{R} \longmapsto \mathbb{R}$ s.t $f''(x) = -f(x)$, with solution set $V$

**Guess**: $\sin(x)$, $\cos(x)$ & their linear combinations...

$$\boxed{F(x) = a\sin(x) + b\cos(x)} \longleftarrow$$

We are told (Problem 4.1.58) that all fns in $V$ are of this form

$$V = \{ a\sin(x) + b\cos(x) \mid a, b \in \mathbb{R} \} \subseteq C^\infty$$

∘ How many solutions are there? → well, infinitely many, but...

↳ By analogy, $(\sin x, \cos x)$ is a "basis" of $V$, and "$\underline{\dim V = 2}$"

**Def** → A linear space / vector space $V$ is a set endowed with a rule for addition and a rule for multiplication with real numbers subject to rules (1−8)

for $u, v, w \in V$, $c, k \in \mathbb{R}$

(1) $u + (v+w) = (u+v) + w$

(2) $v + w = w + v$

(3) $\exists\, n \in V$ s.t $v + n = v \;\forall v$, denoted by $n = 0$

(4) for any $v \in V$ $\exists\, v^*$ s.t $v + v^* = 0$, denoted by $v^* = -v$

(5) $c(v+w) = cv + cw$

(6) $(c+k)v = cv + kv$

(7) $(ck)v = c(kv)$

(8) $1v = v$

ril 6, 2018

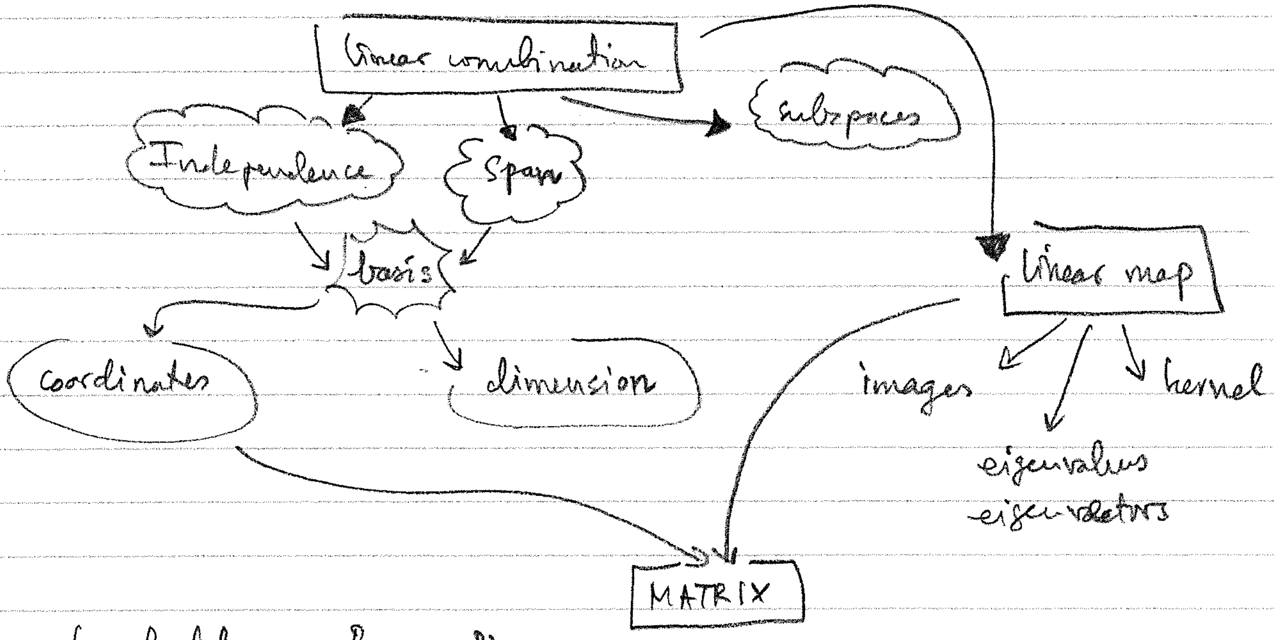
If $x, y$ are sets, then $F(x,y)$ denotes the set of all functions $f: x \to y$

## 4.1 _ Linear Spaces

Def: A linear space ... (previous page)

Example ① $\mathbb{R}^n$ ② $\mathbb{R}^{n\times m}$ ③ $F(\mathbb{R}, \mathbb{R})$ ④ $F(X, \mathbb{R})$ where $X$ is any set
⑤ Sequences (infinite) ⑥ Exotic Example $V = \mathbb{R}^+$
where $x \oplus y = xy$ So $n = 1$
$k \odot v = v^k$

Diagram



Example of Subspace, Bases, Dimension:

① Find basis and dim of 2x2 matrices. $(\mathbb{R}^{2\times 2})$

Write a typical element with parameters as linear combination, with parameters as the coefficient

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

span by construction
independent by 0-1 argument

So $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is a basis of $\mathbb{R}^{2\times 2}$ (

So $\dim(\mathbb{R}^{2\times 2}) = 4$

So $\dim(\mathbb{R}^{n\times m}) = (n \cdot m)$

② Let $P_2 = \{a + bx + cx^2 \,,\ a, b, c \in \mathbb{R}\} \subseteq F(\mathbb{R}, \mathbb{R})$

(a) show that $P_2$ is a subspace of $F(\mathbb{R}, \mathbb{R})$

$$\Rightarrow \begin{cases} 0 \in P_2 \\ P_2 \text{ closed under addition} : a_1 + b_1x + c_1x^2 + a_2 + b_2x + c_2x^2 \in P_2 \\ P_2 \text{ closed under scaler multiplication} \end{cases}$$

(b) Find the basis of $P_2$

↳ $a + bx + cx^2 \Rightarrow 1, x, x^2$ is a basis of $P_2$ by construction.

$1, x, x^2$ are independent

So $\boxed{\dim(P_2) = 3}$

So $\boxed{\dim(P_n) = n+1}$

Let $P$ be set of all polynomials $\Rightarrow$ a subspace of $F(\mathbb{R}, \mathbb{R})$

Is there a finite basis? (NO)

↳ assume basis

$$\begin{cases} f_1 \dots f_n \\ \text{Let } N \text{ be the max of the degrees of those } f_1 \dots f_n \\ \rightarrow x^{N+1} \notin \text{span}(f_1 \dots f_n) \end{cases}$$

So $f_1 \dots f_n$ not a basis

April 9, 2018

Let $P$ be the space of all polynomials, a subspace of $C^\infty$. $P$ doesn't have a (finite) basis

$\boxed{\text{Convention}}$ {All bases considered in this course are assumed to have finite bases}

**Def.** A linear space $V$ is said to have finite-dimensional if it has a (finite) basis, otherwise it is said to be infinite-dimensional

**Theorem**

{ if the number of independent vectors in a linear space $V$ is unbounded, then $V$ is infinitely-dimensional }

→ Proof by contraposition if $n$-dim, then dim bounded by $n$.

**Theorem** | $P$ is infinite-dimensional, $1, x, x^2, x^{n-1}$ are $n$-independent polynomials for any $n$

Other example: $C^{\infty}$, $F(\mathbb{R} \to \mathbb{R})$ is infinite-dimensional

- The space of all infinite sequences $(x_1, x_2, x_3 \dots x_n \dots)$ is infinite-dimensional

  ↳ $(1, 0, 0 \dots), (0, 1, 0 \dots), \dots (0, 0, 0 \dots 1, 0)$ are independent by the 0-1 argument.

