ordinary DIFFERENTIAL EQUATIONS
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MA311

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What is a differential equation?

Loosely speaking, a different equation is an eqn relating a fn to its derivative

A solution (classical) is a sufficiently differentiable function that satisfies the differential eqn

Ex 1 Consider $\frac{dp}{dt} = \frac{1}{2}p$ (1) $P = P(t)$ is the function, also called a dependent variable) and $t$ - the independent variable.

This D.E models population growth

A solution to (1) is $P(t) = e^{t/2}$

Note
$$\frac{dp}{dt} = \frac{d}{dt}\left(e^{t/2}\right) = \frac{1}{2}e^{t/2} = \frac{1}{2}p$$

So $P(t) = e^{t/2}$ is a solution to (1)
Another is $P(t) = 0 \ \forall t$
Another is $P(t) = 5e^{t/2}$

The general solution is $P(t) = Ce^{t/2}$ where $C$ = constant

Ex 2 For a constant $m \neq 0$

$$m\ddot{x} = F(t, x, \dot{x}) \quad (2)$$

Here, $x$ is the dependent va

$x = x(t)$, $\frac{dx}{dt} = \dot{x}$, $\frac{d^2x}{dt^2} = \ddot{x}$

As a physical eqn, $m$ = mass, $x$ = position, $\dot{x}$ = velocity
$\ddot{x}$ = acceleration, $F$ = force ...
(2) is Newton's 2nd law...

Ex 3 Let $I_1 > I_2 > I_3 > 0$.

Consider the following D.E. in the dependent var. $\omega_1, \omega_2, \omega_3$

(3x3)

$$(\star) \begin{cases} I_1 \dot{\omega}_1 = (I_2 - I_3)\omega_2 \omega_3 \\ I_2 \dot{\omega}_2 = (I_3 - I_1)\omega_3 \omega_1 \\ I_3 \dot{\omega}_3 = (I_1 - I_2)\omega_2 \omega_1 \end{cases} \rightarrow \text{A system of D.E}$$

Here, you seek $\omega_1(t), \omega_2(t), \omega_3(t)$ satisfying the system. $\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$

→ This is the 3d Euler equations... They come from rigid body motion (classical dynamics)

$\omega_i$ are angular momenta

$I_i$ are moment of inertia

- Intermediate axis theorem

, 7, 2018

$m\ddot{x} = kx$ general solution : $\boxed{x(t) = c_1 \sin\left(\sqrt{\frac{k}{m}}t\right) + c_2 \cos\left(\sqrt{\frac{k}{m}}t\right)}$

↑ 2 derivatives ↔ 2 sols

(3) $(\star)$ is a system of D.E, models the rotation of a 3D solid body...

But There is NO solution that we can write down for $(\star)$

→ NO closed form solution exists...

Moral We will need to understand properties of solutions to DE even if we can't write solutions down...

↓

(Dynamical System)

Almost all DE have sltns that can't be written down

(4) The following eqn describes vibration

$$\boxed{\frac{\partial^2 u}{\partial t^2} = \Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}$$

This is called the wave eqn, a solution describes wavy thing

A solution is of the form $u(t,x,y) = \sin\left(t + \frac{1}{\sqrt{2}}(x+y)\right)$

another ... $u(t,x,y) = tx + y$

(1)–(3) are called Ordinary Diff. Eq. (ODE)
(4) is called Partial Diff. Eq (PDE)

**ODE**

**Definition** Given a function $F: \mathbb{R}^{n+2} \longrightarrow \mathbb{R}$, an ordinary differential eqn is an eqn of the form

$$F(t, y^{(n)}, y^{(n-1)}, \dots y', y) = 0 \qquad (1_0)$$

then a solution of eqn $(1_0)$ is a sufficiently differential (n-times differentiable) function $y = y(t)$ such that

$$F(t, y^{(n)}(t), y^{(n-1)}(t), \dots, y'(t), y(t)) = 0 \quad \forall t$$

(enough t's)

The order of an ordinary differential eqn is the highest order of derivative appearing in $(1_0)$

e.g. $(1_0)$ is an $n^{th}$ order ODE (as long as $y^{(n)}$ actually appears in it)

Ex $\quad t y^{(3)} + y^{(2)} + \sin(y^2) = 0$ is an ODE $\left(= F(t, y^{(3)}, y^{(2)}, y', y)\right)$

order of this ODE is 3.

We say that an ODE is in standard form if it ~~is~~ is equivalently written as

$$y^{(n)} = f(t, y^{(n-1)}, y^{(n-2)}, \dots y', y) \qquad (1)$$

where $\boxed{f : \mathbb{R}^{n+1} \mapsto \mathbb{R}}$ Until we get to systems, all ODE we consider will be of the form (1)

Ex (1) $y'' = y' + t y^2$ (has "$t$" explicitly)

(2) $y'' = \frac{y'}{1+y^4} \sin(y)$ (easier to understand)

Definition: If the RHS of (1) does NOT explicitly involve $t$, i.e. $y^{(n)} = g(y^{(n-1)}, y^{(n-2)}, \dots y', y)$ $\quad (1_A)$

for $g: \mathbb{R}^n \mapsto \mathbb{R}$, we say that the ODE is "autonomous"

these are the players in dynamical systems

- Our goal is to find or understand properties of solutions to $(1_0)$, $(1)$, $(1_A)$.

{ However, what if I'm handed a fn $y = y(t)$. Can we check to see if it's a solution. What does that mean? }

Ex Consider $\frac{dy}{dt} = \frac{y^2-4}{t^2+4t}$ $(\star)$ $\left\{ \begin{array}{l} 1^{st} \text{ order, ODE, } \underline{\text{NOT autonom}} \\ \text{in standard form} \end{array} \right.$

Are $y_1(t) = 2$, $y_2(t) = t+2$, $y_3 = t$ solutions $\forall$

(1) $y_1' = 0$, $\frac{y_1^2-4}{t^2+4t} = \frac{2^2-4}{t^2+4t} = \frac{0}{t^2+4t} = 0$ $(t \neq 0)$

So $\frac{dy_1}{dt} = 0 = \frac{y_1^2-4}{t^2+4t}$ $\forall t \neq 0$ (ODE makes sense)

So Yes, $y_1$ a s/n

(2) $y_2' = 1$, $\frac{y_2^2-4}{t^2+4t} = \frac{t^2+4t}{t^2+4t} = 1$

So $\frac{dy_2}{dt} = 1 = \frac{y_2^2-4}{t^2+4t}$ $\forall t \neq 0$ Yes, $y_2$ a s/n

(3) $\frac{dy_3}{dt} = 1$, $\frac{y_3^2-4}{t^2+4t} = \frac{t^2-4}{t^2+4t}$ No, $y_3$ NOT a s/n

when $t=-1$, then RH = LH, but $y_3$ still NOT s/n

Although RH = LH for $t=-1$, they aren't equal $\forall t$

So $y_3$ is NOT a s/n to $(\star)$

Sep 10, 2018 Ex Consider $F: \mathbb{R}^3 \mapsto \mathbb{R}$

$F(a,b,c) = 2ac - b$, for $(a,b,c)^T \in \mathbb{R}^3$

This F gives the ODE

$F(t, y', y) = 2ty - y' = 0$ (standard form)

Note This eqn is also expressible in the form $\boxed{y' = 2ty}$

So $y' = 2ty = f(t,y) \longleftarrow$ in standard form
where $f(\alpha,\beta) = 2\alpha\beta$.

Verify $y(t) = 3e^{t^2}$ is a solution.

$y'(t) = 6te^{t^2}$ so $y'(t) = 2ty(t) \quad \forall t \in \mathbb{R}$
$2ty = 6te^{t^2}$

#

Can ODEs $n^{th}$ order of the form

$$F(t, y^{(n)}, y^{(n-1)}, \dots y', y) = 0$$

always be equivalently written in standard form?

Ex $(y'')^2 + 2y''y + y^2 - 4 = 0$

I can solve locally (?) (Implicit function theorem says you can do this (rewrite in SF) as long as $\frac{\partial F}{\partial y^{(n)}} \neq 0$)

Back to an equation

$\frac{dP}{dt} = \frac{1}{2}P$ We observed that $P_1(t) = e^{t/2}$, $P_5 = 5e^{t/2}$ are both solutions. How do we single solution?

↳ Answer: we need to formulate the question appropriately.

Definition of Initial Value Problem (IVP)

Let $y^{(n)} = f(t, y^{(n-1)}, \dots y', y)$ (1) be an ODE. An initial value problem for this eqn is a problem of the form: Given numbers $y_0^{n-1}, y_0^{n-2}, \dots y_0^1; y_0^0$ (n numbers) and a time $t_0$

Find a solution $y = y(t)$ to Eqn (1) such that

$$\begin{cases} y(t_0) = y_0 & y^{(n-1)}(t_0) = y_0^{(n-1)} \\ y'(t_0) = y_0' & \end{cases}$$

This is written

Together this is called an initial value problem

$$\begin{cases} y^{(n)} = f(t, y^{(n-1)}, y^{(n-2)}, \dots y', y) & \longrightarrow \text{the ODE} \\ y^{(n-1)}(t_0) = y_0^{(n-1)}, \dots, \quad y_0'(t_0) = y_0', \quad y(t_0) = y_0 & \longrightarrow \text{initial conditions} \end{cases}$$

Ex Consider this IVP

$$\left.\begin{cases} \dfrac{dP}{dt} = \dfrac{1}{2}P \\ P_0 = P(0) = \pi/6 \end{cases}\right\} (\star)$$

We "argued" that all solutions of $\frac{dP}{dt} = \frac{1}{2}P$ are of the form

$P(t) = Ce^{t/2}$. So, observe that $P(t) = \frac{\pi}{6}e^{t/2}$ solves $(\star)$

because

$$\left.\begin{cases} \dfrac{dP}{dt} = \dfrac{d}{dt}\left(\dfrac{\pi}{6}e^{t/2}\right) = \dfrac{1}{2}\dfrac{\pi}{6}e^{t/2} \\ P(0) = \dfrac{\pi}{6}e^0 = \dfrac{\pi}{6} \end{cases}\right\}$$

Note . $P(t) = 5e^{t/2}$ does not solve IVP $(\star)$

. $P(t) = \pi/6$ does not solve IVP $(\star)$ either

Another Consider

$$\left.\begin{cases} \ddot{x} = -4x \\ x(0) = 1, \quad \dot{x}(0) = 0 \end{cases}\right\}$$

$x_0 = 1$
$\dot{x}_0 = 0$
and $t_0 = 0$ (IVP)

$$x(t) = A\sin(2t) + B\cos(2t)$$
$$x'(t) = 2A\cos(2t) - 2B\sin(2t)$$
$$x'(0) = 2A = 0 \Rightarrow A = 0$$
$$x(0) = B\cos(2t) = 1 = B \Rightarrow B = 1$$

So $\boxed{x(t) = \cos(2t)}$

**Consider the IVP** $\begin{cases} \frac{dy}{dt} = 4t^2 - 3 \\ y(0) = 8 \end{cases}$

$y(t) = \int 4t^2 - 3\,dt = \frac{4t^3}{3} - 3t + C$

$y(0) = C = 8$

so $y(t) = \frac{4}{3}t^3 - 3t + 8$

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**A big & important class of ODE** An $n^{th}$ order ODE of the form

$$a_n(t)\,y^{(n)} + a_{n-1}(t)\,y^{(n-1)} + \dots + a_1(t)\,y' + a_0(t)\,y^0 = g(t)$$

$\uparrow$ driving fn

is called a linear $n^{th}$ order ODE,
here $a_n, a_{n-1}, \dots a_0, g$ are all function of $t$

Ex $t^5\frac{dy}{dt} + \left[e^{t^2} + \tan^{-1}\left(\frac{t}{t^2+1}\right)\right]y = \sin(t)$ is a linear $1^{st}$ order ODE

$4y'' + 2y' + \sin(t)\,y = 0$ is a linear $2^{nd}$ order ODE

**Note** if $g(t) = 0$, then this eqn is said to be homogenous

Ex $y' + \sin(y) = 0$ NOT linear

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**FIRST-ORDER ODE**

Recall a first order ODE in standard from is an eqn of the form

$$\frac{dy}{dt} = f(t,y) \quad (\star)$$

where $f: \mathbb{R}^2 \mapsto \mathbb{R}$ (truthfully, $f: D \mapsto \mathbb{R}$, w/ $D \subseteq \mathbb{R}^2$)

A corresponding initial value problem for $(\star)$ comes by specifying a constant $y_0$ and considering $\begin{cases} \frac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases}$ Here $t_0$ is fixed.

$\boxed{\text{We will focus for a while on these equations}}$

| Recognize some ... Ex | • $\frac{dy}{dt} = t^2 y$ | • $\frac{dy}{dt} = t^2 + y^2$ | • $\frac{dy}{dt} = (\sin t + 2)$ |
|---|---|---|---|
| $f$ | $f(a,b) = a^2 b$ | $f(a,b) = a^2 + b^2$ | $f(a,b) = (\sin a + 2,$ |
| linear? | Yes<br>$y' - t^2 y = 0$<br>$a_1 = 1, a_0 = -t^2$<br>$g(t) = 0$ | NO | Yes<br>$y' - (\sin t + 2) y = 0$<br>$a_1 = 1, a_0 = -(\sin t$<br>$g(t) = 0$ |
| Seperable? | Yes<br>$g(t) = t^2$<br>$h(y) = 1/y$ | No | Yes<br>$g(t) = (\sin t +$<br>$h(y) = 1/y$ |

**Definition**

A $1^{st}$-order ODE is said to be <u>seperable</u> if it can be written in the form

$$\frac{dy}{dt} = \frac{g(t)}{h(y)}$$ where $g$ and $h$ are functions of only 1 variable

<u>More Examples</u>

$$\left.\begin{aligned} \frac{dy}{dt} &= \frac{t}{y^2+1} \\ y\frac{dy}{dt} &= \frac{t}{t^2+1} \end{aligned}\right\}$$ <u>seperable</u>! NOT <u>linear</u>

* $\boxed{\text{Seperable? linear?}}$

| ODE | Sep.? | Linear? |
|---|---|---|
| $-y' = ty$ | Yes | Yes |
| $-y' = ty^2$ | Yes | No |
| $-y' = ty + t^2$ | No | Yes |
| $-y' = \sin(y) + t$ | No | No |

## How to solve SEPERABLE eqn

**Ex** $\quad y' = 2ty$

(1) Seperate : $\quad \frac{1}{y} y' = 2t$

(2) Integrate w.r.t $t$ : $\quad \int \frac{1}{y} y' \, dt = \int 2t \, dt$

(3) Compute $\int g(t)$ :
$$\int 2t \, dt = t^2 + C_1$$

caution!

(4) Use u-sub to do $\int h(y)$

$$\int \frac{1}{y(t)} y'(t) \, dt = \int \frac{du}{u} = \ln(u) + C_2 = \ln|u| + C_2$$

Let $u = y(t) \rightarrow du = y'(t) \, dt$

(5) Identify $\quad \ln|y| + C_2 = t^2 + C_1$

$$\ln|y(t)| = t^2 + C_1 - C_2 = t^2 + C_3$$

$$|y(t)| = e^{t^2 + C_3} = e^{C_3} e^{t^2}$$

So $\quad y(t) = \pm e^{C_3} e^{t^2}$ or i.e $y(t) = Ce^{t^2}$

Our general solution is $\boxed{y(t) = Ce^{t^2}}$

keep track of constants...

1 single constant is ok

Verify $\quad \frac{dy}{dt} = 2t\, Ce^{t^2} = 2t\, y(t)$

So we're "solved" the ODE

## General method for solving seperable ODE's

use u-sub

Say $y' = \frac{g(t)}{h(y)}$ $\rightarrow$ (Seperate) $\rightarrow$ (Integrate) $\rightarrow$ (Integrate) $\rightarrow$ (Identify) $\rightarrow$ (Solve expression)

$hy' = g(t)$ $\quad$ w.r.t $t$ $\quad \int g(t)\, dt$ $\quad \int h(y) y' \, dt$

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## $\boxed{\text{Separable equations}}$ (cont)

Shorthand ... (pretend $y(t)$ & $t$ isn't there)

$$\frac{dy}{dt} = \frac{g(t)}{h(y)} \quad \text{so} \quad \int h(y)\,dy = \int g(t)\,dt$$

Solve $\begin{cases} y^2 \frac{dy}{dt} = t^2 \\ y(1) = 2 \end{cases}$

$$\int y^2 dy = \int t^2 dt$$

$$\frac{1}{3} y^3 = \frac{1}{3} t^3 + C'$$

$$y^3 = t^3 + C$$

$$2^3 = 1 + C \quad \text{so} \quad C = 7$$

So the solution to the IVP is

$$y(t) = \sqrt[3]{t^3 + 7}$$

Careful: Suppose I wrote $y(t) = (t^3)^{1/3} + C$ gives $C = 1$ (wrong, because this doesn't solve the ODE ...