**Example** $V = \{ f \in P_2,\ f(3) = 0 \}$, a subspace of $P_2$

Find a basis and the dim of $V$

by inspection $(x-3)$, $x^2-9$ is a basis ⇒ dim $V = 2$

Systematic approach → write a typical element of $P_2$:

→ $f(x) = ax^2 + bx + c$

→ It is required that $9a + 3b + c = 0$

→ Solve for leading variable

→ $a = -\frac{1}{3}b - \frac{1}{9}c$

→ plug in

↳ $\cancel{\left(-\frac{1}{3}b - \frac{1}{9}c\right)x^2 + bx + c = 0}$

$\cancel{b\left(-\frac{1}{3}x^2 + x\right)}$ $\cancel{c\left(-\frac{1}{9}x^2 + 1\right)}$ → solve for c

basis $(x-3), (x^2-9)$

$= ($[illegible]$)$

then write linear ←

each [illegible]

Linear map

Consider linear spaces $V$ and $W$. A fn $T : V \to W$ is said to be a linear map if

$$\begin{cases} T(x+y) = T(x) + T(y) & \text{for all } x, y \in V \quad (\text{sum rule}) \\ T(kx) = kT(x) & \forall\ x \in V \quad (\text{scaling rule}) \end{cases}$$

Example $D : C^{\infty} \to C^{\infty}$ $D(f) = f'$, the derivative.

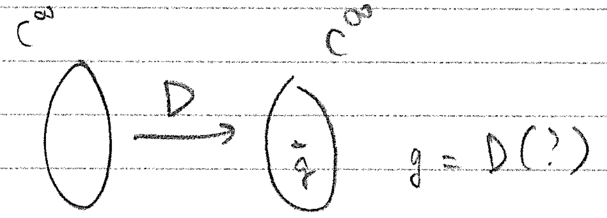
$D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$

ker(D) : $D(f) = 0 = f' \Rightarrow f(x) = k$ for $k \in \mathbb{R}$

$\Rightarrow \ker(D) = \text{span}(1)$

$\hookrightarrow \dim(\ker(D)) = 1$

Im(D)



since $g$ is continuous, it has an antiderivative

$\hookrightarrow G$, so $G' = g = D(G)$, $G \in C^{\infty}$

so $\boxed{\text{Im}(D) = C^{\infty}}$

$\hookrightarrow$ D close to invertible

Example

$\hookrightarrow$ $T : \mathbb{R}^{2\times2} \longmapsto \mathbb{R}^{2\times2}$

$T(A) = S^{-1}AS$ linear map? Yes!

kernel $T(A) = 0 = S^{-1}AS \Rightarrow A = 0$

$\ker(T) = \{0\}$ $\rightarrow$ 1-to-1

$\dim(\text{Im}\,T) = 4 \Rightarrow \text{Im}\,T = \mathbb{R}^{2\times2}$ $\rightarrow$ onto

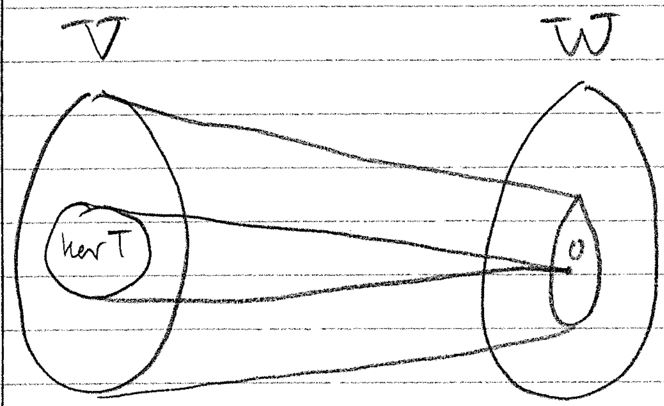
so T invertible.

April 11, 2018

## Rank - Nullity Theorem

If $T : V \mapsto W$ is a linear map, and if $V$ is finite-dimensional then

$$\dim V = \underbrace{\dim \ker T}_{\text{nullity}} + \underbrace{\dim \operatorname{Im} T}_{\text{rank}}$$



## 4.2 · Isomorphism

"Isos": same
"morphe": structure

Coordinates → Consider a linear space $V$ with a basis $\mathcal{B} = (f_1 \dots f_n)$
Then any $f$ in $V$ can be written uniquely as a linear combination

↳ $f = c_1 f_1 + \dots + c_n f_n$

The coefficients $c_1 \dots c_n$ are called the $\mathcal{B}$-coordinates of $f$

$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$ is the $\mathcal{B}$-coordinate vector, denoted with $[f]_{\mathcal{B}}$

We can define the coordinate map: $L_{\mathcal{B}}$

↳ $L : V \mapsto \mathbb{R}$ ; $L(f) = [f]_{\mathcal{B}}$

or

$$\boxed{L(c_1 f_1 + \dots + c_n f_n) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}$$

<u>Claim</u> that $L$ is a linear map, meaning that

$$L(f+g) = L(f) + L(g) \quad \text{and} \quad \boxed{L(kf) = kL(f)} \quad (*)$$

<u>Show $(*)$</u> $\quad f = c_1f_1 + \dots + c_nf_n \rightarrow kf = kc_1f_1 \dots + kc_nf_n$

$$L(kf) = \begin{pmatrix} kc_1 \\ \vdots \\ kc_n \end{pmatrix} = k\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = kL(f)$$

<u>Note</u> $L$ is invertible, its inverse is:

$$L^{-1} : \mathbb{R}^n \rightarrow V, \quad L^{-1}\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = c_1f_1 + \dots + c_nf_n$$

<u>Summary</u>

$$V \longleftrightarrow \mathbb{R}^n$$

← structure preserving ... as linear space

$$\begin{tikzcd} c_1f_1 + \dots + c_nf_n \arrow[r, shift left, "L"] & \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \arrow[l, shift left, "L^{-1}"] \end{tikzcd}$$

<u>Example</u>

$V = P_2$, $\quad \mathcal{B} = (1, x, x^2)$ standard basis

$$L(a + bx + cx^2) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$P_2 \longleftrightarrow \mathbb{R}^3$$

$$\begin{tikzcd} a + bx + cx^2 \arrow[r, shift left, "L"] & \begin{pmatrix} a \\ b \\ c \end{pmatrix} \arrow[l, shift left, "L^{-1}"] \end{tikzcd}$$

isomorphic

$P_2$ and $\mathbb{R}^3$ have the "same structure" as linear spaces

↳ what do we mean by this?

Isomorphism = Invertible + linear

**Definition**

(a) An invertible linear map L from $V \to W$ is called an isomorphism

(b) The space $V$ is said to be isomorphic to $W$ if there exists ~~L invertible~~ an isomorphism $L : V \to W$

→ We write $V \simeq W$

Ex

$P_2 \longmapsto \mathbb{R}^3$

$L(a + bx + cx^2) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is an isomorphism. Therefore $P_2$ is isomorphic to $\mathbb{R}^3$.

More generally, if $V$ is $n$-dimensional, then $V$ is isomorphic to $\mathbb{R}^n$

↳ Proof: Consider a coordinate transformation...