$\boxed{\text{Ex}}$ Newton's Law of cooling $T = T(t)$

Let an object of temperature $t$ sit in a bath of temperature $(T_a)$ (constant) (ambient temperature). Depending on the physical/chemical and geometric make up of the object, there is a constant $K$ for which the evolution of $T(t)$ in time satisfies the ODE

$$\boxed{\frac{dT}{dt} = -k(T - T_a)}$$

Separable

$$\int \frac{dT}{T - T_a} = \int -k\,dt$$

$$\ln|T - T_a| = -kt + C \longrightarrow \text{dropping } |\ | \text{ is o.k after exp}$$

$$T - T_a = Ce^{-kt} \qquad \boxed{T(t) = T_a + Ce^{-kt}}$$

Solve IVP $\begin{cases} T(t) - T_a = Ce^{-kt} \\ T(0) = T_0 \end{cases}$

$$T(t) = (T_0 - T_a)e^{-kt} + T_a$$

$t \to \infty$, $T(t) \to T_a$. Equilibrium. In large time, object meets temp of bath

Ex $K = \log_e(2) = \ln 2$ explains the cooling of coffee in air

If $T_0 = 25°C$
$T_a = 15°C$ Find the solution to Newton's law of cooling

$$\boxed{T(t) = 10\, e^{-kt} (e^{\ln 2})^{-t} + 15 = 10 (2^{-t}) + 15}$$

p 17, 2018 Suppose coffee @ 190°F and you fill 9/10 of a cup with coffee. Add 1/10 a cup of milk @ 40°F

Assume NLC ... with $K = \frac{1}{10} \ln(2) \approx 0.07$, $T_a = 75°F$

If we wish to drink the coffee at 125°F. Is it better to add the milk first then let cool or let cool and add milk so as to minimize the time it takes to start drinking the coffee milk mix.

$\boxed{\text{Strategies}}$ → (1) Add milk (2) Let cool (3) Drink

(1) Let cool (2) Add milk (3) drink

Milk First $T_0 = \frac{9}{10} \cdot 190 + \frac{1}{10} 40 = 175°$

$$T(t) = T_a + (T_0 - T_a) e^{-kt}$$
$$= 75 + 100\, e^{-\frac{1}{10}\ln(2) t} = 75 + 100 \cdot 2^{-t/10}$$

when is $T(t) = 125$? $\to t = -\log_2\left(\frac{125-75}{100}\right) \cdot 10 = 10$

$\boxed{\text{Wait first}}$ → $125 = \frac{9}{10} T(t) + \frac{1}{10} 40°F \Rightarrow T(t) = 134.44$

↓ WINS

! ← ns!

$$T(t) = 134.44 = (190 - 75) \cdot 2^{-t/10} + 75 \to t = 9.52 \text{ min}$$

Sept 18, 2018 When things go wrong. Consider IVP $\begin{cases} \frac{dy}{dt} = 2t\cdot(y-1)^2 \\ y(1) = 1 \end{cases}$

$$\int \frac{dy}{(y-1)^2} = \int 2t\,dt$$

$$\frac{-1}{y-1} = t^2 + C$$

So $y(t) = 1 - \dfrac{1}{t^2+C}$ is a solution to ODE

Find C

$$y(1) = 1 - \frac{1}{1+C} \quad \text{so } c = ?$$

Note Define $y(t) = 1$ for all $t$. Does this satisfy the IVP?

$\boxed{\text{Yes!}}$ (1) $\frac{d}{dt}y(t) = 0 = 2t(y-1)^2 = 2t(1-1)^2$

(2) $y(1) = 1$

So we've verified that this does solve IVP. However, it can't be found using separation of variables.

$\boxed{\text{Moral}}$ – separation of variables is simply a method for finding solutions.
– sometimes it misses them.
– We really need to be careful. How do we know when we've gotten each solution – or can get it...

→ $\boxed{\text{Uniqueness - Existence...}}$

A new method

Consider $\boxed{\dfrac{dy}{dt} + a(t)y = b(t)}$ (☆)

Linear 1st order ODE
$a(t)$, $b(t)$ real-valued, continuous on some interval $I = (\alpha, \beta)$

To solve this, it's useful to consider the associated homogeneous eqn

$$\boxed{\frac{dy}{dt} + a(t)y = 0} \quad (☆☆)$$

(*)(*) separable... Let's solve by new method... $\frac{dy}{dt} + a(t)y = 0$

Separate $\Rightarrow$ $\frac{1}{y}\,dy = -a(t)\,dt$

Integrate $\ln|y| = -\int a(t)\,dt + C$

So assume $A(t) = \int a(t)\,dt$ ($A(t)$ is anti-deriv of $a(t)$)

$\ln|y| = -A(t) + C$

$|y| = Ce^{-A(t)}$

$|ye^{A(t)}| = C$

If $y(t)$ is a solution, it's a differentiable $\to$ continuous and we have shown it satisfies

$|y(t)e^{A(t)}| = C$

So $y(t)e^{A(t)}$ is also continuous

By exercise 5 $\boxed{y(t)e^{A(t)} \text{ is constant}}$

And thus we have a solution $y(t) = Ce^{-A(t)}$ where $A(t) = \int a(t)\,dt$

Q: did I miss solutions here? to (*)(*)

No I didn't but have to wait for proof

What if I choose another antiderivative $\tilde{A}$ of $a(t)$? No!

okay... Now let's solve (*)

$$\frac{dy}{dt} + a(t)y = b(t)$$

(1) Multiply by $e^{A(t)}$ where $A(t)$ is an anti-deriv of $a(t)$

$$\boxed{\frac{dy}{dt}e^{A(t)} + e^{A(t)}a(t)y = e^{A(t)}b(t)}$$

$$\underbrace{\frac{dy}{dt}e^{A(t)} + A'(t)e^{A(t)}\cdot y} = e^{A(t)}b(t)$$

So $$\frac{d}{dt}\left[ye^{A(t)}\right] = e^{A(t)}b(t)$$

So $$e^{A(t)}y(t) = \int b(t)e^{A(t)}\,dt + C$$

or $$\boxed{y(t) = e^{-A(t)}\int b(t)e^{A(t)}\,dt + Ce^{-A(t)}}$$

**Theorem** Let $a(t)$, $b(t)$ be cont. functions on $I = (\alpha, \beta)$, let $A(t)$ be antiderivative of $a(t)$. Then the general solution to

$$\frac{dy}{dt} + a(t)y = b(t) \quad (\star)$$

is given by $y(t) = e^{-A(t)}\int e^{A(t)}b(t)\,dt + Ce^{-A(t)}$

By general solution, I mean all solns are of this f

Don't memorize ... just remember the method ...

Example solve ODE $\boxed{\frac{dy}{dt} + 4y = e^{-3t}}$ $\quad a(t) = 4$, $b(t) = e^{-3t}$

What to multiply by to make LHS like product rule? $e^{4t}$

$$\underbrace{e^{4t}\frac{dy}{dt} + 4e^{4t}y} = e^{-3t}\cdot e^{4t} = e^{t}$$

$$\frac{d}{dt}\left(y(t)e^{4t}\right) = e^{t}$$

So $$y(t) = e^{-4t}\int e^{t}\,dt + Ce^{-4t} = \boxed{e^{-3t} + Ce^{-4t} = y(t)}$$

t 21, wed

<u>Recall</u> $\frac{dy}{dt} + a(t)y = b(t) \to e^{A(t)}\frac{dy}{dt} + e^{A(t)}\frac{dA(t)}{dt}y = b(t)$

<u>So</u> method: multiplying both sides by $\mu(t) = e^{\int a(t)dt} = e^{A(t)}$

So $\frac{d}{dt}\left(ye^{A(t)}\right) = b(t)e^{A(t)}$ $\qquad \mu(t) = e^{\int a(t)dt}$ ↳ int. factor

<u>So</u> $\boxed{y(t) = e^{-A(t)}\int b(t)e^{A(t)}dt + Ce^{-A(t)}}$ (*)

If a & b are continuous on $I = (\alpha, \beta)$ then (*) is the general solution to $\frac{dy}{dt} + a(t)y = b(t)$

<u>Solve the IVP</u>

$$\begin{cases} dy/dt + 4y = e^{-3t} \\ y(0) = 2 \end{cases}$$

<u>Recall</u>

$y(t) = e^{-3t} + Ce^{-4t}$

$y(0) = 1 + C = 2 \Rightarrow C = 1$

So $y(t) = e^{-3t} + e^{-4t}$

<u>Check</u> $\frac{dy}{dt} = -3e^{-3t} - 4e^{-4t} = -4(\underbrace{e^{-4t} + e^{-3t}}_{y}) + e^{-3t}$

So $\frac{dy}{dt} + 4y = e^{-3t}$ (verified) } satisfies IVP

Also, $y(0) = 1 + 1 = 2$

$\boxed{\text{Example}}$ $\begin{cases} y' + y = 10t \\ y(0) = 1 \end{cases}$

$u = t \quad du = dt$

$dv = e^t dt \quad v = e^t$

$e^{\cancel{t^2}}\frac{dy}{dt} + e^{\cancel{t}}y = 10t \cdot e^{\cancel{t^2}}$

$y(t) = e^{-\cancel{t^2}}\int 10t \cdot e^{t}dt + Ce^{-t}$

$= 10e^{-t}\int te^{t}dt + Ce^{-t} = 10e^{-t}\left[te^{t} - \int e^{t}dt\right] + Ce^{-t}$

$$\boxed{y(t) = 10t - 10 + Ce^{-t}}$$

$y(0) = 1 = 0 - 10 + C \rightarrow C = 11 \rightarrow y(t) = 10t - 10 + 11e^{-t}$

The linearity of linear equation (linear algebra with ODE)

Reminder A vector space over $\mathbb{R}$ is a set $V$ which satisfies a number of properties:

+ For any $v \in V$, $1 \cdot \vec{v} = \vec{v}$
+ For any $a, b \in \mathbb{R}$, $\vec{v} \in V$, $(a+b)\vec{v} = a\vec{v} + b\vec{v}$
+ ~~For $v \in V$~~, $\exists\ 0_V$, called the zero vector for which $\vec{0}_V + \vec{v} = \vec{v} = \vec{v} + \vec{0}_V \quad \forall \vec{v} \in V$
+ For all $\vec{v} \in V$, $\vec{v} + (-1)\vec{v} = \vec{0}_V$
+ (associativity), (commutativity) of vector addition with scalar multiplication

Example $\mathbb{R}^d = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix}, a_i \in \mathbb{R} \text{ for } i = 1 \dots d \right\}$ is a vector space

Here, $\vec{0}_{\mathbb{R}^d} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

Ex $$\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2-1 \\ 3-2 \\ 2+4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix}$$

Similarly $$-4 \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -8 \\ -12 \\ -8 \end{pmatrix}$$

Example Consider an interval $I = (0,1)$ & define

$$C^0(I) = \{ f = \text{fn on } I \text{ which are continuous} \}$$
$$= \{ f : I \rightarrow \mathbb{R} \mid f \text{ continuous} \}$$

Ex $Id(t) = t$, $abv(t) = |t|$, $sin(t)$, $0(t) = 0$ $\Bigg\} \in C^0(0,1)$

We add functions $(f+g)(t) = f(t) + g(t)$ → addition in $C^0$

scalar multiply $(4f)(t) = 4(f(t))$

$\boxed{\text{Fact}}$ $\boxed{C^0(I) \text{ is a vector space equipped with this addition, multiplication, and zero function...}}$

Another example $\boxed{C'(I) = \left\{ f: I \mapsto \mathbb{R} \;\middle|\; f \text{ is differentiable } \& \; f' \in C^0(I) \right\}}$

Fact $C'(I)$ is a vector space

Note $\sin(t) \in C'(I)$ (Because $\sin'(t)$ exists and $\in C^0(I)$)

Note
$f(t) = |t - \frac{1}{2}| \notin C^1(I)$, but $f \in C^0(I)$

Recall Differentiability implies continuity → $\boxed{C'(I) \subset C^0(I)}$

Is $C^1(I)$ a subspace of $C^0(I)$

t 24, 2019 We've seen that $C^0(I)$ and $C'(I)$ are vector spaces over $\mathbb{R}$

Let's return to the general theory of linear algebra.

Def. A function $T: V \mapsto W$ is called a linear operator if the following properties hold:

(1) For $\forall \vec{v}_1, \vec{v}_2 \in V$, $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$

(2) For $\vec{v} \in V$, $k \in \mathbb{R}$, $T(k\vec{v}) = kT(\vec{v})$

An important object for a linear operator is its kernel – the s

$$\boxed{\ker(T) = \{ \vec{v} \in V \mid T(\vec{v}) = \vec{0} \}}$$

Recall $\boxed{\ker(T) \subset V \text{ and } \ker(T) = \{\vec{0}\} \text{ iff } T \text{ is one-to-one}}$

Ex Let $V = \mathbb{R}^3$, $W = \mathbb{R}^2$

Consider $T: V \mapsto W$ given by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ z+x \end{pmatrix}$

Verify this is linear... $T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix}$

$$= \begin{pmatrix} -y_1 \\ z_1+x_1 \end{pmatrix} + \begin{pmatrix} -y_2 \\ z_2+x_2 \end{pmatrix} = \begin{pmatrix} -y_1+y_2 \\ z_1+x_1+z_2+x_2 \end{pmatrix}$$

Also $T\begin{pmatrix} cx_1 \\ cy_1 \\ cz_1 \end{pmatrix} = \begin{pmatrix} cy_1 \\ cx_1+cz_1 \end{pmatrix} = c\begin{pmatrix} -y_1 \\ z_1+x_1 \end{pmatrix} = c\,T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$

Find $\ker(T)$ $\quad T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ x+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $\begin{cases} y = 0 \\ x+z = 0 \end{cases}$

So $\ker(T) = \left\{ \begin{pmatrix} x \\ 0 \\ -x \end{pmatrix}, x \in \mathbb{R} \right\} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Is $T$ injective? $\boxed{\text{No}}$ $\quad \ker T \neq \{\vec{0}\}$, $\ker T \neq \emptyset$

$\boxed{\text{Example}}$ Let $V = C^1(I)$, $W = C^0(I)$, $I = (\alpha, \beta) \subseteq \mathbb{R}$

Define a function $D: C^1(I) \mapsto C^0(I)$

by $D[y](t) = y'(t)$ for $t \in I$ $\forall$ $y \in C^1(I)$

Verify $D$ is a linear operator

<u>Claim</u> $D$ is a linear operator

Given $y_1, y_2 \in C'(I)$, $k \in \mathbb{R}$

☑ $D[y_1 + y_2](t) = y_1'(t) + y_2'(t)$

$$\overset{\|}{(y_1+y_2)'} \;\; = \;\; D[y_1](t) + D[y_2]t$$

☑ $D[k y_1](t) = [k(y_1)]' = k y_1' = k D[y_1](t)$

<u>So</u> $D$ is ~~anlinear~~ operator

<u>Observe that</u> $\ker(D) = \{ y \in C'(I) \mid D[y](t) = 0 \;\; \forall t \in I \}$

$= \{ y \in C'(I) \mid y'(t) = 0 \;\; \forall t \in I \}$

$=$ set of solution to $y'(t) = 0$

<u>On the other hand</u> $\ker(D) =$ set of constant functions

<u>So</u> the set of solutions to $y'(t) = 0$ is the set of constants

<u>Ex</u>

Let $I = (\alpha, \beta) \in \mathbb{R}$. and $a, \cancel{b} \in C^0(I)$

<u>Define</u> $L : C'(I) \longmapsto C^0(I)$ by

$$L[y](t) = y'(t) + a(t)\, y(t) \qquad \forall t \in I$$

We can easily check that $L[y](t)$, or, $L$, is an linear operator

↳ It is called a $1^{st}$ order ODE operator

$\boxed{\text{Note}}$ $\ker(L) = \{ y \in C'(I) \mid L[y](t) = 0 \;\; \forall t \in (\alpha, \beta) \}$ homogeneous

$= $ set of solutions to $y'(t) + a(t)\, y(t) = 0$

Sept 26, 2012 We saw, for $I = (\alpha, \beta)$ and $a(t) \in C^0(I)$, $L : C^1(I) \to C^0(I)$ defined by:

$$L[y](t) = y'(t) + a(t)\,y(t) \qquad (t \in I)$$

for each $y \in C^1(I)$

$\boxed{\text{Proposition}}$ $L : C^1(I) \to C^0(I)$ is a linear operator, called a first-order linear differential operator. Further $\ker(L)$ is exactly the set of solutions to

$$y' + a(t)\,y = 0 \qquad (*)$$

i.e. $y \in \ker(L)$ iff $y$ solves $(*)$

$\boxed{\text{Remark}}$ Our knowledge of Linear Algebra informs our study of ODE

~~Collor~~ $\boxed{\text{Corollary}}$ If $y$ is any solution to $(*)$, then $cy$ is a solution to $(*)$ $\forall\, c \in \mathbb{R}$.