Theorem

↳ Isomorphism is an equivalence relation

→ (a) Reflexive $V \simeq V$ since: $id : V \mapsto V$ isomorphism

→ (b) Symmetric if ~~V is iso~~ $V \simeq W$, then $W \simeq V$

If $T$ is an isomorphism $V \simeq W$

then $T^{-1} : W \longmapsto W$ is also an isomorphism

↳ We need to show that

$T^{-1}$ is linear (so that it's an isomorphism)

↳ Consider sum rule $T^{-1}(f+g)$

$$= T^{-1}\left(T(T^{-1}(f)) + T(T^{-1}(g))\right)$$

(since $T$ _ linear)

$$= T^{-1}\left(T\left(T^{-1}(f) + T^{-1}(g)\right)\right)$$

$$= T^{-1}(f) + T^{-1}(g)$$

So $T^{-1}(f+g) = T^{-1}(f) + T^{-1}(g)$

→ (c) Transitivity : If $V \simeq W$ , and $W \simeq U$ then $V \simeq U$

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[rr, bend right, "F \circ L"'] & W \arrow[r, "F"] & U \end{tikzcd}$$

Claim $F \circ L$ is an isomorphism from $V \longmapsto U$

Proof

$F \circ L$ invertible, linear

Scaling rule

$$F(L(kf)) = F(fL(f)) = fF(L(f))$$

Apr 17, 2018 An isomorphism is an invertible linear map

Theorem | a linear map $T: V \to W$ is an isomorphism iff $\ker T = \{\vec{0}\}$ and $\operatorname{Im} T = W$

Theorem | Two finite-dimensional spaces $V$ and $W$ are isomorphic $\Leftrightarrow \dim V = \dim W$

Proof Suppose $V \simeq W$ (isomorphic)

↳ $\exists$ an isomorphism $T: V \longmapsto W$

by rank-nullity theorem

$\ker T = 0$

$\dim \operatorname{Im} T = W$

$$\dim V = \dim \ker T + \dim \operatorname{Im} T = 0 + \dim W$$

So $\dim V = \dim W$

Converse Suppose $\dim W = \dim V = n$

Note $V \simeq \mathbb{R}^n$ (coordinate...)

$W \simeq \mathbb{R}^n \Rightarrow \mathbb{R}^n \simeq W$ (sym)

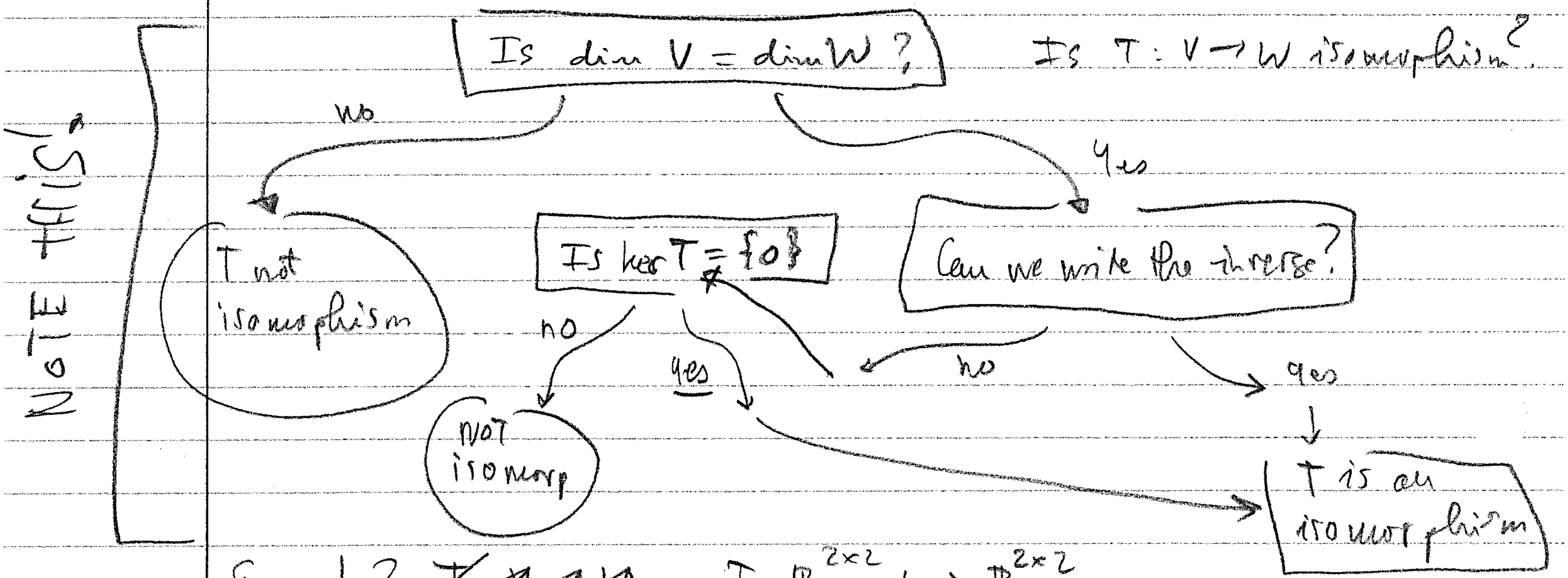
So $V \simeq W$ (transitivity)

not onto

Example 1 Is $T: \underbrace{P_2 \to \mathbb{R}^2}$ : $T(f(x)) = \begin{pmatrix} f(1) \\ f(2) \end{pmatrix}$ an iso morphism?

$\dim P_2 = 3$
$\dim \mathbb{R}^2 = 2$ → T not isomorphism

or counter example → $f(x) = (x-1)(x-2)$



Example 2 : $T: \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$

$B = T(A) = S^{-1}AS$ where $S = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Solve for A $A = SBS^{-1} = T^{-1}(B)$ ✓ T is an isomorphism

Example 3 → Is $T: P_2 \to \mathbb{R}^3$

$T(f(x)) = \begin{pmatrix} f(1) \\ f(2) \\ f(3) \end{pmatrix}$ isomorphism?

$\ker T = \{ f(x) \in P_2 : f(1) = f(2) = f(3) = 0 \} = \{0\}$

$\mathrm{Im}\, T = \mathbb{R}^{\cancel{3}2}$

Since the rank nullity → $\dim P_2 = \dim \cancel{\ker T}^{0} + \dim \mathrm{Im}\, T$

But $\dim P_2 = \dim \mathbb{R}^3 = 3$ → $\boxed{\mathrm{Im}\, T = \mathbb{R}^3}$ → T is isomorph

## The matrix of a linear map

↳ Consider linear map $T : P_2 \mapsto P_2$

$$T(f) = f' + f''$$

$$T(a + bx + cx^2) = b + 2cx + 2c = (b+2c) + (2cx)$$

$\mathfrak{B} = (1, x, x^2)$

standard basis of $P_2$

$$\begin{tikzcd} f(x) = a + bx + cx^2 \arrow[r, "T"] \arrow[d] & (b+2c) + 2cx \arrow[d] \\ {[f(x)]_{\mathfrak{B}} = (a, b, c)^T} \arrow[r, "{B = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}}"'] & {[T(f(x))] = \begin{pmatrix} b+2c \\ 2c \\ 0 \end{pmatrix}} \end{tikzcd}$$

So $[T(f(x))]_{\mathfrak{B}} = B[f]_{\mathfrak{B}}$

$\operatorname{Im} B = \operatorname{span} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$     eigenvalue of $B = \{0\}$

$\operatorname{Im} T = \operatorname{span}(1, 2+2x) = P_1$     So os eigenvalues of $T$

$\ker B = \operatorname{span}(\vec{e}_1)$

$\ker T = \operatorname{span}(1) = P_0$

April 16, 2018 → Consider a linear map $T : V \mapsto V$ and a basis $\mathfrak{B} = (f_1, \dots f_n)$ of $V$
Then the $\mathfrak{B}$-matrix $B$ of $T$ is defined by

$$[T(f)]_{\mathfrak{B}} = B[f]_{\mathfrak{B}} \quad \text{for all } f \in V$$

We can construct $B$ column by column

$$B = \begin{pmatrix} [T(f)]_B & \cdots & [T(f_n)]_B \end{pmatrix}$$

**Proof**

$$f = c_1 f_1 \cdots c_n f_n$$

$$T(f) = c_1 T(f_1) + \cdots + c_n T(f_n)$$

$$[T(f)]_B = c_1 [T(f)]_B + \cdots + c_n [T(f)]_B = B [f]_B$$

**Revisit** $T(f) = f' + f''$ $\quad \mathcal{B} = (1, x, x^2)$

$$B = \begin{pmatrix} [T(1)]_B \,, & [T(x)]_B \,, & [T(x)]_B \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

**We write**

$$B = \begin{array}{cccc} T(1) & T(x) & T(x^2) & \\ \left(\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \left.\begin{matrix} 2 \\ 2 \\ 0 \end{matrix}\right) & \begin{matrix} 1 \\ x \\ x^2 \end{matrix} \end{array}$$

So $T(x^2) = 2 + 2x$

**Example** $T : V \mapsto V$ $\quad \mathcal{B} = (f_1, f_2)$ basis of $V$

$$T(f_1) = a f_1 + b f_2$$
$$T(f_2) = c f_1 + d f_2$$

**B of T is?** $\quad$ ~~Bc=~~

$$\boxed{B = \begin{array}{ccc} T(f_1) & T(f_2) & \\ \left(\begin{matrix} a \\ b \end{matrix}\right. & \left.\begin{matrix} c \\ d \end{matrix}\right) & \begin{matrix} f_1 \\ f_2 \end{matrix} \end{array}}$$

## When is B diagonal?

B is diagonal $\Leftrightarrow$ $b = c = 0$

$\hookrightarrow$ $\Leftrightarrow$ $T(f_1) = af_1$ and $T(f_2) = df_2$

$\Leftrightarrow$ $f_1, f_2$ are eigenvectors of $T$

//

**Theorem** Consider a linear map $T : V \mapsto V$ and a basis $\mathcal{B}(f_1 \dots f_n$

Then the $\mathcal{B}$ matrix $B$ of $T$ is diagonal $\Leftrightarrow$

$f_1 \dots f_n$ are eigenvectors of $T$.

//

**Example** $T(f(x)) = f(1+2x)$ $P_2$ to $P_2$

For $\mathcal{B} = (1, x, x^2)$, Find $\mathcal{B}$-matrix $B$ of $T$, use it to find eigenvalues & eigenfunctions of $T$.

$$B = \left( [T(1)]_{\mathcal{B}} \ \ [T(x)]_{\mathcal{B}} \ \ [T(x^2)]_{\mathcal{B}} \right)$$

$$= \left( [1]_{\mathcal{B}} \ \ [1+2x]_{\mathcal{B}} \ \ [(1+2x)^2]_{\mathcal{B}} \right)$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{pmatrix} \Rightarrow \lambda = 1, 2, 4$$

$\boxed{\text{Eigenvectors of } B = ?} \longrightarrow$

1-eigenvectors $\rightarrow$ $\boxed{\text{find } \ker(B - \lambda I)}$

<u>1 - eigenvectors</u>: $\ker(B - I_3) = \ker\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$

$= \text{span}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$   (look for relations)

<u>2 - eigenvectors</u> $\ker(B - 2I_n) = \ker\begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{pmatrix}$

$= \text{span}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

<u>4 - eigenvectors</u> $\ker(B - 4I_n) = \ker\begin{pmatrix} -3 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{pmatrix}$

$= \text{span}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

<u>So</u> what ~~are~~ is a basis of $P_2$ consisting of eigenfunctions of T?

$$(1 \;\; x \;\; x^2)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1$$

$$(1 \;\; x \;\; x^2)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 + x$$

$$(1 \;\; x \;\; x^2)\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 1 + 2x + x^2 = (1+x)^2$$

$$T(1+x) = f(1+2x) = 1 + 2(1+x) = 2x+2 = 2(1+x)$$

$$T((1+x)^2) = 4(1+x)^2$$

<u>So</u> $\mathcal{C}$-matrix C of T $(\mathcal{C} = (1, 1+x, (1+x)^2))$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

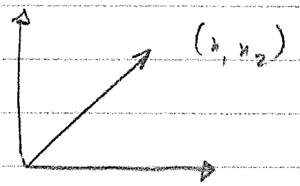
#

April 23, 2018

## $\boxed{\text{Chapter 5: Euclidean Geometry in } \mathbb{R}^n}$

<u>in $\mathbb{R}^2$</u>: Length

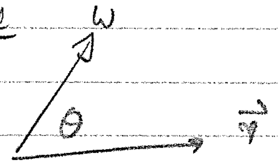
$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2} = \sqrt{\vec{x} \cdot \vec{x}}$$



$$\boxed{\|\vec{x}\|^2 = \vec{x} \cdot \vec{x}}$$

$\vec{v}, \vec{w} \in \mathbb{R}^n$, $\vec{v}, \vec{w} \neq \vec{0}$

<u>Angle</u>



$$\boxed{\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos\theta}$$

$$\theta = \cos^{-1}\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right)$$

<u>Geometry in $\mathbb{R}^n$</u>

<u>Define</u> <u>Length</u> $\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$ for $\vec{x} \in \mathbb{R}^n$

<u>Angle</u>

$$\measuredangle(\vec{v}, \vec{w}) = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right) \text{ for } \vec{v}, \vec{w} \in \mathbb{R}^n \setminus \{\vec{0}\}$$

Need to show $\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|} \in [-1, 1]$

need to ~~prove~~ the Cauchy - Schwartz - inequality $|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\|\|\vec{w}\|$

- $\vec{v}, \vec{w}$ in $\mathbb{R}^n$ are said to be orthogonal if $\vec{v} \cdot \vec{w} = 0$

- Vectors $\vec{v}_1, \ldots \vec{v}_m$ in $\mathbb{R}^n$ are said to be <u>orthonormal</u> if they are <u>orthogonal</u> <u>unit</u> vectors

$$\boxed{\vec{v_i}\cdot\vec{v_j} = \begin{cases} 0 & \text{if } i\neq j \\ 1 & \text{if } i=j \end{cases}}$$

T/F n of the orthonormal vectors in $\mathbb{R}^n$, must form a basis?

↳ Suffices to show that they're independent

↳ Consider a relation $c_1\vec{v_1} + c_k\vec{v_k} + c_n\vec{v_n} = \vec{0}$

Show $c_k = 0$

So $c(\vec{v_1}\cdot\vec{v_k}) + \dots + c_k(\vec{v_k}\cdot\vec{v_k}) + \dots c_n(\vec{v_n}\cdot\vec{v_k}) = \vec{0}\cdot\vec{v_k}$

So $\boxed{c_k = 0}$

Examples of orthonormal basis (ONB)

$\mathbb{R}^n$ : $\vec{e_1},\dots,\vec{e_n}$   $\mathbb{R}^2$ : $\begin{bmatrix}0.6\\0.8\end{bmatrix}, \begin{bmatrix}0.8\\-0.6\end{bmatrix}$

$\mathbb{R}^3$ $\frac{1}{3}\begin{bmatrix}2\\2\\1\end{bmatrix}, \frac{1}{3}\begin{bmatrix}2\\-1\\-2\end{bmatrix}, \frac{1}{3}\begin{bmatrix}-1\\2\\-2\end{bmatrix}$

**Theorem of Pythagoras**

Consider $\vec{x}, \vec{y}$ in $\mathbb{R}^n$ $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ iff $\vec{x}\cdot\vec{y} = 0$

Proof $\|\vec{x}+\vec{y}\|^2 = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) = \vec{x}\cdot\vec{x} + \vec{y}\cdot\vec{y} + 2\vec{x}\cdot\vec{y}$

$= \|\vec{x}\|^2 + \|\vec{y}\|^2 + 0$

**Definition**

A linear map $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^n \to \mathbb{R}^n$ is said to be an orthogonal map if it preserves length, meaning that the

$\|T(\vec{x})\| = \|\vec{x}\| \ \forall \vec{x} \in \mathbb{R}^n$

weird → unfortunate term

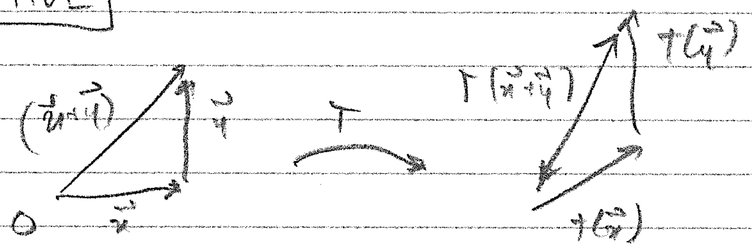
Examples $\mathbb{R}^2 \to \mathbb{R}^2$: rotation; Reflections about lines in $\mathbb{R}^2$, about lines/planes in $\mathbb{R}^3$...