$\boxed{\text{So, constant multiples of solutions are solutions}}$

↳ This does not hold for non-linear ode

$\boxed{\text{Q: what about bases?}}$

Proposition Let $L$ be as above, i.e. $L[y] = y' + a(t)y$

And let $A(t)$ be an antiderivative of $a(t)$ (FTC).

Then $\{e^{-A(t)}\}$ is a basis for $\ker(L)$

$\boxed{\text{In particular, } \ker(L) \text{ is one-dimensional}}$

Remark Antiderivatives always exist! Given $a(t) \in C^0(I)$, let $t_0 \in I$ then ~~$A(t)$~~ $A(t) = \int_{t_0}^{t} a(s)\,ds$ is an antiderivative in view of FTC part (1).

**Proof** Note $\frac{d}{dt}\left(e^{-A(t)}\right) + a(t)e^{-A(t)} = -a(t)e^{-A(t)} + a(t)e^{-A(t)}$
$= 0$

So $e^{-A(t)} \in \ker(L)$

It remains to show that any solution $w = w(t) \in \ker(L)$ is of the form

$$w(t) = Ce^{-A(t)}$$

Let $w(t) \in \ker(L)$. Consider $f(t) = \frac{w(t)}{e^{-A(t)}} = w(t)e^{A(t)}$

$$\frac{d}{dt}f(t) = w'(t)e^{A(t)} + w(t)a(t)e^{A(t)}$$

$$= e^{A(t)}\left[\underbrace{w'(t) + a(t)w(t)}_{0}\right] = 0 \quad \forall t \in I$$

$\boxed{\text{So, by MVT. We have that } f \text{ is identically constant}}$

So, $\exists C$ s.t. $C = f(t) = w(t)e^{A(t)} \quad \forall t \in I$

Thus $w(t) = Ce^{-A(t)} \quad \forall t \in I$

So every element of $\ker(L)$ is a constant multiple of $e^{-A(t)}$

So $\ker(L)$ has $\{e^{-A(t)}\}$ as a basis

So $\boxed{\dim(\ker(L)) = 1}$

This is a statement about uniqueness. All solutions look like $\underbrace{Ce^{-A(t)}}$

Sept 28, 2018

$\boxed{\text{Slope fields}}$ Consider a first-order ODE

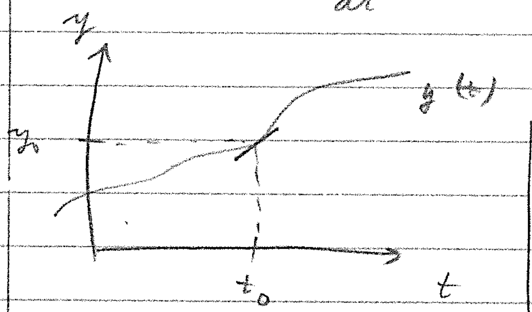
$$\frac{dy}{dt} = f(t,y) \quad (1) \longrightarrow \text{general}$$

What do solutions look like, graphically?

Suppose $y = y(t)$ is a solution to (1), and for some $(t_0, y_0) \in \mathbb{R}^2$

We can consider $f(t_0, y_0)$ (which we compute) and draw a small line segment at $(t_0, y_0)$ with slope $f(t_0, y_0)$. Then, as $y$ is a solution, the eqn

$$\frac{dy}{dt}(t_0) = f(t_0, y_0)$$ means that at $t_0$, the graph of $y$ is tangent to this mini-tangent



So, globally, for any $(t,y) \in \mathbb{R}^2$, a solution $y = y(t)$ to the diff. eq (1) must have its graph tangent to all mini tangent lines.

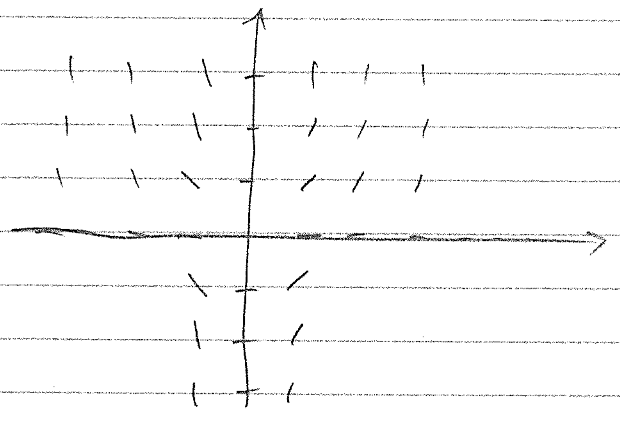
A slope field for $f(t,y)$ is made by plotting mini tangent lines with slope $f(t,y)$ at a number of points $(t,y) \in \mathbb{R}^2$

Given a slope field, a solution is simply a function which "fits" the slope field

Ex $\frac{dy}{dt} = y^2 t = f(t,y)$

Slope field

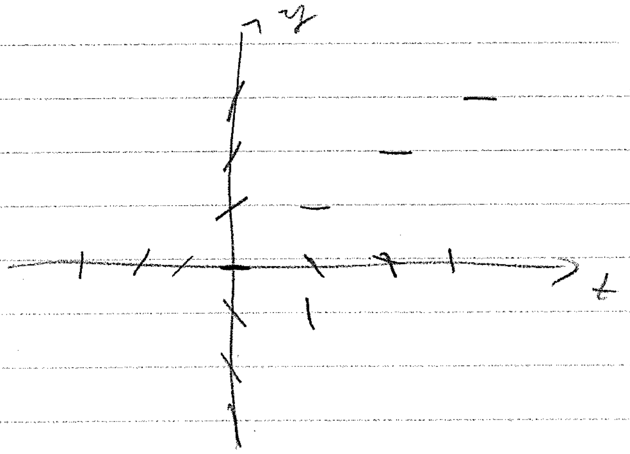
| $t$ | $y$ | $y^2t$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |
| -1 | 1 | -1 |
| 1 | -1 | +1 |



$y = 0$ or

$$\frac{-1}{y} = \frac{1}{2}t^2 + c \qquad y = \frac{-1}{\frac{1}{2}t^2 + C}$$

Ex $\frac{dy}{dt} = y - t$





Slope field

| t | y | y-t |
|---|---|---|
| 0 | y | y |
| t | 0 | -t |

Solution

$$y(t) = e^{+t}\int -te^{-t}\,dt$$

$$= e^{t}\left[-te^{-t} - \int -e^{-t}\,dt\right] = -t - 1 + Ce^{t}$$

-1, 2018 We saw that $\frac{dy}{dt} = y^2 t$. We found $y(t) = \frac{2}{t^2+C}$ was a solution to this ODE. We also missed a solution using sep. of var. $y(t) = 0 \;\forall t$.

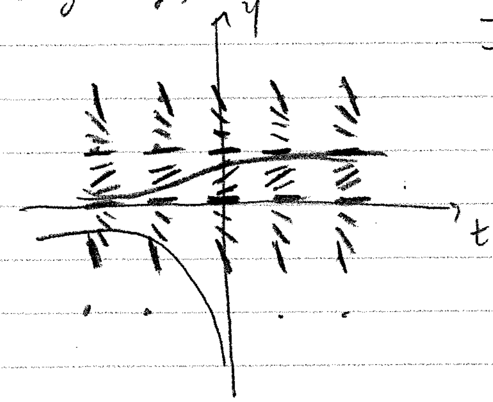
litative servation: Just looking at slope field and $\boxed{\text{solution curves}}$, expect
$$\lim_{t\to\infty} y(t) = 0$$
We expect all solution to drift toward the constant solution $y = 0$

- looking slope field.. Analytically, $\forall C \in \mathbb{R}$, $\lim_{t\to\infty} \frac{2}{t^2+C} = 0$

Called "Equilibrium behavior".

$\boxed{\text{Ex}}$ Consider $\frac{dy}{dt} = y(1-y)$

| t | y | y(1-y) |
|---|---|---|
| | 1 | 0 |
| | 0 | 1 |
| | 2 | -2 |
| | -1 | -2 |





It appears that solutions drift toward $y = 1$ or away from $y = 0$

Are these ($y = 0, 1$) constant solutions?

$\boxed{\text{Yes}}$

So, once again, this ODE has equilibrium behavior. Solutions seem to all drift away from $y=0$ and towards $y=1$

Example
$$\begin{cases} \frac{dy}{dt} = y(1-y) \\ y(0) = y_0 \end{cases}$$
Just from slope field
$$\lim_{t\to\infty} y(t) = \begin{cases} 1 & \text{if } y_0 > 0 \\ 0 & \text{if } y_0 = 0 \\ -\infty & \text{if } y_0 < 0 \end{cases}$$

Solution to IVP is just a curve passing through $(t_0, y_0)$

**Def** Consider ODE $\frac{dy}{dt} = f(t,y)$ where we assume $f$ is a continuous function. If $y(t) = \text{constant} = y_0$ for all $t$ solves this ODE, we call the solution an equilibrium solution.

**Proposition** Consider $\frac{dy}{dt} = f(t,y)$. Then $y(t) = y_0$ is an equilibrium solution iff $f(t, y_0) = 0 \ \forall t$

**Proof** Let $y_0$ be a number for which $f(t, y_0) = 0 \ \forall t$. Set $y(t) = y_0 \ \forall t$ and observe that
$$\frac{dy}{dt} = \frac{d}{dt}(f(t,y_0)) = 0 = f(t, y(t)) \ \forall$$
So $y(t) = y_0$ is an eq. solution

Conversely, if $y(t) = y_0$ is an eq. solution
$$\hookrightarrow f(t, y_0) = \frac{dy}{dt} = \frac{dy_0}{dt} = 0$$
, so $y_0$ is a number

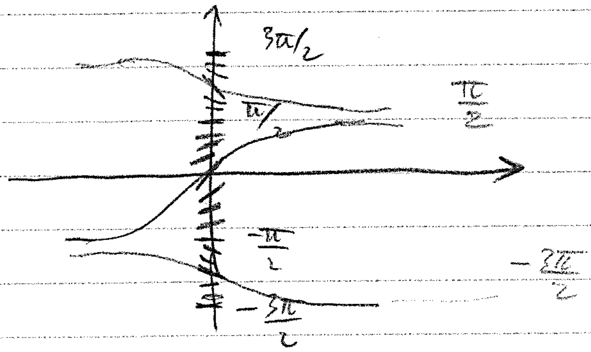
**Def 2** If $y_0$ is a number s.t. $f(t, y_0) = 0 \ \forall t$, we say that $y_0$ is an eq. value or eq. point

$\hookrightarrow y_0$ is an eq. value $\Leftrightarrow y(t) = y_0$ is an eq. solution

⇒ Can we always observe interesting behaviors at equiv values?

$\frac{dy}{dt} = \cos(y)$   Eqv values - draw slope field $y_0 = \frac{\pi}{2} + k\pi$

$\forall k \in \mathbb{Z}$




$\underbrace{\begin{array}{l} D \circ K = I_d \\ K \circ D = I_d - y(t_0) \end{array}}$ } same

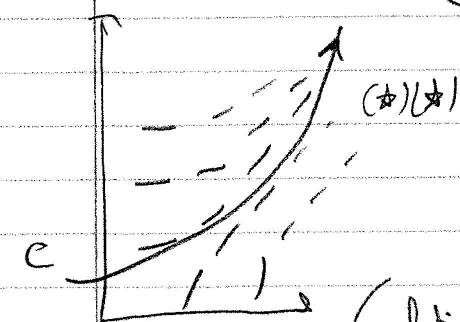
Rank - Nullity theorem doesn't apply here (infinite dimensional)

Consequence of Rank Nullity theorem
↳ If $V$ finite dimensional, and $A$ & $B$ are linear operators on $V$
$A: V \to V$, $B: V \to V$
If $AB = I$, then $B = A^{-1}$
$BA = I$

$\boxed{\text{Back to slope fields}}$

Consider ODE $\frac{dy}{dt} = f(t, y)$ (★) and let's draw its associated slope field

A curve C in the slope field (★★) which is smooth and its tangent line at each point align with the mini-tangent line of the slope field is called
↳ "An integral curve for (★★) for (★)




(★)(★)

When $y$ is a solution to (★), the graph of $y$ is an integral curve

↳ can talk abt integral curve → (geometric)

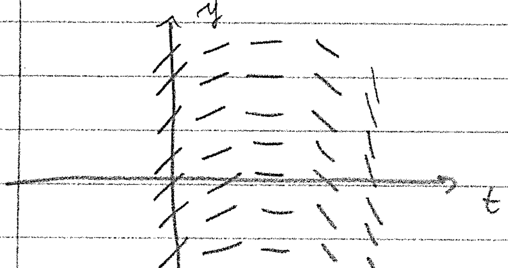
(analytic) → solution simultaneously

Now, geometry of IVP $\rightarrow$ Consider $\begin{cases} dy/dt = f(t,y) \\ y(t_0) = y_0 \end{cases}$

Suppose that $y$ is a solution to IVP. Then because it satisfies $\star$, its graph is an integral curve and b/c it also satisfies the initial condition $y(t_0) = y_0$, its graph must pass thru point $(t_0, y_0)$ ~~that are the~~

So $\boxed{\text{Solution to IVP } (\star\star\star) \Leftrightarrow \text{Integral curve to } (\star\star) \text{ pass thru } (t_0, y_0)}$

Special cases of slope fields $\frac{dy}{dt} = g(t) \rightarrow$ no dep on $y$



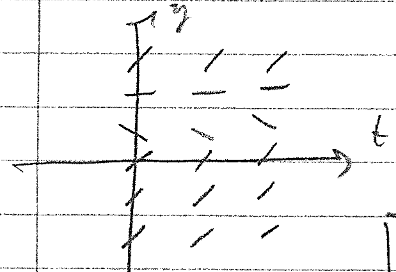

Note that we have vertical translation symmetry. (no surprise)

Solutions are vertical translation of each other. B/c $\frac{dy}{dt} = g(t)$

$\rightarrow y(t) = \int g(t)\,dt + C$ $\leftarrow$ families of antiderivatives

Autonomous

$\hookrightarrow$ A 1st order ODE of form $\frac{dy}{dt} = g(y)$. No dependence on $t$.




Slope field and integral curves have horizontal translation symmetry.