Nonexample → a projection onto a line in $\mathbb{R}^2$

$$\|\text{proj}_L(\vec{x})\| \le \|\vec{x}\|$$

**T/F** If $T(\vec{x}) = A\vec{x}$ is an orthogonal map, then it preserves orthogonality, meaning that $T(\vec{v}) \perp T(\vec{w})$ if $\vec{v} \perp \vec{w}$

**TRUE**



Congruent triangles

↳ $\perp$ preserves...

April 25, 2018

Aside

**Transpose of a matrix**

Ex $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ then $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

Def

↳ For an $n \times m$ matrix $A$, the transpose $A^T$ is the $m \times n$ matrix whose $ij^{th}$ entry is the $ji^{th}$ entry of the original matrix

Properties

① $(A+B)^T = A^T + B^T$

② $(A^T)^T = A$

③ $kA^T = (kA)^T$

④ $(AB)^T = B^T A^T$ → Proof the $ij^{th}$ entry of $(AB)^T$ is the $ji^{th}$ entry of $AB = (j^{th}$ row of $A) \cdot (i^{th}$ col. $B)$

Same for $B^T A^T$. $ij^{th}$ element of $B^T A^T$ $= (j^{th}$ row of $A)(i^{th}$ column of $B)$

⑤ If $A$ invt, then $A^T$ invertible

square

$$\boxed{(A^T)^{-1} = (A^{-1})^T}$$

**Proof** : $AA^{-1} = I_n \Rightarrow (AA^{-1})^T = I_n^T = I_n$

$\hookrightarrow (A^{-1})^T A^T = I$ So $(A^{-1})^T = (A^T)^{-1}$

and both of them are invertible

---

Back to orthogonal

**Theorem** For an $n \times n$ matrix $A$, the following are equivalent

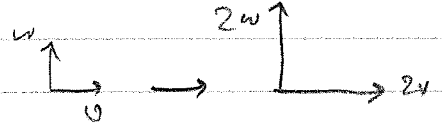
(1) $A$ is an orthogonal matrix
(2) $A$ preserves length (meaning that $\|A\vec{x}\| = \|\vec{x}\| \ \forall \vec{x} \in \mathbb{R}^n$)
(3) the columns $\vec{v}_1 \dots \vec{v}_n$ of $A$ are orthonormal

$\boxed{T/F}$ If $A$ preserves orthogonality, then it is an orthogonal matrix

FALSE Counterexample : $A = 2I \rightarrow$ does not preserve length, but

preserves ~~not~~ orthogonality

the only counterexample.



Q Is $A = \frac{1}{3}\begin{bmatrix} 2 & 1 & -2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Example

. Is $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}$ orthogonal? No. $A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

. Is $A = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ orthogonal?

$\vec{e}_1 \cdot \vec{e}_2 = 0$ , $(A\vec{e}_1)(A\vec{e}_2) = \frac{8}{9} \neq 0 \rightarrow$ does not preserve orthogonality

$\Rightarrow$ fails to be orthogonal matrix

**Theorem**

If $A$ is an orthogonal matrix, then the columns of $A$ form an orthonormal basis of $\mathbb{R}^n$

since the columns are $A\vec{e}_1 \dots A\vec{e}_n$ ∎

**Is the converse true?**

**Proof** → $A = \left(\overbrace{\vec{v}_1 \ \vec{v}_2 \dots \ \vec{v}_n}^{\text{assume form ONB}}\right)$ $\qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_n \end{pmatrix}$

**Show** $\|A\vec{x}\|^2 = \|\vec{x}\|^2$

$$\hookrightarrow \left\|\left(\vec{v}_1 \dots \vec{v}_n\right)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right\|^2 = \left\| x_1\vec{v}_1 + x_2\vec{v}_2 + \dots x_n\vec{v}_n \right\|^2$$

$$= \|x_1 v_1\|^2 + \dots + \|x_n\vec{v}_n\|^2 \qquad \searrow \text{dot product } \vec{v}_j \vec{v}_i = 0$$

$$= x_1^2 + \dots + x_n^2 = \|x_n\|^2 \qquad \overset{\uparrow}{\text{since ONB}}$$

**Theorem (cont)**

(4). $\boxed{A^TA = I_n}$

(5) $\boxed{A^{-1} = A^T}$

$$A = \frac{1}{3}\begin{pmatrix} 2 & 1 & -2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{pmatrix} \Rightarrow A^TA = \frac{1}{9}\begin{pmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 & -2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Leftrightarrow$ columns of A are ON

**April 27, 2018**

(6) **Proof** that (3) $\Leftrightarrow$ (4) 3

$\vec{v}_1, \vec{v}_2, \vec{v}_3 \dots$ are orthogonal $\Leftrightarrow$ $A^TA = I_n$

$$A^TA = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix}\begin{bmatrix} \vec{v}_1 \dots \vec{v}_n \end{bmatrix} = \begin{bmatrix} \vec{v}_1\vec{v}_1 & \vec{v}_1\vec{v}_2 & \dots \\ \vec{v}_2\vec{v}_1 & \ddots & \\ \vdots & & \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots\, 1 \end{bmatrix} = I_n$$

iff $\vec{v}_1, \dots \vec{v}_n$ are orthonormal

$\underset{\perp}{\text{ortho}}\ \underset{\|\,\| = 1}{\text{normal}}$

(6) A preserves the dot product . $\vec{v}\cdot\vec{w} = (A\vec{v})(A\vec{w})$ , $\forall\, \vec{v},\vec{w}\in\mathbb{R}^n$

↳ $(A\vec{v})\cdot(A\vec{w}) = (A\vec{v})^T(Aw) = \vec{v}^TA^TA\vec{w} = \vec{v}^T I\vec{w} = \boxed{\vec{v}^T\vec{w} = \vec{v}\cdot\vec{w}}$

show (6) ⇒ (2) by 6

$$\|A\vec{v}\|^2 = (A\vec{v})\cdot(A\vec{v}) \overset{\text{by } 6}{=} \vec{v}\cdot\vec{v} = \|v\|^2$$

**Example** what are the possible (real) $\lambda$ of orthogonal matrix?

$$A\vec{v} = \lambda\vec{v} \qquad \vec{v}\neq 0$$

preserves length → $\|\vec{v}\| = \|A\vec{v}\| = \|\lambda\vec{v}\| = |\lambda|\,\|\vec{v}\|$ so $|\lambda|=1$ so $\boxed{\lambda=\pm1}$

**Example** $A = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ $\lambda$ are $\pm1$

$A = I_n$ $\lambda=1$

$A = -I_n$ $\lambda=-1$

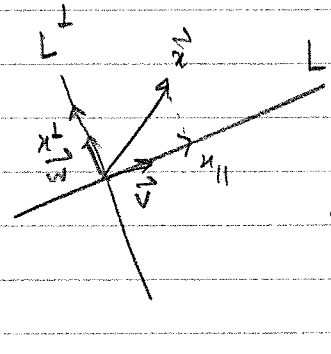
$A = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ no real $\lambda$ $(\lambda^2+1)=0$

(no real $\lambda$ with even sizes)

(if odd sizes..., then there exists $\lambda$)

**Reflections**

Special case



$A\vec{v} = \vec{v}$

$A\vec{w} = -\vec{w}$

$\exists$ matrix $B$ of $T(\vec{x}) = A\vec{x}$

$$B = \begin{array}{cc} A\vec{v} & A\vec{w} & \\ \left(\begin{array}{c}1\\0\end{array}\right. & \left.\begin{array}{c}0\\-1\end{array}\right) & \begin{array}{c}\vec{v}\\\vec{w}\end{array}\end{array}$$

## Reflections in general

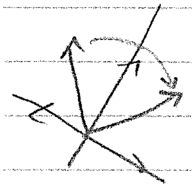
Consider a subspace $\bar{W}$ of $\mathbb{R}^n$. Define the orthogonal complement

$$\bar{W}^{\perp} = \{\vec{w} \in \mathbb{R}^n : \vec{w}\cdot\vec{v} = 0 \quad \forall\ \vec{v} \in \bar{W}\}$$

**Properties** (a) $\bar{W}^{\perp}$ is a subspace of $\mathbb{R}^n$

(b) $\bar{W}^{\perp} \cap \bar{W} = \{\vec{0}\}$

(c) $(\bar{W}^{\perp})^{\perp} = \bar{W}$



(d) $\dim \bar{W}^{\perp} + \dim \bar{W} = n$

(e) Every $\vec{x}$ in $\mathbb{R}^n$ can be written uniquely as $\vec{x} = \vec{x}_{\parallel} + \vec{x}_{\perp}$ where $\vec{x}_{\parallel} \in \bar{W}$ and $\vec{x}_{\perp} \in \bar{W}^{\perp}$

Define Reflection L in $\bar{W}$ is defined as

$$L(\vec{x}) = L(\vec{x}_{\parallel} + \vec{x}_{\perp}) = \vec{x}_{\parallel} - \vec{x}_{\perp}$$ , and orthogonal

$$\{\vec{x} : L(\vec{x}) = \vec{x}\} = \bar{W}$$

$$\{\vec{x} : L(\vec{x}) = -\vec{x}\} = \bar{W}^{\perp}$$

T/F? A Reflection matrix A must satisfy $A^T = A$ (we say that A is a symmetric matrix)

$$\left.\begin{array}{l} A \text{ orthogonal} \Rightarrow A^T = A^{-1} \\ \text{But since } A^2 = I_n \end{array}\right\} \Rightarrow A = A^{-1} = A^T$$

<u>T/F</u> A reflection matrix A must be similar to a <u>diagonal matrix</u>

↳ equivalently, is there a basis of $\mathbb{R}^n$ consisting of eigenvectors of A?

Let $\vec{v}_1 \dots \vec{v}_p$ be a basis of $V$ & $\vec{w}_1 \dots \vec{w}_q$ basis of $V^\perp$

<u>Then</u> $\vec{v}_1 \dots \vec{v}_p, \vec{w}_1 \dots \vec{w}_q$ form a basis of $\mathbb{R}^n$ (TRUE)

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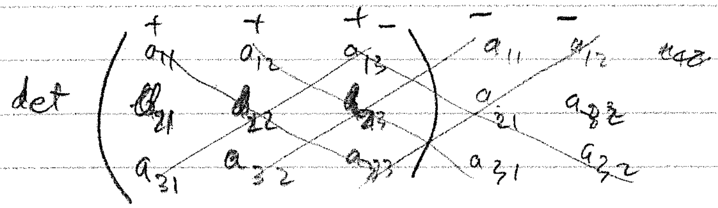
April 30, 2018

$\boxed{\text{DETERMINANTS}}$ <u>2×2</u> $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\det A = ad - bc$

<u>Theorem</u> $\boxed{\det A \neq 0 \iff A \text{ is invertible}}$

<u>3×3</u>

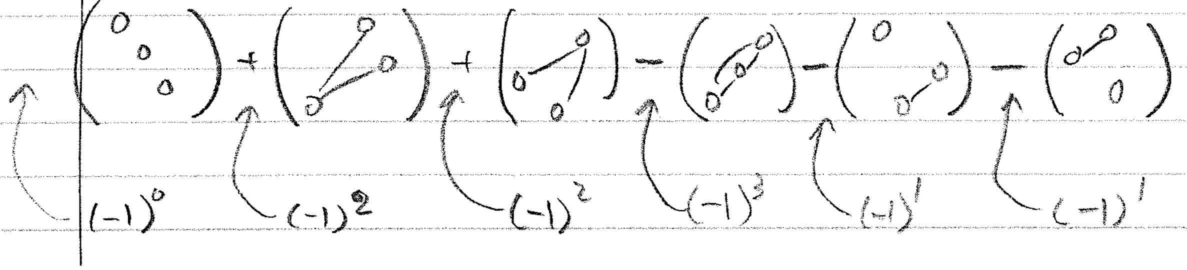


$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

<u>Def</u> $\boxed{\text{A pattern in an } n \times n \text{ matrix is a way to choose } n \text{ entries of the matrix, 1 in each row and in each column}}$

With a pattern P we associate the product of its entries, prod(P)

We can write $\boxed{\det A = \sum_{\substack{P \\ \text{all patterns}}} \pm \operatorname{prod}(P)}$



**Def** 2 entries in a pattern are said to form an inversion (Fehlstand) if one of them is to the right and above the other.

**Def** The sign as $\boxed{\text{sign}(P) = (-1)^{\#\text{ inversions}}}$

For $3\times 3$ $\boxed{\det A = \sum_P \text{sign}(P)\cdot\text{prod}(P)}$

**Def** determinant of $n\times n$ matrix $A$ as

$$\boxed{\det A = \sum_P \text{sign}(P)\,\text{prod}(P)}$$

**Ex** $\det\begin{pmatrix} 0 & 0 & \text{\textcircled{2}} & 0 \\ \text{\textcircled{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{\textcircled{4}} \\ 0 & \text{\textcircled{5}} & 0 & 0 \end{pmatrix}$ Only one pattern ~~pro~~ product, count inversions

nonzero

$\det A = (-1)^3\, 5.4.3.2 = -5! = -120$

**Ex** $\det\begin{pmatrix} \text{\textcircled{1}} & 2 & 3 & 4 \\ 0 & \text{\textcircled{2}} & 3 & 4 \\ 0 & 0 & \text{\textcircled{3}} & 4 \\ 0 & 0 & 0 & \text{\textcircled{4}} \end{pmatrix} = 4!$ (no inversions ...)

**Theorem** The determinant of a triangular matrix (upper or lower) is the product of its diagonal entries

$\det\begin{pmatrix} 0 & 0 & 0 & \text{\textcircled{5}} \\ \text{\textcircled{1}} & 3 & 0 & 4 \\ 2 & 4 & \text{\textcircled{6}} & 3 \\ 0 & \text{\textcircled{5}} & 0 & 2 \end{pmatrix}$ $\det A = (-1)^4\, 5\cdot 5\cdot 6\cdot 1 = 150$

Is the determinant a linear map? **NO**

$\det\ \mathbb{R}^{n\times n} \to \mathbb{R}^n$

C.E : $\det(2I_3) = 8 \neq 2\cdot 1$

Is $T \; \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \det\begin{pmatrix} 1 & x & 4 \\ 2 & y & 5 \\ 3 & z & 6 \end{pmatrix}$$ a linear map?

$\boxed{\text{YES!}}$ $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underset{\uparrow \text{linear}}{(6y - 5z) + x(15-12) + 4(2z - 3y)}$

→ We never multiply the variables together...