$\boxed{\text{Observation}}$ If given a solution $y = y(t)$ we expect

$y_c(t) = y(t+c)$ to also be a solution to an autonomous eqn

Check

$$\frac{dy_c}{dt} = \frac{dy(t+c)}{dt} = \left[\frac{dy}{dt}\overset{\text{eval}}{(t+c)}\right] \cdot 1$$

$$= g(y(t+c)) \rightarrow \underline{y(t+c) \text{ a solution}} \rightarrow (y_c \text{ a solution})$$

$$= g(y_c(t))$$

A consequence is to solve IVP for autonomous eqn
→ only need to focus on $t_0 = 0$

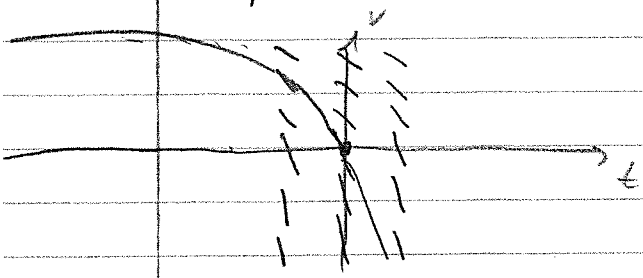
$\boxed{\text{One restriction}}$ → need $y$ differentiable $\forall t$

<u>An application</u> → consider skydiver, mass $m$, jumping from a plane
The motion / velocity of the skydiver is
modelled by

$$m\frac{dv}{dt} = -mg + cv(t)$$ where $c$ depends on friction & density of the air...

$$\frac{dv}{dt} = -g + \frac{c}{m}v \quad (c>0)$$ → autonomous

Slope field, let $c/m = 1$, $g = 10$



IVP $t_0 = 0$
$v(0) = 0$

5, 2018

$\boxed{\text{Vocabulary}}$ → Consider an autonomous O.D.E (first order)

$$\frac{dy}{dt} = h(y) \quad (*)$$

Suppose that $y_0$ is an equilibrium value for $(*)$, i.e $h(y_0) = 0$

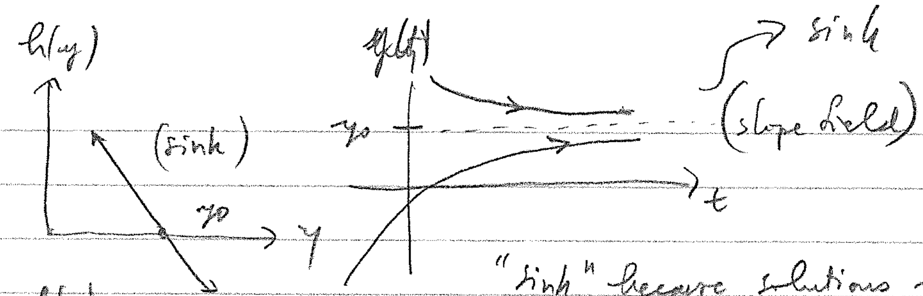
Also, $y(t) = y_0$ is the corresponding equilibrium solution

We say that $y_0$ (both the value & solution) is a <u>sink</u> or a <u>stable</u> eq.

if $h(y) < 0$ for $y > y_0$ (close to $y_0$)
<u>and</u>
$h(y) > 0$ for $y < y_0$ (close to $y_0$)

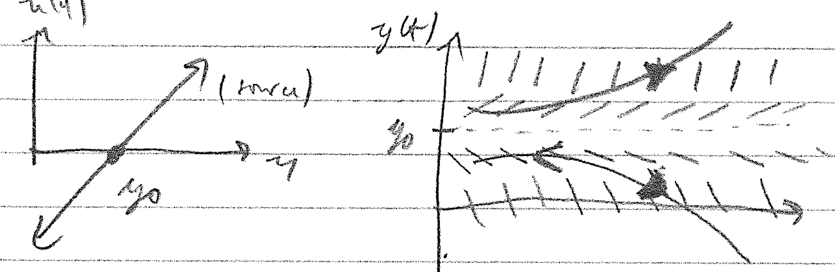
We say that $y_0$ is a <u>source</u> if the opposite is true ...
Otherwise, $y_0$ is called a <u>node</u>. Further, if $y_0$ = source/node, it's said to be unstable

Ex

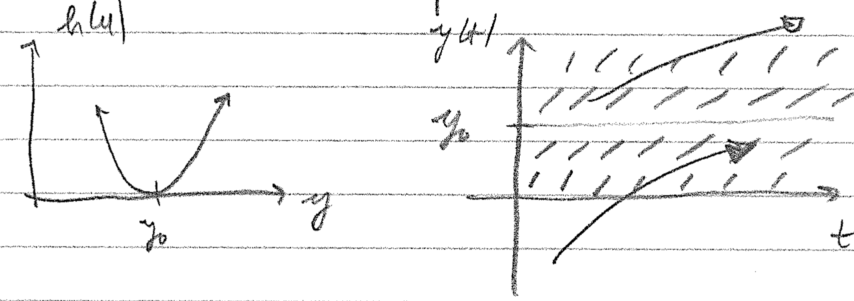



"sink" because solutions not at $y_0$ go to

Ex




Ex




$\boxed{\text{Proposition}}$

Let $y_0$ be an equilibrium value for $\frac{dy}{dt} = h(y)$

Assume $h$ is differentiable at $y_0$. If $\frac{\partial h}{\partial y}(y_0) > 0$, then $y_0$ is a source. If $\frac{\partial h}{\partial y}(y_0) < 0$, then $y_0$ is a sink.

If $\frac{\partial h}{\partial y}(y_0) = 0$, nothing can be said.

$\boxed{\text{Ex}}$ Skydiver

Model $m\ddot{y} = mg - cv$

autonomous 1st order ODE linear

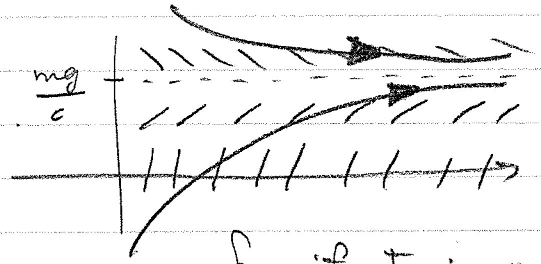
Since $v = \dot{y}$, $\dot{v} = \ddot{y}$ so $m\dot{v} = mg - cv$ or $\boxed{\dot{v} = g - \frac{c}{m}v}$

$h(v)$

Eq. value $h(v) = 0 = g - \frac{c}{m}v$

$\hookrightarrow v_T = \frac{gm}{c}$.

Note that $\frac{\partial h}{\partial v} = \frac{\partial}{\partial v}\left(g - \frac{c}{m}v\right) = \frac{-c}{m} < 0$ so $\boxed{v_T \text{ is a sink}}$



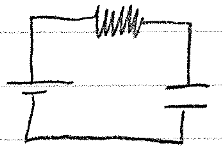

(Integral Curve)

So, if I jump out of a plane, I solve

$$\begin{cases} \dot{v} = g - \frac{c}{m}v \\ v(0) = v_0 \end{cases} \qquad \lim_{t\to\infty} = V_T = \frac{gm}{c} \quad \left(\begin{array}{c}\text{terminal} \\ \text{velocity}\end{array}\right)$$

In fact solution looks like $\boxed{V(t) = V_T + (V_0 - V_T)e^{-\frac{c}{m}t}}$

Example 6 RC circuit



$V_c = ?$

Electric circuit theory gives $RC\frac{dv_c}{dt} + V_c = \underset{\text{known}}{V_l(t)}$

Assume $V_l(t) = V_l = \text{constant}$

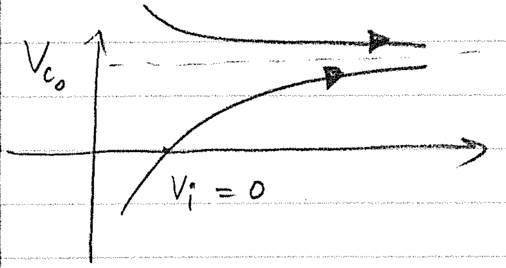
$$\frac{dv_c}{dt} = \frac{V_l - V_c}{RC} = h(v_c)$$

$$0 = h(v_{c_0}) = \frac{V_l - V_c}{RC} \text{ so } V_{c_0} = V_l \leftarrow \text{makes sense}$$

Is this a sink or source?

$$\frac{\partial h}{\partial v} = \frac{-1}{RC} < 0 \rightarrow (\text{sink})$$

$$\begin{cases} \frac{dv}{dt} = \frac{V_l - V_c}{RC} \\ V_c(0) = V_i \end{cases}$$



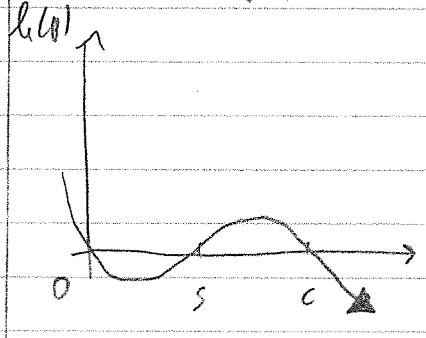
Oct 8, 2018

**Final application** (Logistic Equation)

- $P(t)$ = population @ time $t$ of an "isolated species". We want to ha[ve] a model taking into account
- If population too big, food is scarce → growth of $P$ negative
- If population too small, hard to find mates → growth of $P$ negat[ive]

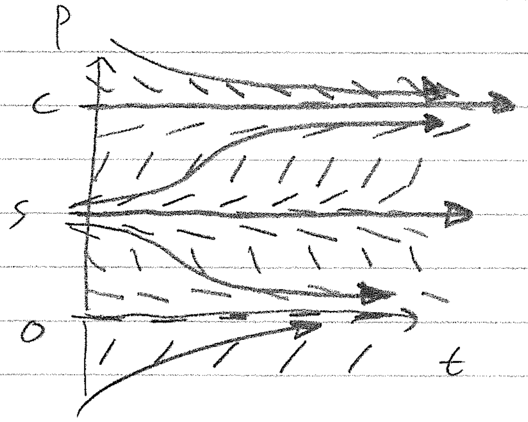
**Model** $\frac{dP}{dt} = kP\left(1-\frac{P}{c}\right)\left(\frac{P}{s}-1\right) = h(P)$

$k$ ↑ growth rate; $c$ ↑ carrying cap; $s$ ↑ sparcity param




3 eq values

0 & c are sinks
S is source




**Verify** $\frac{\partial h}{\partial p} = \frac{\partial}{\partial p}\left[\frac{k}{cs}\, p\,(c-p)(p-s)\right]$

$= -k < 0$ if $P = 0$ (sink)

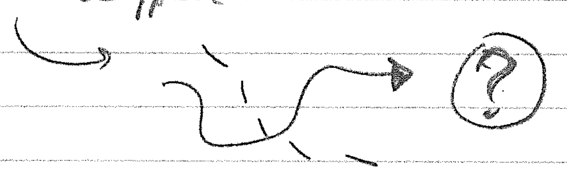
$= \frac{-k}{s}(c-s) < 0$, if $P = c$ (sink)

$= \frac{k}{c}(c-s) > 0$ if $P = s$ (source)

So if $P$ is a solution to ~~the~~ the IVP, $\lim_{t\to\infty} P(t) = \begin{cases} 0 & \text{if } P_0 < \\ s & \text{if } P_0 = \\ c & \text{if } P_0 > \end{cases}$

$\boxed{Q1}$ What happens if there are predators?

$\boxed{Q2}$ Why can't this happen?



$\boxed{\text{Existence / Uniqueness}}$ (Picard Lindelöff Theorem)

Consider $\begin{cases} \frac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases}$ with $f$ minimally continuous ... and its slope field

ALWAYS ← ($Q1_A$) When do we know we can find solution to IVP?
($Q1_G$) when does there exist an integral curve passing through $(t_0, y_0)$

($Q2_A$) if there is a solution to IVP, when do we know it's the only one? That is, is it possibly exist $y$, $\tilde{y}$ both satisfy ODE but $y \neq \tilde{y}$
($Q2_G$) If $\exists$ integral curve thru $(t_0, y_0)$. then do we know it's the only one?

Q1 $\boxed{\text{Ex}}$ Consider $\begin{cases} \frac{dy}{dt} = \frac{y}{t} \\ y(0) = 1 \end{cases}$ can this be solved?

NOT ALWAYS

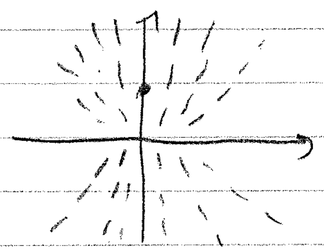
No proof Suppose $y = y(t)$ is a solution, it is a continuous fn and $\frac{dy}{dt}$ is also cont ($y \in C^1$)

So $t \frac{dy}{dt}(t) = y(t)$ $\forall t$ or $\forall t$ near $0$

Thus $1 = y(0) = \lim_{t \to 0} y(t) = \lim_{t \to 0} t \frac{dy}{dt}(t) = 0 \frac{dy}{dt}(0) = 0$

So contradiction!

Graphically



$\boxed{\text{Ex}}$ Consider $\begin{cases} dy/dt = 3y^{2/3} \\ y(0) = 0 \end{cases}$

Sep. of vars

$$\int \frac{1}{3} y^{-2/3}\, dy = \int dt \rightarrow \boxed{y^{1/3} = t + C}$$

So $y = (t+c)^3 = t^3$.

Also $\tilde{y} = 0$ is a solution (eq)

Picard - Lindelof theorem

Oct 10, 2018

Ex $\boxed{\text{Theorem}}$ Let $f = f(t,y)$ be defined, continuous & have continuous partial derivative $\frac{\partial f}{\partial y}$ on the rectangle

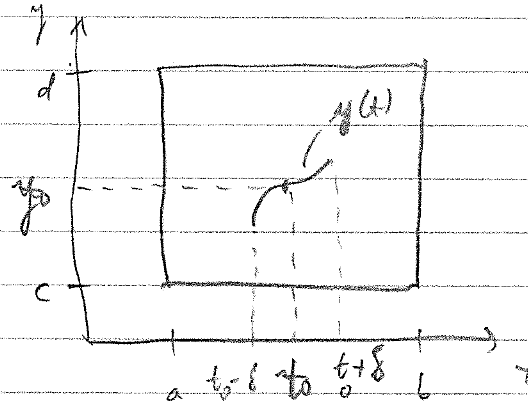
$$R = \{(t,y) : a \leq t \leq b,\ c \leq y \leq D\} = [a,b] \times [c,\ldots \subseteq \mathbb{R}^2$$

If $(t_0, y_0) \in \text{Interior}(R)$, i.e. $a < t_0 < b$, $c < y_0 < d$

then the IVP $\begin{cases} \frac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases}$ admits a unique solution. More precisely, $\exists\ \delta > 0$ and a unique function $y \in C^1(t_0 - \delta, t_0 + \delta)$ s.

$y(t_0) = y_0$, and

$\left.\frac{dy}{dt}\right|_t = f(t, y(t))\ \forall\ t \in (t_0-\delta, t_0+\ldots$




Comment Proof goes by something called "Picard iterat...

One notices that $y(t)$ solves (*) iff

$$y(t) = \int_{t_0}^{t} f(s, y(s))\, ds + y_0$$

integral eqn

So, one defines a sequence of functions $\{y_n\}$ by the following recursive method (Picard iterates...)

$$y_1(t) = \int_{t_0}^{t} f(s, y_0)\, ds + y_0, \quad y_n(t) = \int_{t_0}^{t} f(s, y_{n-1})\, ds. \longrightarrow$$

**Idea** If you can use the hypotheses to show that $\lim_{n\to\infty} y_n(t)$ exists. If $y(t) = \lim_{n\to\infty} y_n(t)$ exists and is in $C^I$, then

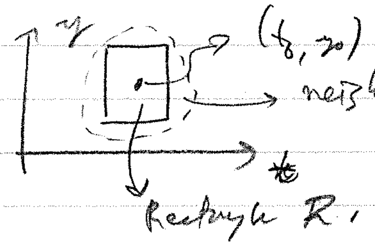
$$y(t) = \lim_{n\to\infty} y_n(t) + y_0$$

$$= \lim_{n\to\infty} \int_{t}^{t_0} f(s, y_{n-1}(s))ds + y_0$$

$$= \int_{t_0}^{t_n} f(s, y_n(s))ds + y_0 = y(t)$$

**Vocab** We say that $f(t,y)$ or $(*)$ satisfies the hypotheses of the Picard Lindeloff theorem at an initial point $(t_0, y_0)$ if in fact $f$ & $\frac{\partial f}{\partial y}$ continuous in a neighborhood of $(t_0, y_0)$

**Idea**




**Examples** (1) Does $\frac{dy}{dt} = y - t$ satisfy the hypotheses of Picard-Lindeloff theorem at $(0,0)$

(1) Is $f(t,y) = y - t$ continuous on near $(0,0)$? Yes!
↳ just a polynomial

(2) Is $\frac{\partial f}{\partial y}$ also continuous (if it exists)?

$$\frac{\partial f}{\partial y} = 1 \text{ (continuous) near } (0,0)$$

By P-L, there's no other solution

So there exists a unique solution $y(t)$ to the IVP $\begin{cases} \frac{dy}{dt} = y - t \\ y(0) = 0 \end{cases}$

Find solution $y(t) = t + 1 + ce^t$

$y(0) = 0 = 1 + C \Rightarrow C = -1$

$\Rightarrow \boxed{y(t) = t + 1 - e^t}$

By P-L → unique

By P-L, in fact, there's no other solution!

Note $f(t,y) = y - t$ satisfies the P-L theorem at every initial point $(t_0, y_0)$

Conclusion I can solve any IVP of the form $\begin{cases} \frac{dy}{dt} = y - t \\ y(t_0) = y_0 \end{cases}$ unique

$\boxed{\text{Ex}}$ $\begin{cases} \frac{dy}{dt} = y/t \\ y(0) = 1 \end{cases}$

What about P-L? Initial point is $(0,1)$. Here $f(t,y) = \frac{y}{t}$. Note that this function is not continuous on any neighborhood containing $(0,y)$

Oct 12 2018 Ex (what went wrong example) $\begin{cases} \frac{dy}{dt} = 3y^{2/3} \\ y(0) = 0 \end{cases}$ ← Recall we found 2 dis solutions $y(t) = t^3$, $y(t) = 0$

Since ∃ 2 solutions, the hypotheses of P-L can't be met.