⇒ $\boxed{\text{"Determinant is linear in the rows and in the columns"}}$

May 2, 2018

$\underline{\textcircled{T}}$/F $\boxed{\det A = \det(A^T) \; ?}$

Ex

$$4\times4 \; A = \begin{pmatrix} 1 & \textcircled{2} & 3 & 4 \\ 5 & 6 & 7 & \textcircled{8} \\ \textcircled{9} & 8 & 7 & 6 \\ 5 & 4 & \textcircled{3} & 2 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 5 & \textcircled{9} & 5 \\ \textcircled{2} & 6 & 8 & 4 \\ 3 & 7 & 7 & \textcircled{3} \\ 4 & \textcircled{8} & 6 & 2 \end{pmatrix}$$

Take

↳ $\text{sgn}(P)\,\text{prod}(P) = \text{sgn}(P^T)\,\text{prod}(P^T)$

TRUE for all P

So $\boxed{\det A = \det(A^T)}$

Determinants in terms of Gaussian elimination

How do elementary row operations affect the determinant?

row div

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \div k \xrightarrow[\text{div}]{\text{row}} \begin{pmatrix} a/k & b/k \\ c & d \end{pmatrix} = B$$

$$\det B = \frac{ad}{k} - \frac{bc}{k} = \frac{1}{k}\det A$$

So $\boxed{\det B = \frac{1}{k}\det A}$

row swap

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{\text{swap}} B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$ $\boxed{\det B = -\det A}$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow[\text{subtraction}]{\text{row}} B = \begin{pmatrix} a & b \\ c - ka & d - kb \end{pmatrix}$$

$$\boxed{\det B = \det A}$$

<u>Proof</u>

$$\det \begin{pmatrix} \vdots \\ v_j + k v_i \\ v_i \end{pmatrix} = \det \begin{pmatrix} \vdots \\ v_j \\ \vdots \\ v_i \end{pmatrix} + \underbrace{k \det \begin{pmatrix} \vdots \\ v_i \\ \vdots \\ v_i \end{pmatrix}}_{0} = \det \begin{pmatrix} \vdots \\ v_j \\ \vdots \\ v_i \end{pmatrix}$$

2 same row

$$A \xrightarrow[\text{reduction}]{\text{row}} B$$ such that det B easy to find

s swaps
<u>div</u> by $k_1 \dots k_r$

$$\boxed{\det B = (-1)^s \frac{1}{k_1 k_2 \dots k_r} \det A}$$

<u>So</u>

$$\boxed{\det A = (-1)^s k_1 \dots k_r \det B}$$

<u>Example</u>

$$\det \begin{pmatrix} 0 & 3 & 6 \\ 1 & 4 & 7 \\ 2 & 5 & 4 \end{pmatrix} ?$$

Swaps : 1
div : 3

$$\begin{pmatrix} 1 & 4 & 7 \\ 0 & 3 & 6 \\ 2 & 5 & 4 \end{pmatrix} \div 3$$

$$\begin{pmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 2 & 5 & 4 \end{pmatrix} -1(I)$$

$$\begin{pmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & -3 & -10 \end{pmatrix} +3II$$

$$\begin{pmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\det A = (-1)^1 . 3 \det B$$
$$= (-1)^1 . 3 . (-4) . 1 . 1$$
$$= \boxed{12}$$

$\boxed{\det A = (-1)^s k_1 \ldots k_r \det B}$

If let $B = \text{rref}(A)$ $\quad A \xrightarrow[\text{divide } k_1 \ldots k_r]{s \text{ swaps}} \text{rref}(A)$

$\boxed{\det A = (-1)^s k_1 \ldots k_r \det B}$ $\quad$ or $\det(\text{rref} A)$

If $A$ invertible $\Rightarrow$ rref $A = I$ $\to \det(\text{rref} A) = 1 \to \det A \neq 0$

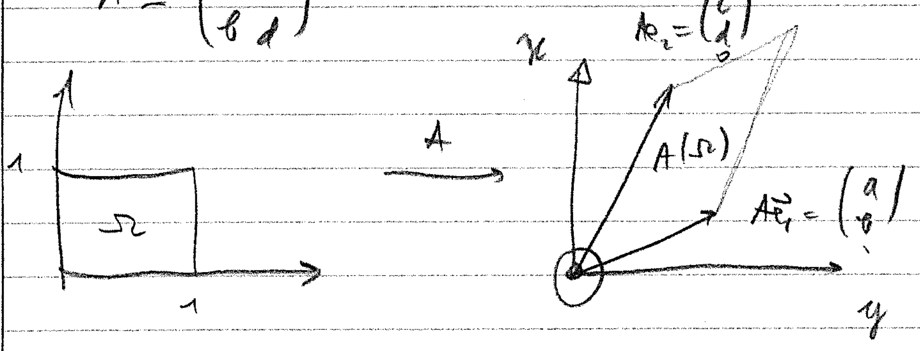
If $A$ fails to be invertible $\Rightarrow$ rref = upper triangular with at least 1 zero on the diag $\to \det(\text{rref} A) = 0$

$\Rightarrow \underline{\det A = 0}$.

$\boxed{A \text{ invertible} \iff \det A \neq 0}$

$\boxed{\text{Geometrical meaning of the det in } 2\times 2 \text{ case}}$

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$



area of $A(\Omega)$

$= \| A\vec{e}_1 \times A\vec{e}_2 \|$

$= \left\| \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \times \begin{pmatrix} c \\ d \\ 0 \end{pmatrix} \right\|$

$= \left\| \begin{bmatrix} 0 \\ 0 \\ ad - bc \end{bmatrix} \right\|$

$\underline{\text{So}}$ $A(\Omega) = |(ad - bc)| = |\det A|$

$\underline{\text{So}}$ $\boxed{A(\Omega) = |\det A| \, \Omega}$

<u>One can show</u> that $|\det A|$ is the ratio factor on parallelograms

Sign of determinant → orientation of image

For 2x2 matrices, A, B, what is the relationship among det A, det B, det BA

$$\boxed{\det(BA) = (\det B)(\det A)}$$

Claim this is true for all n x n matrices A, B (proof w/ rref)

T/F $\boxed{\det(A^{-1}) = ?}$

$$\boxed{\det A^{-1} = \frac{1}{\det A}}$$

$$AA^{-1} = I$$

$$\det(AA^{-1}) = \det A \cdot \det A^{-1} = 1$$

$$\boxed{\det A^{-1} = \frac{1}{\det A}} = (\det A)^{-1}$$

T/F $\boxed{A \sim B, \text{ then } \det A = \det B}$

AS = SB for some invertible S

$\hookrightarrow \det(AS) = \det A \cancel{\det S} = \cancel{\det S} \det B \rightarrow \boxed{\det A = \det B}$

May 7, 2018

$\boxed{\text{Finding } \lambda}$

Consider n×n matrix A

$\lambda$ is an eigenvalue for A $\iff A\vec{v} = \lambda\vec{v}$ for some nonzero $\vec{v} \in \mathbb{R}^n$

$\iff (A - \lambda I_n)\vec{v} = 0$ for some nonzero $\vec{v} \in \mathbb{R}^n$

$\iff \ker(A - \lambda I_n) \neq \{\vec{0}\}$

$\iff A - \lambda I_n$ fails to be invertible

$\iff \det(A - \lambda I_n) = 0$

**Theorem 1** $\boxed{\lambda \text{ is eigenvalue of } n\times n\ A \text{ iff } \det(A-\lambda I_n)=0}$

Ex

$$A=\begin{pmatrix}1&0&0\\2&4&0\\3&5&6\end{pmatrix} \rightarrow \det(A-\lambda I_n)= (1-\lambda)(4-\lambda)(6-\lambda)=0$$

$$\Leftrightarrow \lambda\in\{1,4,6\}$$

**Theorem 2** $\boxed{\lambda \text{ of diag matrix are its diagonal entries}}$

Ex

$$A=\begin{pmatrix}0&0&0\\1&0&-1\\0&1&2\end{pmatrix} \qquad \det(A-\lambda I_3)=\det\begin{pmatrix}-\lambda&0&0\\1&-\lambda&-1\\0&1&2-\lambda\end{pmatrix}$$

$$= (-\lambda)^2(2-\lambda) \;-\lambda$$

$$\Leftrightarrow 2\lambda^2 - \lambda^3-\lambda = 0$$

$$\Leftrightarrow (-\lambda)(\lambda^2-2\lambda+1)=0$$

$$\Leftrightarrow (-\lambda)\ (\lambda-1)^2=0$$

eigenvalues $\{0,1\}$

#

**Theorem 3**

(a) $\det(A-\lambda I_n)$ is a polynomial in $\lambda$ of degree $n$, called the "characteristic polynomial" of $A$, denoted by

$$\cancel{F(A,\lambda)} = \boxed{F_A(\lambda)= \det(A-\lambda I_n)}$$

(b) eigenvalues of $A$ are the zeros of $F_A(\lambda)$

Describe some of the coefficients of $F_A(\lambda)$

$$3\times 3 \ :\ F_A(\lambda)=\det(A-\lambda I_3)=\begin{pmatrix}a_{11}-\lambda&a_{12}&a_{13}\\a_{21}&a_{22}-\lambda&a_{23}\\a_{31}&a_{32}&a_{33}-\lambda\end{pmatrix}$$

$$=(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) + \underbrace{r(\lambda)} \rightarrow \in P_1$$

$$=(-\lambda)^3 + (a_{11}+a_{22}+a_{33})(-\lambda)^2 + \ldots + \det(A)$$

⑨

The sum of diagonal entries of $n \times n$ matrix $A$ is called its trace

$$\boxed{tr(A) = \text{sum of diag-entries of } A = \sum_i a_{ii}}$$

__Theorem 5__ If $A$ is an $n \times n$ matrix, then

$$\boxed{f_A(\lambda) = (-\lambda)^n + tr(A)(-\lambda)^{n-1} + \dots + \det A}$$

__Revisit Ex 2__

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \qquad f_A(\lambda) = -\lambda(\lambda-1)^2$$

$\lambda = 1$ is said to have algebraic multiplicity 2, meaning that the factor $(1-\lambda)$ appears twice in polynomial (but not 3 times...)

__Aside__ Find ~~gemu~~ $\text{gemu}(1) = \dim \underbrace{\ker(A - I_3)}_{\text{eigenspace } E_1}$

$$= \dim\left(\ker\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix}\right)$$

$$= \underline{1}$$

$$\text{almu}(1) = 2$$

$$\text{but gemu}(1) = \underline{1}$$

__Theorem 6__ (proof in book)

$$\boxed{1 \leq \text{gemu}(\lambda) \leq \text{almu}(\lambda) \quad \forall \lambda \text{ eigenvalue of } A}$$

gemu$(\lambda)$ cannot be 0 (duh...)

__Example 6__ $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$f_A(\lambda) = \det(A - \lambda I_3) = \det\begin{pmatrix} -\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix}$$

$$= \lambda^2(1-\lambda) + (1-\lambda)$$

$$= (1-\lambda)(\lambda^2+1) = 0 \Leftrightarrow \boxed{\lambda = 1} \rightarrow \text{almu}(1) = 1$$

<u>and</u> non-real eigenvalues ...

<u>May 8, 2018</u> Reading Assignment page 283 - 286

<u>Theorem 7</u> An n × n matrix A has <u>at most</u> n eigenvalues, even if they are counted with their algebraic multiplicity, almu

<u>Theorem 8</u> if n is odd, then n × n matrix has at least 1 (real) eigenvalue

<u>Theorem 9</u>

When is an n × n matrix diagonalizable?

<u>Know</u> A diagonalizable $\Leftrightarrow \exists$ an eigenbasis for A

$$\Leftrightarrow \sum_\lambda \text{gemu}(\lambda) = n = \sum_\lambda \text{almu}(n)$$

<u>Remarks</u>

$f_A(\lambda)$ splits iff $\sum_\lambda \text{almu}(\lambda) = n$

$$\left( \begin{array}{l} \text{gemu}(\lambda) \leq \text{almu}(\lambda) \quad \forall \lambda \\ \sum_\lambda \text{gemu}(\lambda) \leq \sum_\lambda \text{almu}(\lambda) \leq n \end{array} \right)$$

$\Leftrightarrow$ gemu$(\lambda)$ = almu$(\lambda)$ $\forall \lambda$
and $f_A(\lambda)$ splits ...

<u>Theorem 9</u>

$$\boxed{\begin{array}{l} n \times n \text{ matrix } A \text{ diagonalizable} \\ \quad \text{iff} \quad \text{gemu}(\lambda) = \text{almu}(\lambda) \quad \forall \lambda \\ \quad\quad \underline{\text{and}} \quad f_A(\lambda) \text{ splits} \end{array}}$$

## Determinant & Trace & Eigenvalues

**Special case** $A = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$$\left\{ \begin{array}{l} \det A = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_n \\ \operatorname{tr} A = \lambda_1 + \lambda_2 + \dots + \lambda_n \end{array} \right\}$$

**Theorem 10** Let $A$ be $n \times n$ matrix with eigenvalues $\lambda_1, \dots \lambda_n$ listed with their almus ...

then $\left\{ \begin{array}{l} \det A = \lambda_1 \dots \lambda_n \\ \operatorname{tr} A = \lambda_1 + \dots + \lambda_n \end{array} \right.$

$$f_A(\lambda) = \det(A - \lambda I_n)$$
$$= (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\dots)(\lambda_n - \lambda)$$

$$\boxed{\text{Let } \lambda = 0 \rightsquigarrow f_A(0) = \det A = \lambda_1 \dots \lambda_n}$$

$$\operatorname{tr} A = \lambda_1 + \dots + \lambda_n$$

$$f_A(\lambda) = (-\lambda)^n + (-\lambda)^{n-1} \operatorname{tr}(A) + \dots + \det A \qquad (\text{duh} \dots)$$
$$= (-\lambda)^n + (-\lambda)^{n-1}(\lambda_1 + \dots + \lambda_n) + \dots + \det A$$

So $\boxed{\operatorname{tr} A = \sum_{i=1}^{n} \lambda_i}$

Ⓣ/Ⓕ → If $A$ is similar to $B$ then they have the same characteristic polynomial ...

$$\hookrightarrow B = S^{-1} A S \rightsquigarrow \underbrace{\det(B - \lambda I_n)}_{f_B(\lambda)} = \det(S^{-1} A S - \lambda I_n)$$
$$= \det(S^{-1}(A - \lambda I_n) S)$$
$$= \cancel{\det S^{-1}} \cdot \det(A - \lambda I_n) \cancel{\det S}$$
$$= \det(A - \lambda I_n) = f_A(\lambda)$$

**Show** $= \det(A - \lambda I_n) = f_A(\lambda)$

Theorem 11

$$\boxed{\text{If } A \sim B \text{ then } f_A(\lambda) = f_B(\lambda)}$$

Thus $A \sim B$ have the same eigenvalues with the same almu

Also

$\det A = \det B$

and $\operatorname{tr} A = \operatorname{tr} B$

$$\boxed{\text{T/F if } f_A(\lambda) = f_B(\lambda) \text{, then } A \sim B}$$

$\boxed{\text{FALSE}}$ $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$f_A(\lambda) = f_B(\lambda) = (1-\lambda)^2$

But $A \nsim B$ because $I_n$ only $\sim I_n$