$f(t,y) = 3y^{2/3}$. $\frac{\partial f}{\partial y} = 3 \cdot \frac{2}{3} y^{-\frac{1}{3}}$ not cont. @ $y = 0$

For any $(t_0, y_0)$ such that $y_0 \neq 0$, this function does in fact satisfy the conditions / hypotheses of P-L theorem → ∃ unique solution.

(1) P-L theorem tells us when our search for solutions isn't fruit (this ties up with prescription for numerical approximation of solutions)
(2) It further puts our slope field analysis on a rigorous footing. It says when ∃ an integral curve.
(3) Guessing can therefore be an effective method. If, by hook or crook, you find a solution, P-L tells you if it's the only one & you can stop looking

(4) What P-L does NOT do:
↳ P-L doesn't give formula for a solution
For that you have to use method / guess or numerical approximation

(5) With P-L, I can start saying THE solution instead of "a" solution for IVP's.

$\boxed{\text{A new method}}$ (Exact Equations)

First, background: Consider a function $f: \mathbb{R}^n \to \mathbb{R}^m$ and point $\vec{x}_0$ in $\mathbb{R}^n$. We say that $f$ - differentiable at $\vec{x}_0 \in \mathbb{R}^n$ if $\exists$ a linear transformation $D$
$D: \mathbb{R}^n \to \mathbb{R}^m$ for which

$$\lim_{\vec{h} \to \vec{0}} \frac{\overbrace{\|f(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) - D\vec{h}\|}^{\mathbb{R}^m\text{-norm}}}{\underbrace{\|\vec{h}\|}_{\mathbb{R}^n\text{-norm}}} = 0$$

Ex

$f(x) = x^2 : \mathbb{R} \mapsto \mathbb{R}$  $n = m = 1$  $x_0 = 1$

$$\lim_{h \to 0} \frac{|f(x_0+h) - f(x_0) - Dh|}{|h|} = \lim_{h \to 0} \frac{|(1+h)^2 - 1^2 - Dh|}{|h|}$$

$$= \lim_{h \to 0} \left| \frac{2h + h^2 - Dh}{h} \right|$$ $Dh =$ Constant$\cdot h$ for $\mathbb{R}^1$, $D = 2$

$$= \lim_{h \to 0} \left| \frac{h^2}{h} \right| = 0$$ (true if $D = 2$)

$$\boxed{D = \left.\frac{df}{dx}\right|_{x_0 = 1}}$$ → unique linear operator

Fact if $f$ is differentiable at $x_0$, $D = D_{x_0}$ is unique is called the derivative of $f$ at $x_0$
Denoted $D_{x_0} f$

Fact In Euclidean words, if $f$ is diff. at $\vec{x}_0$

$\frac{\partial f^j}{\partial x_i}(x_0)$ for $i = 1, 2, 3 \ldots n$, $j = 1, 2, 3, \ldots m$ exist

and

$$Df_{x_0} = \begin{pmatrix} \frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f^m}{\partial x_1} & \cdots & \frac{\partial f^m}{\partial x_n} \end{pmatrix}$$

Ex $f: \mathbb{R}^2 \mapsto \mathbb{R}^2$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + y^2 \\ \sin y \end{pmatrix} \to Df = \begin{pmatrix} 2x & 2y \\ 0 & \cos y \end{pmatrix}$$

Oct 17, 2018 We Recall A function $f: \mathbb{R}^n \mapsto \mathbb{R}^m$ $f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f^1 \\ \vdots \\ f^m \end{pmatrix}$ where $f^i: \mathbb{R}^n \to$

differentiable at $\vec{x}_0 \in \mathbb{R}^n$ if $\exists$ matrix $Df(\vec{x}_0)$ $(m \times n)$ called the Jacobian matrix such that

$$\boxed{\lim_{\vec{h} \to 0} \frac{\| f(x_0 + h) - f(x_0) - Df(x_0)(\vec{h}) \|}{\| \vec{h} \|} = 0}$$

norm

Fact if $f$ differentiable at $\vec{x}_0$, then all of $f'$ partial derivatives exist and

$$Df(\vec{x}_0) = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \vdots & & \\ \frac{\partial f^m}{\partial x^1} & \cdots & \frac{\partial f^m}{\partial x^n} \end{pmatrix}$$

Ex Consider $f: \mathbb{R}^2 \mapsto \mathbb{R}^2$, given by $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sin x_2 \\ x_1^2 + x_2^2 \end{pmatrix}$ for $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in$

Let's show that $f$ is diff. @ $\vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Let's form the candidate for the Jacobian mtx @ $(1, 0)^T$

$$Df\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & \cos x_2 \\ 2x_1 & 2x_2 \end{pmatrix}\Bigg|_{\binom{1}{0}} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$

Consider $\tilde{h} = (h_1, h_2)^T \to f(\tilde{x}_0 + \tilde{h}) - f(\tilde{x}_0) - Df(x_0)(\tilde{h})$

$$= f\begin{pmatrix} 1+h_1 \\ 0+h_2 \end{pmatrix} - f\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$= \begin{pmatrix} \sin h_2 \\ (1+h_1)^2 + h_2^2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} h_2 \\ 2h_1 \end{pmatrix}$$

$$= \begin{pmatrix} \sin h_2 - h_2 \\ h_2^2 + h_1^2 \end{pmatrix}$$

$$T \stackrel{\Delta}{=} \frac{\| f(\tilde{x}_0 + \tilde{h}) - f(\tilde{x}_0) - Df(x_0)\tilde{h} \|}{\|h\|} = \frac{\sqrt{(\sin h_2 - h_2)^2 + (h_2^2 + h_1^2)^2}}{\sqrt{h_2^2 + h_1^2}}$$

So $$0 \le \lim_{h\to 0} T \le \frac{|\sin h_2 - h_2|}{\sqrt{h_2^2 + h_1^2}} + \frac{|h_1^2 + h_2^2|}{\sqrt{h_2^2 + h_1^2}}$$

$$\le \lim_{h\to 0} \left| \frac{\sin h_2 - h_2}{h_2} \right| + \sqrt{h_2^2 + h_1^2} = 0$$

So $\lim_{h\to 0} T = 0$ by squeeze theorem $\to f$ diff. @ $x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

**Theorem** (Chain Rule) Let $f: \mathbb{R}^n \mapsto \mathbb{R}^m$ & $g: \mathbb{R}^m \mapsto \mathbb{R}^k$. Consider $g \circ f : \mathbb{R}^n \mapsto \mathbb{R}^k$ defined by $g \circ f(\vec{x}) = g(f(\vec{x}))$. If $f$ is diff. @ $\tilde{x}_0$ & $g$ is diff. @ $f(\tilde{x}_0)$, then $g \circ f$ diff @ $\tilde{x}_0$ and

$$D g \circ f(\tilde{x}_0) = Dg(f(\vec{x}_0)) \cdot Df(\vec{x}_0)$$

matrix multiplication

Note

Df is a $(m \times n)$ matrix and Dg is a $(k \times m)$ matrix

So $Dg\,Df$ is well-defined & the product is a $(k \times n)$ matrix

Remark If: $f: \mathbb{R} \mapsto \mathbb{R}$ & $g: \mathbb{R} \to \mathbb{R} \Rightarrow g \circ f: \mathbb{R} \mapsto \mathbb{R}$
It's easy to see in these cases

$$(g\circ f)'(x_0) = Dg\circ f = D_g\, Df = Dg \cdot f'(x_0) = g'(f(x_0))\, f'(x_0)$$

this is 1D chain rule

1×1 matrix

Oct 19, 2018

**Theorem** Chain Rule If: $f: \mathbb{R}^n \mapsto \mathbb{R}^m$ & $g: \mathbb{R}^m \mapsto \mathbb{R}^k$ are diff. then $(g\circ f)$ diff.

$$\boxed{D(g\circ f) = Dg(f)\cdot Df}$$

Let $y$ be a diff. fn of $x \Rightarrow y = y(x)$ & and let $\Psi: \mathbb{R}^2 \mapsto \mathbb{R}$ diff. and define

$$h: \mathbb{R} \mapsto \mathbb{R} \text{ by } h(x) = \Psi(x, y(x))$$

Compute $h'$?

Observe $h(x) = \Psi(f(x)) = (\Psi\circ f)(x)$ where $f: \mathbb{R} \to \mathbb{R}^2$

$$f(x) = (x, y(x))^T$$

So $\Psi\circ f: \mathbb{R} \mapsto \mathbb{R}$

$$D\Psi = \begin{pmatrix} \frac{\partial \Psi}{\partial x} & \frac{\partial \Psi}{\partial y} \end{pmatrix} \qquad (\Psi: \mathbb{R}^2 \mapsto \mathbb{R})$$

$$Df = \begin{pmatrix} \frac{\partial f^1}{\partial x} \\ \frac{\partial f^2}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial x} \\ \frac{\partial y}{\partial x} \end{pmatrix} \qquad (f: \mathbb{R} \mapsto \mathbb{R}^2)$$

Chain Rule
$$D(\Psi\circ f) = \begin{pmatrix} \frac{\partial \Psi}{\partial x} & \frac{\partial \Psi}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial f^1}{\partial x} \\ \frac{\partial f^2}{\partial x} \end{pmatrix} = \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx}$$

A diff. eqn $\boxed{M(x,y) + N(x,y)\frac{dy}{dx} = 0}$ is called exact if

$M(x,y) = \frac{\partial\psi}{\partial x}(x,y)$ & $N(x,y) = \frac{\partial\psi}{\partial y}(x,y)$ for some nice fn $\psi$

Ex Is $2y + 2xy' = 0$ exact? Solve for $\psi$.

Yes! $\psi = 2xy$

Ex Is $2y^2x + \cos yx \; y' = 0$ exact?

$\psi = x^2y^2 + g(y) = \frac{1}{x}\sin(yx) + h(x) \rightarrow$ NO

If eqn is exact, we can solve easily!

$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0$ for some $\psi : \mathbb{R}^2 \mapsto \mathbb{R}$

i.e. $\psi(x, y(x)) = C$, So our solution must be gotten from $\psi = C$
$\rightarrow$ solve for $y(x)$

Ex Consider $2y + 2x\frac{dy}{dx} = 0$

Note this is exact $\rightarrow$ $\psi(x,y) = 2xy = C$

So $y(x) = \frac{C}{2x}$

If $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is exact for $\psi(x,y)$, Then the solution is found by solving $\psi(x,y) = C$ for $y$

Ex $(2x+y) + (x^2 + \cancel{1})\frac{dy}{dx} = 0$, $y(1) = 0$

$\psi(x,y) = yx^2 + y$ ~~x~~ + ~~...~~ No!

$= x^2 + yx + g(y)$

~~$x^2 + 2y = C \rightarrow y(x) = \frac{C - x^2}{x}$~~

~~Since $y(1) = 0 \Rightarrow y(x) = \frac{1 - x^2}{x}$~~

Oct 22, 2018

**Def** Let $\overset{o}{R} = (a,b)\times(c,d)$ be a rectangle (open). A first-order ODE of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$ is said to be exact if

$\exists$ a differentiable function $\psi : \overset{o}{R} \mapsto \mathbb{R}$ s.t.

$$M(x,y) = \frac{\partial \psi}{\partial x}, \quad \frac{\partial \psi}{\partial y} = N(x,y)$$

(☆) We observed if $M(x,y) + N(x,y)\frac{dy}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0$ for this the eqn

$$\left.\psi(x,y) = C \text{ iff } M(x,y) + N(x,y)\frac{dy}{dx} = 0 \right\} \text{ by}$$

virtue of chain rule.

Ex Consider $2xy + x^2\frac{dy}{dx} = 0$.

This is exact. $\underline{\psi(x,y) = x^2 y}$, $= \int \frac{\partial \psi}{\partial x}dx + g(y)$

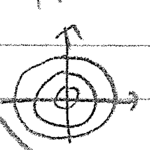
$$= \int \frac{\partial \psi}{\partial y}dy + h(x)$$

Solution: $x^2 y = C = \psi(x,y) \rightarrow \boxed{y = \frac{C}{x^2}} \quad (x \neq 0)$

$\boxed{Q1}$ If the eqn is exact, how in general do you solve $\psi(x,y) =$ for $y = y(x)$? Question is local solvability. That's what is important for IVP.

Ex

If $\psi(x,y) = x^2 + y^2$, then $\psi(x,y) = x^2 + y^2 = C$ gives circles as integral curves.



← not graphs

**Fact** (corollary to implicit function theorem)

Given an initial point $(x_0, y_0)$. If $\frac{\partial \psi}{\partial y}(x_0, y_0) \neq 0$, then $\exists$ a fn $C^1$ $y = y(x)$ s.t.

$$\psi(x, y(x)) = C \quad \forall x \text{ near } x_0$$

This condition should make us happy.

**Note**
$$\begin{cases} M(x,y) + N(x,y)\dfrac{dy}{dx} = 0 \\ \dfrac{dy}{dx} = \dfrac{-M(x,y)}{N(x,y)} = \dfrac{-M(x,y)}{\frac{\partial \Psi}{\partial y}(x,y)} \end{cases}$$

What does this mean? Often $\Psi(x,y)$ cannot easily be solved for $y$. So writing $\Psi(x,y) = C$ is good enough for solution

→ This is called an implicit solution

$\boxed{Q2}$ This still leaves us with the business of figuring out if $M + N\frac{dy}{dx} = 0$ is exact.

For this, we have the following theorem:

**Theorem** Let $M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ be continuous on rectangle $\mathring{R}$: $(a,b)\times(c,d)$
then $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is exact iff
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ on } \mathring{R}$$

Proof of necessity Suppose that $M + N\frac{dy}{dx} = 0$ is exact, then

$M = \frac{\partial \Psi}{\partial x}$, $N = \frac{\partial \Psi}{\partial y}$ for some nice fn $\Psi$. Then

$$\frac{\partial M}{\partial y} = \frac{\partial^2 \Psi}{\partial y \partial x} = \frac{\partial^2 \Psi}{\partial x \partial y} = \frac{\partial N}{\partial x}$$ (equality of mixed partials for smooth fn)

{ The Converse is slightly difficult to prove. It relies on the algebraic topology of $\mathring{R}$. (counting holes... simply connectedness)

→ DeRahm Cohomology)

$\underline{Ex}$ $2xy + (x^2+y^2)\frac{dy}{dx} = 0$

Is this exact? $\frac{\partial M}{\partial y} = 2x \quad \frac{\partial N}{\partial x} = 2x \Rightarrow$ Exact!

$\Psi(x,y) = \int 2xy\,dx + g(y) = \int (x^2+y^2)\,dy + h(x)$

$= x^2y + g(y) = x^2y + \frac{1}{3}y^3 + h(x)$ let this be 0

$g(y) \downarrow \frac{1}{3}y^3$

So $\Psi(x,y) = x^2y + \frac{1}{3}y^3 = C$ can be solved for y, but hard → leave in implicit form.

$\underline{Ex}$ $(2x+y) + (x^2+1)\frac{dy}{dx} = 0$

Not exact because $\frac{\partial M}{\partial y} = 1 \neq 2x = \frac{\partial N}{\partial x}$ → stop searching

Method for exact eqn won't work. But this is linear → can solve

$$\frac{dy}{dx} + \frac{1}{x^2+1}y = \frac{-2x}{x^2+1}$$

$\boxed{\text{General Method for solving}}$

Given $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

(1) Determine if it's exact $\left(\frac{\partial M}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial x}\right)$

Yes → Find $\Psi$ s.t. $\Psi = C$ by partial integration → $y(x)$ / implicit form

No → Stop. Try another method

Oct 14, 2008 $\boxed{\text{EXAM}}$

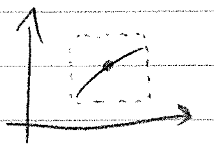
(T, F), Exact, Application, Linear, Solving ODE

- Picard-Lindelöff (why do we need this?)
- Linear algebra ~ ODE (spaces, linear, calculation w FOLDO)

$C^1 \supset C^0$ → computational

analytic solution? what does mean for ODE to have solution...

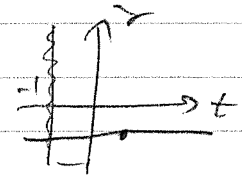
To satisfy hyp of P-L : $f$, $\frac{\partial f}{\partial y}$ cont on $R$ containing $(t_0, y_0)$



conclusion $\exists!$ a unique solution $y$ whose graph passes thru $(t_0, y_0)$ & lives inside $R$

Suppose $f(t,y) = \frac{t^2}{y+1}$ $y = -1$ is the problem

Everything nice except at $y = -1$



given any $(y_0, t_0)$ s.t $y_0 \neq -1$

Suppose given $(t_0, y_0)$ s.t $y_0 \neq 1$

↳ can draw a Rectangle around $(t_0, y_0) \to$ unique soln thru $(t_0, y_0)$

But if $(t_0, y_0)$ s.t. $y_0 = -1$

$\to$ no way can we draw a rectangle that $f$, $\frac{\partial f}{\partial y}$ cont over the entire $R$.

If $R = \mathbb{R}^2$ $\boxed{\text{"Every point has a solution but might not } \forall t\text{"}}$

## Second order & higher order ODE

Recall : A second order ODE is one of the form $y'' = f(t, y', y)$

where $f = f(t, a, b) : \mathbb{R}^3 \mapsto \mathbb{R}$ (domain $D \subseteq \mathbb{R}^3$)

E.g : $\boxed{\text{Newton 2}}$

How many integration constants (I.C.) do we expect?

Expect 2.

"Justify"

Free fall : $y'' = a = -g$ . $y'(t) = v(t) = \int -g\, dt = -gt + V_0$

so $y(t) = \frac{-gt^2}{2} + V_0 t + y_0$

initial v ↑ initial y ↑

**Theorem:** $\boxed{\text{P-L for order 2 ODE}}$

Consider IVP: $\underset{(*)}{y'' = f(t, y', y)}$, $\overbrace{y(t_0) = y_0,\ y'(t_0) = y_0'}^{2\text{ I.C.}}$

If $f(t, a, b)$ and $\frac{\partial f}{\partial a}$ and $\frac{\partial f}{\partial b}$ are continuous on an open region (perhaps a cube) containing $(t_0, y_0', y_0)$, then $(*)$ has a unique solution $\underline{y \in C^2(I)}$ where $I$ is an open interval containing $t_0$ $\longrightarrow$ twice differentiable, with both derivs continuous.

**Remark** The proof of this is a trivial consequence of the $1^{st}$-order version of P-L for system

$\boxed{\text{Ex}}$ $y'' = (y')^2 + y \sin t \rightarrow f(t, a, b) = a^2 + b \sin t \rightarrow$ satisfies P-L at any point $(t_0, y_0, y_0')$. $\rightarrow$ can't get solutions by hand th

We focus on some simple $2^{nd}$ order ODE. Def A second order linear ODE is an ODE of the form

$$y'' + p(t) y' + q(t) y = r(t) \qquad (*)(*)$$

If $r(t) = 0$, $(*)(*)$ homogeneous. $r(t) \neq 0 \rightarrow (*)(*)$ inhomogeneous and $y'' + py' + qy = 0$ is called the homogeneous counterpart of $(*)(*)$

$\boxed{\text{Ex}}$ $y'' + \left(\frac{-2t}{1-t^2}\right) y' + \frac{\alpha(\alpha-1)}{(1-t^2)} y = 0$

Legendre's diff eqns

Also, $t^2 y'' + t y' + (t^2 - \nu^2) y = 0$

This called Bessel's equation. Arises in the study of the beating of a drum.

Given $y'' + py' + qy = r(t)$

- If $p, q, r \in C^0$, then $f$ is continuous & $\frac{\partial f}{\partial a} = -p(t)$, $\frac{\partial f}{\partial b} = -q(t)$ also cts

Theorem (Linear P-L)

If $p, q, r \in C^0(J)$, where $J$ is an open interval containing $t_0$, then
$$\begin{cases} y'' + py' + qy = r \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \end{cases}$$ has a unique solution $y \in C^2(I)$, $t_0 \in I \subseteq J$

Quick check $y'' + y = 0$ $y(0) = 1$, $y'(0) = 0$

By guessing, we see that $y(t) = \cos(t)$ is a solution.
By P-L, it's the only solution.

To build a satisfactory theory for solving linear 2nd-order IVP, we lean on linear algebra...

We focus first on homogeneous equations. Given $p, q \in C^0$. We define a linear operator

$$L[y](t) = y''(t) + p(t)\,y'(t) + q(t)\,y(t)$$

By linearity of $D[y] = y'$ & $D^2[y] = y''$ → $L$ is a linear operator $C^2 \to C^0$

Further, we observe:

$$\ker(L) = \{ y \in C^2 : L[y] = 0 \}$$

$$= \text{set of solutions to } y'' + p(t)y' + q(t)y = 0$$

{ Since $\ker(L) \subseteq C^2$, if $y_1, y_2$ satisfy $y'' + py' + qy = 0$, then $y(t) = c_1 y_1(t) + c_2 y_2(t)$ also satisfies the ODE $\forall\, c_1, c_2$ }
→ principle of superpositions )--

Oct 29, 2018

Given $2^{nd}$ order linear ODE

$$y'' + py' + qy = r \quad , \ q, p, r \in C^0(I), \ I \text{ open}$$

Associated with this is the homogeneous counterpart

$$y'' + py' + qy = 0 \qquad (\Delta)$$

and a $2^{nd}$ order linear diff. operator $L : C^2(I) \to C^0(I)$ defined by

$$L[y](t) = y''(t) + p(t)y'(t) + q(t)y(t)$$

We say that

$$y \text{ solves } (\Delta) \iff y \in \ker(L) = \{ y \in C^2 : L[y](t) = 0 \}$$

Since $\ker(L)$ is a subspace of $C^2(I)$, if $y_1, y_2 \in \ker(L)$, then $\forall c \in \mathbb{R}$, $c_1 y_1 + c_2 y_2 \in \ker(L)$

Prop. Superposition: if $y_1, y_2$ solve $(\Delta)$, then $\forall c_1, c_2 \in \mathbb{R}$, then

$$y := c_1 y_1 + c_2 y_2 \text{ solves } (\Delta)$$

<u>Note</u> this isn't true for the inhomogeneous case

<u>Ex</u> $y'' + y = t \qquad I = (-\infty, \infty) = \mathbb{R}$

Homogeneous counterpart: $y'' + y = 0$

<u>Note</u> $y_1(t) = \cos(t)$
$y_2(t) = \sin(t)$ solve the homogeneous eqn

<u>Also note</u> $\tilde{y}_1(t) = t$ , $\tilde{y}_2(t) = t + \cos(t)$

So, by prop $y = c_1 y_1 + c_2 y_2$ solves $(\Delta)$

<u>Note</u> $\tilde{y}_1'' + \tilde{y}_1 = t'' + t = t$ , So $\tilde{y}_1$ solves the inhomogeneous eqn

<u>Also</u> $\tilde{y}_2'' + \tilde{y}_2 = -\cos t + t + \cos t = t$ solves inhomogeneous

<u>But</u> $\tilde{y}_1 + \tilde{y}_2 = \cos t + 2t$

plug into ODE $\Rightarrow (-\cos t) + (\cos t + 2t) = 2t \neq t$

<u>So</u> $(\tilde{y}_1 + \tilde{y}_2)$ does <u>not</u> solve inhomogeneous eqn

The reason being solutions to the inhomogeneous eqn don't form a subspace
They form an affine space

$\boxed{\text{The principle of superposition only works for homogeneous eqn}}$

In what follows, for a while, we'll focus on homogeneous equations.
In fact, we will work with

$$y'' + py' + qy = 0 \iff L[y] = y'' + py' + qy$$

and assume $y_1$ & $y_2$ are two known solutions.

Ex $L[y] = y'' + y \qquad y_1(t) = \cos t, \quad y_2(t) = \sin t$.

Goal: figure out how to (if we can) solve IVP
Note: We have a general solution (with perhaps integration constants) with which we can solve every IVP.

Let's suppose that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is our candidate for a general sol

For a given IVP, $\begin{cases} y'' + py' + qy = 0 \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \end{cases} \qquad y_0, y_0' = \text{constants} \in \mathbb{R}$

So $\begin{cases} c_1 y_1''(t) + c_2 y_2''(t) + p(c_1 y_1'(t) + c_2 y_2'(t)) + q(c_1 y_1(t) + c_2 y_2(t)) = 0 \\ c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0' \end{cases}$

$$\to \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} = A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = W(t_0) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$\boxed{\text{I can solve this system for } c_1, c_2 \text{ iff } \det(A) \neq 0}$

or iff $\overline{W}(t_0)$ has an inverse

$$\to \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \left(\overline{W}(t_0)^{-1}\right) \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

or $\quad W(t_0) = \det(\overline{W}(t_0)) \neq 0$

$\overline{W} \to$ Wronskian matrix
$W \to$ Wronskian determinant

Oct 31, 2018

Ex
$$\begin{cases} y'' + y = 0 \\ y(0) = y_0 = 1/2 \\ y'(0) = -1 \end{cases} \qquad W = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

General solution: $y(t) = c_1 \cos t + c_2 \sin t$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \left( W(t_0) \right)^{-1} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} = 1 \begin{pmatrix} \cos t_0 & -\sin t_0 \\ \sin t_0 & \cos t_0 \end{pmatrix} \begin{pmatrix} 1/2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 \\ -1 \end{pmatrix}$$

So $\boxed{y(t) = \frac{1}{2} \cos t - \sin t}$

**Theorem** Let $p, q \in C^0(I)$, let $y_1, y_2 \in \ker(L)$ where $L[y] = y'' + py' + qy$.

Given any IVP
$$\begin{cases} L[y] = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$$

If $w_{y_1, y_2}(t_0) = \det(W_{y_1 y_2}(t_0)) = y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0) \neq 0$

then the IVP can be solved by setting

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

or $$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \left( W_{y_1, y_2}(t_0) \right)^{-1} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

$\boxed{Q}$ When is $y(t) = c_1 y_1(t) + c_2 y_2(t)$ a general solution to the ODE

↳ in other words, when do I know that any IVP (for any $t_0$) can be solved by a linear combo of $y_1, y_2$?

**Abel's Theorem** Let's work through a calculation (it will lead us to Abel's theorem)

Given $y_1, y_2 \in \ker(L)$

$$\frac{d}{dt} w_{y_1,y_2}(t) = \frac{d}{dt}\left(y_1 y_2' - y_1' y_2\right)$$

$$= \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'}$$

$$= y_1 y_2'' - y_1'' y_2 \qquad (\circledast)$$

Recall $y_1, y_2 \in \ker(L) \rightarrow$
$$y_1'' + p y_1' + q y_1 = 0$$
$$y_2'' + p y_2' + q y_2 = 0$$

so
$$\begin{cases} y_1'' = -p y_1' - q y_1 \\ y_2'' = -p y_2' - q y_2 \end{cases} \qquad \binom{*}{*}$$

So $\binom{*}{*}, (\circledast) \Rightarrow$
$$\frac{d}{dt} w_{y_1,y_2}(t) = y_1\left(-p y_2' - q y_2\right) - y_2\left(-p y_1' - q y_1\right)$$

$$= -p y_1 y_2' - \cancel{q y_1 y_2} + p y_2 y_1' + \cancel{q y_1 y_2}$$

$$= p\left(y_2 y_1' - y_2' y_1\right)$$

$$\boxed{\frac{d}{dt} w_{y_1,y_2}(t) = -p\, w_{y_1,y_2}(t)}$$

So, we've just shown that if $y_1, y_2 \in \ker(L)$, then $w_{y_1,y_2}(t)$ solves the first order ODE

$$\boxed{x' + px = 0}$$

And also note that $\tilde{x} = 0$ is an equilibrium solution

**Abel's Theorem**

By P-L $\longrightarrow$

> Given $y_1, y_2 \in \ker(L)$, the wronskian $w_{y_1,y_2}(t) = y_1 y_2' - y_1' y_2$ satisfies the ODE $\dot{x} + px = 0$. Thus, if $w_{y_1,y_2}(t_0) \neq 0$ at some $t_0$, then $w_{y_1,y_2}(t) \neq 0 \ \forall t$

Proof: The calculation combined with $y_1, y_2 \in \ker(L)$ shows that $x' + px = 0$ is satisfied by $W_{y_1,y_2}$, B/c $p \in C^0$. Thus, P-L shows that $x' + px = 0$ has unique solution $\Rightarrow$ 0 equ. solution
So if $W(t_0) \neq 0$ at $t_0$, then $W(t) \neq 0$ $\forall t$,

**Corollary** Let $y_1, y_2 \in \ker(L)$. If $W_{y_1,y_2}(t) \neq 0$ @ any $t$, then

$y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a general solution to

$L[y] = y'' + py' + qy = 0$. In particular, any initial value problem can be solved by specifying $c_1, c_2$

Nov 2, 2018 **Aside** Theorem (P-L for linear ODE)

$$\begin{cases} y'' + py' + qy = r \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$$

Let $r, p, q \in C^0(I)$ and consider for any $t_0 \in I$, $y_0, y_0' \in \mathbb{R}$. Then $\exists$ a unique solution $y \in C^2(I)$ $\longrightarrow$ Things are very good because solutions exist for all $t \in I$ ( $C^2$ for all $t \in I$)

Recall $\boxed{\text{Theorem (Abel)}}$. Let $p, q \in C^0(I)$ and $y_1, y_2 \in \ker(L)$
$y_1, y_2 \in C^2(I)$ which solves $L[y] = y'' + py' + qy = 0$

If $W_{y_1,y_2}(t) \neq 0$ for some $t \in I$, then $W_{y_1,y_2}(t) \neq 0$ $\underline{\forall t \in I}$

and for any $t_0 \in I$, $y_0', y_0 \in \mathbb{R}$, the IVP

$$\begin{cases} y'' + py' + qy = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$$

is solved uniquely by putting

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

where

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \left(W_{y_1,y_2}(t_0)\right)^{-1} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

## Linear Algebraic Review

A set of elements $\{v_1 \dots v_n\}$ in a vector field $V$ is said to be linearly independent if

$$\sum_{i=1}^{n} a_i v_i = 0 \quad \text{implies} \quad a_i = 0 \ \forall\ 1 \le i \le n$$

~~Ex~~ $\boxed{Ex}$ $\cosh(t) = \dfrac{e^x + e^{-x}}{2}$ $\quad \sinh(t) = \dfrac{e^x - e^{-x}}{2}$

Consider $V = C^2(\mathbb{R})$. Consider $\{e^t, e^{-t}, \cosh t\}$

Linear independence? No But $\{e^t, e^{-t}\}$ is a linearly ind in $C^2(\mathbb{R})$

Proof

$$c_1 e^t + c_2 e^{-t} = 0 \Rightarrow c_1 = c_2 = 0 \quad \forall t$$

$$\frac{d}{dt} \rightarrow c_1 e^t - c_2 e^{-t} = 0$$

$\boxed{\text{Observe}}$ for $y_1(t) = e^t$, $y_2 = e^{-t}$ $\quad \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix} = W_{y_1, y_2}(t)$

So, this calculation suggests that the linear independence of $y_1$ & $y_2$ has sth to do with $\omega_{y_1 y_2}(t)$.

$\boxed{\text{Proposition}}$

Let $y_1$ & $y_2$ be elements of $C^1(I)$ (could be $C^2(I)$). Define

$$W_{y_1, y_2}(t) = \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$$ to be the Wronskian of $y_1, y_2$ @ $t \in I$.

If $W_{y_1 y_2}(t)$ is invertible, i.e., if $\omega_{y_1 y_2}(t) \neq 0$ for ~~some~~ $t \in I$ then $\{y_1, y_2\}$ is a linearly independent list.

Proof Suppose $c_1 y_1(t) + c_2 y_2(t) = 0 \ \forall t \in I$

$$\rightarrow c_1 y_1'(t) + c_2 y_2'(t) = 0 \ \forall t \in I$$

So $\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. By hyp $\exists\, t \in I$ s.t. $\omega \neq 0$

$$\rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = (W_{y_1 y_2}(t))^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

& $y_1, y_2$ linearly ind

Nov 5, 2018

What's missing? (1) How do I know exist?
(2) If they exist, how do we find them?

Theorem

Let $p, q \in C^0(I)$. Consider ODE $y'' + py' + qy = 0$ (☺)
$\exists$ solutions $y_1, y_2 \in C^2(I)$ to (☺) for which
$w_{y_1 y_2}(t) \neq 0$ at some (and hence all) $t$.

That is, $\exists\ y_1, y_2$ s.t $y = c_1 y_1 + c_2 y_2$ is a general solution to ☺

Such a pair of solutions is called a fundamental generating set of solutions.

Proof Consider two IVP
$t_0 \in I$

$$\begin{cases} y'' + py' + qy = 0 \\ y(t_0) = 1 \\ y'(t_0) = 0 \end{cases}$$

and

$$\begin{cases} y'' + py' + qy = 0 \\ y(t_0) = 0 \\ y'(t_0) = 1 \end{cases}$$

By P-L, $\exists\ y_1$ to (1) and $y_2$ to (2) solutions. These are necessary member of $C^2(I)$

Observe that

$$w_{y_1, y_2}(t_0) = y_1(t_0)\, y_2'(t_0) - y_1'(t_0)\, y_2(t_0)$$

$$= 1.1 - 0.0 = 1 \neq 0$$

$\cancel{}$

$\boxed{\text{Back to Linear Algebra}}$

(1) We know what linear indep. is...
(2) Span. Given a vector space $V$ & a collection $\{V_1 \ldots V_n\}$ in $V$ we say that $\{\ldots\}$ spans $V$ if $\forall \vec{V} \in V$, $\exists\ c_1 \ldots c_n \in$

s.t $\vec{V} = \sum_i c_i \vec{V_i}$

**Def** $\boxed{\begin{array}{l}\text{let } V \text{ be vector space and } \{V_1 \dots V_n\} \text{ a collection of vectors in } V \\ \text{if } \{V_1 \dots V_n\} \text{ spans } V \text{ and are lin indep then} \\ \qquad \{V_1 \dots V_n\} \text{ is a basis}\end{array}}$

**Facts** the # of elements in any basis for a vector space is the same.

**Def** $\left\{\begin{array}{l}\text{The number of elements in any such collection is called} \\ \text{the dimension of the vector space.} \\ \qquad n = \dim(V)\end{array}\right.$

**Def** $\boxed{\begin{array}{l}\text{Given } V \text{ and } W. \text{ A map } T: V \to W \text{ is linear if} \\ T(\alpha V_1 + \beta V_2) = \alpha T(V_1) + \beta T(V_2) \quad \forall\, \alpha, \beta \in \mathbb{R} + V_1, V_2 \in V\end{array}}$

surjective • $T$ is onto if $\mathrm{Ran}(T) = \mathrm{Im}(T) = \{w \in W : \exists\, v \in V \text{ s.t } T(v) = w\} = W$

injective • $T$ is 1-to-1 if $\forall\, V_1, V_2 \in V$, $T(V_1) = T(V_2) \Rightarrow V_1 = V_2$

**Fact** $T$ inj $\Leftrightarrow \ker(T) = \{0\}$

$\boxed{\text{A linear operator } T \text{ is called an isomorphism if it's bijective}}$

$\left\{\begin{array}{l}\text{2 vector spaces } V \text{ and } W \text{ is said to be isomorphic if there } \exists \text{ an} \\ \quad \text{isomorphism } T: V \to W\end{array}\right\}$

$\boxed{\textbf{Fact} \quad \text{If } V \text{ is } n\text{-dim}, V \text{ is isomorphic to } \mathbb{R}^n}$

$\boxed{\mathbb{R}^n, \ \mathbb{P}^n(I)}$

let $I$ be an interval $P_n(I) = \{\text{set of polynomials}\}$ is a $(n+1)$ dim vector space

In fact

$E_n : \mathbb{R}^{n+1} \to P_n(I)$ given by

$$E_n\begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = a_0 + a_1 t + \dots a_n t^n \text{ is an isomorphism}$$

**Non-ex** $C^\infty(I)$, $C^0(I)$, $C^2(I)$ ... , $\mathbb{R}^{\mathbb{N}}$ = {set of sequences ...

Nov 7, 2018

Recall $y \in \ker(L)$ iff $y \in C^2(I)$ solves $L[y]=0$

Last: provided $p, q \in C^0(I)$, $\exists\, y_1, y_2 \in \ker(L)$ for which
$y = c_1 y_1 + c_2 y_2$ is a general solution
We said $\{y_1, y_2\}$ is a fundamental generating set of solutions to $L[y]=0$

In fact, because $W_{y_1 y_2} \neq 0$ implies that $y_1, y_2$ linearly independent
$\hookrightarrow$ $\{y_1, y_2\} \subseteq \ker(L)$ and this is a linearly indep. set of elements ...

Corollary $\dim(\ker(L)) \geq 2$

Theorem $\boxed{\text{the pair } \{y_1, y_2\} \text{ is a basis of } \ker(L)}$

We need to show $y_1, y_2$ span $\ker(L)$, i.e., $\forall y \in \ker(L)$, $\exists\, c_1, ...$
st $y = c_1 y_1 + c_2 y_2 \in \ker$.

So, let $y \in \ker(L)$. In particular, $y$ solves $L[y]=0$ and takes values
$y(t_0) = y_0$, $y'(t_0) = y_0'$. So, $y$ solves the IVP $L[y]=0$, $y(t_0) ...$
$y'(t_0) = y_0'$. And since $W_{y_1, y_2} \neq 0$. $\exists\, c_1, c_2$ s.t. $c_1 y_1 + c_2 ...$
~~because~~ $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \left(W_{y_1, y_2}(t_0)\right)^{-1} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$ solves IVP

Since solutions are unique $y = c_1 y_1 + c_2 y_2 \rightarrow$ ~~so~~ $y_1, y_2$ span

Corollary $\boxed{\text{Given } p, q \in C^0(I), \text{ the subspace } \ker(L) \text{ of } C^2(I) \text{ is } ...}$

Corollary Given any linearly independent pair of elements $y_1, y_2 \in \ker ...$
$W_{y_1, y_2} \neq 0$, $\{y_1, y_2\}$ is necessarily a basis
$\rightarrow$ Every element of $\ker(L)$ is a linear combination of $y_1, y_2$

Moral { Any way I find 2 solutions $y_1, y_2$ (lin. indep...) to $L[y]=0$ automatically, they form a basis. & so every IVP can be solved with them.

Terminology → { A pair $\{y_1, y_2\}$ which is a fundamental generating set of solutions to the eqn $L[y]=0$ is the same as the basis of $\ker(L)$.

$\boxed{\text{Reduction of order}}$ Suppose that I have $y_1$ is a solution to $L[y]=0$

Goal Find another linearly indep solution. You already know how to do this with Abel's identity

$$y_2 = y_1(t)\int \frac{e^{-P(t)}}{y_1^2(t)}\,dt \qquad P' = p$$

Another way

Suppose that $y_2(t) = v(t)\,y_1(t)$

We make an ansatz, which is to suppose that a lin. indep solution $y_2$ exists and is of this form

So, in this case $\quad 0 = L[v(t)\,y_1(t)]$

$$\begin{aligned}
0 &= (vy_1)'' + p(vy_1)' + q(vy_1)\\
&= (v'y_1 + vy_1')' + p(v'y_1 + vy_1') + q(vy_1)\\
&= v''y_1 + 2v'y_1' + \cancel{vy_1''} + p(v'y_1 + \cancel{vy_1'}) + q(\cancel{vy_1})\\
&= v''y_1 + 2v'y_1' + pv'y_1
\end{aligned}$$

So $\quad 0 = v''y_1 + (2y_1' + py_1)v' \quad$ Suppose $\Phi = v' \Rightarrow \Phi' = v''$

So $\quad 0 = \Phi' y_1 + (2y_1' + py_1)\Phi$

or $\quad \Phi' = -\dfrac{(2y_1' + py_1)}{y_1}\Phi = -\left(2\dfrac{y_1'}{y_1} + p\right)\Phi$

$$\Phi' + \left(2\frac{y_1'}{y_1} + p\right)\Phi = 0 \rightarrow \mu(t) = \exp\left[\int\left(2\frac{y_1'}{y_1} + p\right)dt\right]$$

So $\mu(t) = e^{2\ln(y_1) + P(t)} = y_1^2 \cdot e^{P(t)}$

So $\Phi = \frac{1}{\mu(t)} \cdot C = C\frac{e^{-P(t)}}{y_1^2}$

So $v(t) = \int \Phi\, dt = \int \frac{Ce^{-P(t)}}{y_1^2}dt$

Hence $$\boxed{y_2(t) = y_1(t)\int \frac{Ce^{-P(t)}}{y_1^2(t)}dt}$$

Ask: is $y_2(t)$ lin. ind of $y_1(t)$? Yes!

Moral: It's enough to know one solution $y_1$ → find the other by reduction of order OR Find it via Abel's identity

$$y_2(t) = y_1(t)\int \frac{e^{-P(t)}}{y_1^2(t)}dt$$

where $P(t) = \int p(t)\,dt$

Next: Producing Solutions

Easiest case: linear, homogeneous, constant coeffs.

$$L[y] = ay'' + by' + cy = 0 \quad , a, b, c \text{ constants}, a \neq 0$$

[...]n 9, 2012

Need Basis for ker(L)

Make ansatz: suppose $y \in \ker(L)$ can be written as $y(t) = e^{rt}$, r is constant

So

[...]haracteristic equ → $\boxed{r^2 + br + c = 0}$ $\quad r = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$

Cases (1) $\boxed{b^2 - 4c > 0}$, so $r$ is real-valued and hence we have 2 solutions

$$y_1(t) = e^{\frac{-b+\sqrt{b^2-4c}}{2} t}, \quad y_2(t) = e^{\frac{-b-\sqrt{b^2-4c}}{2} t}$$

So, $y_1(t) = e^{\lambda_1 t}$, $y_2 = e^{\lambda_2 t}$ are solutions.

$$W_{y_1,y_2}(t) = \det\begin{pmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} \end{pmatrix} = (\lambda_2 - \lambda_1) e^{(\lambda_1+\lambda_2)t}$$
$$= -\sqrt{b^2-4c}\; e^{(\lambda_1+\lambda_2)t} \neq 0$$

So $\boxed{y_1, y_2 \text{ linearly indep.}}$

Theorem Consider ODE $L[y] = y'' + by' + cy = 0$
If $b^2 - 4c > 0$, then $y_1 = e^{\lambda_1 t}$, $y_2 = e^{\lambda_2 t}$ form a basis for $\ker(L)$
i.e., $\{y_1, y_2\}$ are fundamental generating set of solutions...

$\boxed{\text{Ex}}$ $L[y] = y'' - 5y' + 6y = 0$

$$y_1(t) = e^{4/2\, t} = e^{2t}, \quad y_2(t) = e^{5+1/2\, t} = e^{3t}$$

$\boxed{\text{General Case}}$

$$L[y] = y'' + by' + cy = 0$$

cis(x)

Char. poly: $r^2 + br + c = 0$

Suppose $\boxed{b^2 - 4c < 0}$ $\lambda_1 = \frac{-b}{2} + \sqrt{b^2-4c}/2 = \frac{-b}{2} + i\frac{\sqrt{4c-b^2}}{2}$

We worry about $i$

Aside Def For $x \in \mathbb{R}$, we define $e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$

(Euler's Identity) $= \cos x + i \sin x$

5

Jan 12, 2017

Given $z = x + iy \Rightarrow e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$

$\boxed{\text{Back to ODE}} \longrightarrow$ Consider ODE $y'' + by' + cy = 0$

Ansatz: $y(t) = e^{rt}$ $\quad$ $y(t)$ solves ODE iff $r^2 + br + c = 0$

$b^2 > 4c \rightarrow \boxed{y = e^{\pm \frac{1}{2}\sqrt{b^2-4c}} e^{-b/2}}$

$b^2 < 4c \Rightarrow y = e^{-b/2 t} e^{\pm i/2 \sqrt{4c^2 - b^2}\, t}$

$$\Rightarrow \boxed{y = e^{-(b/2)t}\left(\cos\left(t\frac{\sqrt{4c-b^2}}{2}\right) \pm i \sin\left(t\frac{\sqrt{4c-b^2}}{2}\right)\right)}$$

Let $\alpha = \frac{-b}{2}$, $\beta = \frac{1}{2}\sqrt{4c-b^2}$

$$\Rightarrow \boxed{y(t) = e^{\alpha t} e^{\pm i \beta t} = e^{\alpha t}\left(\cos(\beta t) \pm i \sin(\beta t)\right)}$$

Consider $y'' - 2y' + 2y = 0 \Longrightarrow r^2 - 2r + 2 = 0$

$$r = 1 \pm \frac{i\sqrt{8-4}}{2} = 1 \pm i$$

So $\boxed{\begin{aligned} y(t) &= e^{t} e^{\pm i t} \\ &= e^t\left(\cos(1) \pm i \sin(1)\right)\end{aligned}}$

Check $y_1$ $\quad y_1'' - 2y_1' + 2y_1$

$= \cancel{e^t\left[-\cos t - i \sin t + i 2 \sin t + 2\cos + 2\cos t - i 2 \sin t\right]}$

$y_1(t) = e^t e^{\pm it}$ $\quad y_1'(t) = ie^t e^{it} + e^t e^{it}$

$y_1''(t) = ie^t e^{it} + (-1)\cancel{e^t} e^{it} + \cancel{e^t} e^{it} + ie^t e^{it}$

$= 2ie^t e^{it}$

So $y_1'' - 2y_1' + 2y_1 = 0$

<u>Observation</u> <u>Note</u> $ker(L)$ is a subspace. So, $\tilde{y}_1(t) = \frac{y_1(t) + y_2(t)}{2} \in ker(L)$

$$\tilde{y}_1(t) = \frac{1}{2}\left(2e^{\alpha t}\cos(\beta t)\right) = e^{\alpha t}\cos(\beta t) \in ker(L)$$

Similarly $\tilde{y}_2(t) = \frac{y_1(t) - y_2(t)}{2i} = +e^{\alpha t}\sin(\beta t) \in ker(L)$

we now produce 2 solutions $\tilde{y}_1$ & $\tilde{y}_2$ of the original ODE.

Check the Wronskian $\neq 0$

$$\det\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = \det\begin{pmatrix} e^{\alpha t}\cos\beta t & e^{\alpha t}\sin\beta t \\ \alpha e^{\alpha t}\cos\beta t - \beta e^{\alpha t}\sin\beta t & \alpha e^{\alpha t}\sin\beta t + \beta e^{\alpha t}\cos(\beta t) \end{pmatrix}$$

$$= \alpha e^{2\alpha t}\sin\beta t\cos\beta t + \beta e^{2\alpha t}\cos^2\beta t - \alpha e^{2\alpha t}\sin\beta t\cos\beta t + \beta e^{2\alpha t}\sin^2\beta t$$

$$= \boxed{\beta e^{2\alpha t} \neq 0}$$ (since we assumed $\beta \neq 0$)

**Theorem** Consider $y'' + by' + cy = 0$. Suppose that $b^2 < 4c$. Let $\alpha = \frac{-b}{2}$ & $\beta = \frac{1}{2}\sqrt{4c - b^2}$. Then,

$y_1(t) = e^{\alpha t}\cos\beta t$ & $y_2(t) = e^{\alpha t}\sin\beta t$ form a fundamental generating set of solutions.

What happens when $b^2 = 4c$ or $r = \frac{-b}{2} \pm 0$ ?

$\Rightarrow$ gives the only solution to $r^2 + br + c = 0$

Obviously, $e^{rt} = e^{-b/2\, t}$ is a solution

So, this ansatz only gives 1 solution & we need 2. Abel's identity gives that

$$\{e^{rt}, te^{rt}\}$$ is a basis for $ker(L)$

Nov 14, 2012

$\boxed{\text{Theorem}}$ let's consider the 2nd order ODE $L[y] = y'' + by' + cy = 0$

Case 1 If $b^2 - 4c > 0$, set $\lambda_1 = \frac{-b}{2} + \frac{\sqrt{b^2-4c}}{2}$

Then $y_1(t) = e^{\lambda_1 t}$, $y_2(t) = e^{\lambda_2 t}$ form a basis for $\ker(L)$

Case 2 if $b^2 - 4c > 0$. Set $\alpha = \frac{-b}{2}$, $\beta = \frac{\sqrt{4c-b^2}}{2}$

Then $y_1(t) = e^{\alpha t}\cos\beta t$, $y_2(t) = e^{\alpha t}\sin(\beta t)$ form a basis for $\ker(L)$, i.e. form a fundamental generating set of solutions...

Case 3

$b^2 - 4c = 0$, $\lambda = \frac{-b}{2} \rightarrow y_1 = e^{\lambda t} = e^{-b/2}$, $y_2(t) = te^{-bt/2}$ form a fundamental generating set of solutions...

Solve IVP
$$\begin{cases} y'' + 6y' + 9y = 0 \\ y(0) = 1, \ y'(0) = -1 \end{cases}$$

Case 3 $b = 6$, $c = 9 \rightarrow b^2 - 4c = 0$

So $y_1 = e^{-6/2 t} = e^{-3t}$, $y_2 = te^{-3t}$

Verify $y_2$
$$y_2' = e^{-3t} - 3te^{-3t}$$
$$y_2'' = -3e^{-3t} - 3e^{-3t} + 9te^{-3t} = -6e^{-3t} + 9te^{-3t}$$

So $L[y_2] = -6e^{-3t} + 9te^{-3t} + 6e^{-3t} - 18te^{-3t} + 9te^{-3t} = 0$

Solve IVP $\rightarrow$ seek $c_1, c_2$:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \left(W_{y_1,y_2}\right)^{-1} \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} \cdots \end{pmatrix}$$

But more simply

$$\begin{cases} 1 = c_1 \\ -3c_1 + c_2 = -1 \end{cases} \rightarrow$$

$$\rightarrow \begin{cases} c_1 = 1 \\ c_2 = 2 \end{cases} \rightarrow \boxed{y(t) = e^{-3t} + 2te^{-3t}}$$

$\hookrightarrow (1+2t)e^{-3t}$

$\boxed{\text{Inhomogeneous}}$ Theory First Consider the linear $2^{nd}$ order inhomogeneous ODE

$$y'' + py' + qy = r$$

Thru the lens of linear algebra : $L[y] = r$

$\boxed{\text{Big Idea}}$ Suppose $y_{p_1}$ & $y_{p_2}$ solve ODE. Compute $L[y]$ when $y = y_{p_1} - y_{p_2}$

$$L[y_{p_1} - y_{p_2}] = L[y_{p_1}] - L[y_{p_2}] = r - r = 0$$

So $y_{p_1} - y_{p_2} \in \ker(L)$. If $\{y_1, y_2\}$ form a basis for $\ker(L)$, then

$$y_{p_1} - y_{p_2} = c_1 y_1 + c_2 y_2$$

So $\boxed{y_{p_1} = c_1 y_1 + c_2 y_2 + y_{p_2}}$ $\longrightarrow$ parallel w/ linear algebra...

Moral Any 2 solutions $y_{p_1}, y_{p_2}$ to $L[y] = r$ differ at most, by an element in $\ker(L)$, i.e. a solution $y_h$ to $L[y_h] = 0$

Consider $\rightarrow L[y] = y'' + py' + qy = r$. Let $y_p$ be a solution $\rightarrow L[y_p] = r$. Then any & every solution $y$ to ODE is given by

$$y = y_p + y_h \quad \text{where } y_h \in \ker(L)$$

Moral To understand all solutions to ODE, it is enough to know 1 solution $y_p$, called a particular solution & also understand $\ker(L)$ (know a basis)

If, somehow, you know a solution to inhomogeneous equation $y_p$ $L[y] = y'' + py' + qy = r$, then every solution to the ODE is of the form

$$y = y_p + y_h$$ - $y_p$ is called a "particular" solution, and $y_h$ a "homogeneous solution".

**Theorem** Let $y_1, y_2$ be a fundamental generating set for the homogeneous equation

$L[y] = y'' + py' + qy = 0$. Then, if $y_p$ is some solution to the inhomogeneous equation, then

$y(t) = C_1 y_1(t) + C_2 y_2(t) + y_p$, is a general solution to the ODE, i.e.,

All solutions can be gotten by an appropriate choice $C_1, C_2$

**Moral** All I need is a single particular solution.

Ex Consider $y'' - y = t$

$y_h = C_1 e^t + C_2 e^{-t}$

$y_p = -t$

(1) Solve homogeneous $\to y_h = C_1 e^t + C_2 e^{-t}$

$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$\{e^t, e^{-t}\}$ is fund. gen. set to ODE.

(2) Find any particular soln to inhomogeneous...

Guess $y_p(t) = -t$ satisfies this...

(3) Combine. By theorem, the general solution is $\boxed{y(t) = C_1 e^t + C_2 e^{-t} -}$

Check that $\frac{3}{2}e^t - \frac{3}{2}e^{-t} - t$ solves I.V.P $y(0) = 0$, $y'(0) = 2$

**Undetermined Coefficients** (Guessing)

Goal: Find a particular solution to inhomogeneous ODE of the form $L[y] = y'' + py' + qy = r(t)$, where $r$ is very special

List $r(t) = \sum_{i=0}^{n} a_i t^i$, $r(t) = e^{\lambda t}$, $r(t) = A\cos(\lambda t) + B\sin(\lambda t)$

$r(t) =$ products / sums

| $r(t)$ | Guess |
|---|---|
| $\sum_{i=0}^{n} a_i t^i = p(t)$ | $Q(t) = \sum_{i=0}^{n} b_i t^i$ (same degree as $P(t)$) |
| $e^{\lambda t}$ | $Ae^{\lambda t}$, then find $A$. |
| $A\cos(\lambda t) + B\sin(\lambda t)$ | $C\cos(\lambda t) + D\sin(\lambda t)$ |
| Products / Sums | Product / Sums... |

**Q** $\boxed{\text{What if } r \in \ker(L)\,?}$

$\boxed{\text{If } r(t) \in \ker(L)\text{, then multiply guess by } t}$

Nov 19, 2018 Find a GS to $y'' + 3y' + 2y = e^{5t}$

(1) Solve homogeneous eqn $y'' + 3y' + 2y = 0$ $\lambda = -1, -2$

$\rightarrow y_h(t) = c_1 e^{-t} + c_2 e^{-2t}$

(2) Inhomogeneous eqn $y_p = Ae^{5t}$. Plug in

$\rightarrow 25A + 15A + 2A = 1 \rightarrow A = \frac{1}{42}$

$\rightarrow \boxed{y(t) = \frac{1}{42} e^{5t} + c_1 e^{-t} + c_2 e^{-2t}}$

**Problem** $y'' + y = \cos t$

We know $\cos t$ & $\sin t$ form a fundamental generating set of solutions to homogeneous problem $y'' + y = 0$

Note $r(t) = \cos t \in \ker(L) \Rightarrow$ Strange guess & says. $\boxed{y_p(t) = At\cos t + Bt\sin t}$

$y_p' = A\cos t - At\sin t + B\sin t + Bt\cos t$

$$y_p'' = -A\sin t - A\sin t - At\cos t + B\cos t + B\cos t - Bt\sin t$$
$$= -2A\sin t + 2B\cos t - At\cos t - Bt\sin t$$

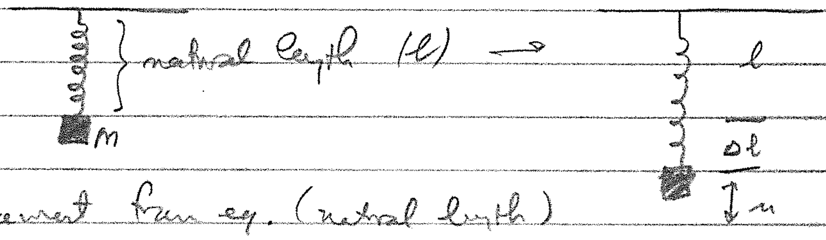
So $y_p'' + y_p = -2A\sin t + 2B\cos t = \cos t \Rightarrow A = 0,\ B = \frac{1}{2}$

Thus, our particular solution is $y_p(t) = \frac{1}{2} t\sin t$

So GS: $\boxed{y(t) = \frac{1}{2} t\sin t + C_1\cos t + C_2\sin t}$

## Mechanical Vibrations

- Mass on spring




Def: $u$ = downward displacement from eq. (natural length)
$u = u(t)$ $\qquad \Delta l$ : downward displacement due to $g$

Hooke's law: Force at rest $\propto$ stretch

$\boxed{k\Delta l = mg}$ , $k$ = spring constant

(1) Force of gravity $F_g = mg$
(2) Spring force : $F_s = -k(\Delta l + u)$
(3) Force of resistant (damping force) : $-c\frac{du}{dt}$ (opposes motion)

Newton's 2nd law $\rightarrow$ $F_g + F_s + F_d + \underset{\substack{\uparrow \\ \text{driving force}}}{F(t)} = m\frac{d^2u}{dt^2}$

$$mg - k(\Delta l + u) - c\frac{du}{dt} + F(t) = m\frac{d^2u}{dt^2}$$

$$\rightarrow \underbrace{mg - k\Delta l}_{0} - ku - c\frac{du}{dt} + F(t) = m\frac{d^2u}{dt^2}$$

$$\rightarrow \boxed{\frac{d^2u}{dt^2} + \frac{k}{m}u + \frac{c}{m}\frac{du}{dt} = \frac{F(t)}{m}}$$

or $\boxed{\ddot{u} + \frac{c}{m}\dot{u} + \frac{ku}{m} = \frac{F(t)}{m}}$ Eq'n of harmonic motion

for $F(t) \neq 0 \Rightarrow$ "driven". $F(t)$ is driving force ...

Nov 26, 18 $\boxed{\text{Case study}}$ $F(t) \equiv 0$, $c = 0$ (no driving, no damping)

free vibration

$\rightarrow m\ddot{u} + ku = 0$ or $\boxed{\ddot{u} + \frac{k}{m}u = 0}$

introduce $\omega_0 = \sqrt{\frac{k}{m}}$ $\boxed{\ddot{u} + \omega_0^2 u = 0}$

By theory, general solution is $\boxed{u(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)}$

$\boxed{\text{Study IVP}}$ $\begin{cases} \ddot{u} + \omega_0^2 u = 0 \\ u(0) = u_0 \\ \dot{u}(0) = 0 \end{cases}$ $u_0 > 0 \rightarrow \boxed{u(t) = u_0\cos(\omega_0 t)}$

$\boxed{T = \frac{2\pi}{\omega_0}, \quad f = \frac{2\pi}{T} = \omega_0}$ $\rightarrow \omega_0$ is called the natural freq. of the oscillator

$\omega_0 = \sqrt{\frac{k}{m}}$

$\boxed{\text{Damped, free vibration}}$ $F(t) = 0$

$\boxed{m\ddot{u} + c\dot{u} + ku = 0}$ $\rightarrow \ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = 0$

$\hookrightarrow r^2 + \frac{c}{m}r + \frac{k}{m} = 0 \rightarrow r = \frac{-c/m \pm \sqrt{(c/m)^2 - 4(k/m)}}{2}$

$r = \frac{-c}{2m} \pm \frac{1}{2}\sqrt{\left(\frac{c}{m}\right)^2 - (2\omega_0)^2}$

$\boxed{r = \frac{-c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km}}$

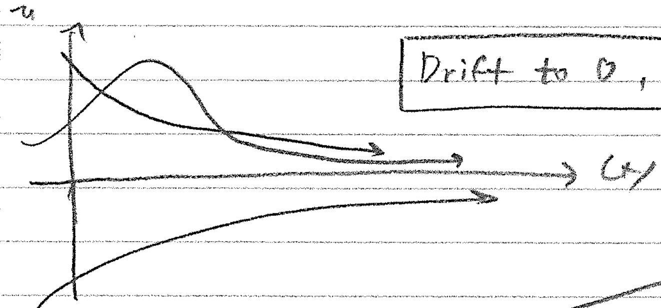
over damped

$\boxed{\text{Cases 1}}$ $c^2 > 4km$

$\rightarrow$ Solution are $u(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$, $r_1, r_2 = \frac{-c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km}$

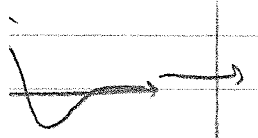
$0 > \;= \frac{-1}{2m}\left(c - \sqrt{c^2 - 4km}\right) \in \frac{-1}{2m}(0, c$

$$r_2 = \frac{-c}{2m} - \frac{1}{2m}\sqrt{c^2-4km} = \underbrace{\frac{-1}{2m}\left(c + \sqrt{c^2-4km}\right)}_{<0} \in \frac{-1}{2m}(c, 2c)$$



$\boxed{\text{Drift to } 0\text{, no oscillation}}$

~~[illegible]~~ critically damped

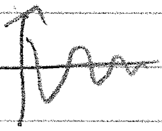


$\boxed{\text{Case 2} \quad c^2 = 4km}$ Here $\boxed{u(t) = c_1 e^{-\frac{c}{2m}t} + c_2 t e^{-\frac{c}{2m}t}}$

$\boxed{\text{Case 3} \quad c^2 < 4km}$ Set $\alpha = \frac{-c}{2m}$, $\beta = \frac{+1}{2m}\sqrt{4km - c^2}$

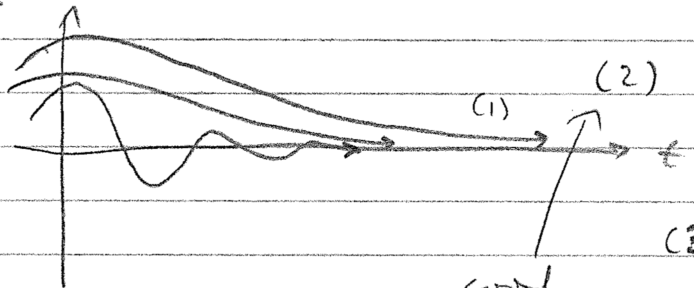
Then solutions: $u(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t$

underdamped



$$= e^{\alpha t}\left(c_1 \cos\beta t + c_2 \sin\beta t\right)$$

Possible solutions $u(t)$ formula I $\to$ (1) ~~[illegible]~~ = $c^2 > 4km$ overdamped



(2) $c^2 = 4km$ Critically damped

(3) $c^2 > 4km$ (under damped

good

bad, shock not working

Nov 28, 2018

$\boxed{\text{Forced Vibration}}$

Consider $m\ddot{u} + ku = F(t) \neq 0$, with $F(t) = F_0 \cos(\omega t)$

$\omega \neq \omega_0 = \sqrt{k/m}$

(1) First, find homogeneous solutions $u_h(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$

(2) Seek particular solution: $u_p(t) = A\cos(\omega t) + B\sin(\omega t)$

$\boxed{\text{Find } A \,\&\, B}$ $\quad u_p'(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$

$$u_p''(t) = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)$$
$$= -\omega^2 u_p(t)$$

$$m u_p''(t) + k u_p(t) = F_0\cos(\omega t) \Rightarrow \begin{cases} -\omega^2 m A + kA = F_0 \\ -\omega^2 m B + kB = 0 \end{cases}$$

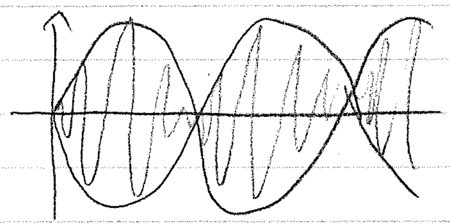
$$\to A = \frac{F_0}{k-\omega^2 m}, \quad B = 0$$

$$\Rightarrow A = \frac{F_0/m}{\omega_0^2-\omega^2}, \quad B = 0$$

So, general solution, $\boxed{u(t) = \left[C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)\right] + \frac{F_0/m}{\omega_0^2-\omega^2}\cos(\omega t)}$

For varying I.C., you obtain different interesting phenomena. One of them is called beats. $u(0)$ & $u'(0) = 0$

$C_1$ & $C_2$ yields $u(t) = \dfrac{2F_0/m}{(\omega_0^2-\omega^2)}\sin\left(\dfrac{\omega_0-\omega}{2}t\right)\sin\left(\dfrac{\omega_0+\omega}{2}t\right)$



$\boxed{\text{What if } \omega = \omega_0?}$

$\to$ seek the general solution to $mu'' + ku = F_0\cos(\omega_0 t)$

$\to F(t) = F_0\cos(\omega_0 t) \in \ker(L)$

So, a particular solution $u_p(t) = t\left(A\cos(\omega_0 t) + B\sin(\omega_0 t)\right)$

$$u_p'(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \omega_0 t\left(-A\sin(\omega_0 t) + B\cos(\omega_0 t)\right)$$

$$u_p''(t) = -\omega_0 A\sin(\omega_0 t) + \omega_0 B\cos(\omega_0 t) - \omega_0 A\sin(\omega_0 t) + \omega_0 B\cos(\omega_0 t)$$
$$+ t\omega_0^2\left(-A\cos(\omega_0 t) - B\sin(\omega_0 t)\right)$$

$$= -2\omega_0\left(A\sin(\omega_0 t) - B\cos(\omega_0 t)\right) - t\omega_0^2\left(A\cos(\omega_0 t) + B\sin(\omega_0 t)\right)$$

$$m\ddot{u}_p(t) + ku_p = -2m\omega_0\big(A\sin(\omega_0 t) - B\cos(\omega_0 t)\big) - mt\omega_0^2\big(A\cos(\omega_0 t) + B\sin(\omega_0 t)\big) + tk\big(A\cos(\omega_0 t) + B\sin(\omega_0 t)\big)$$

$k/m$

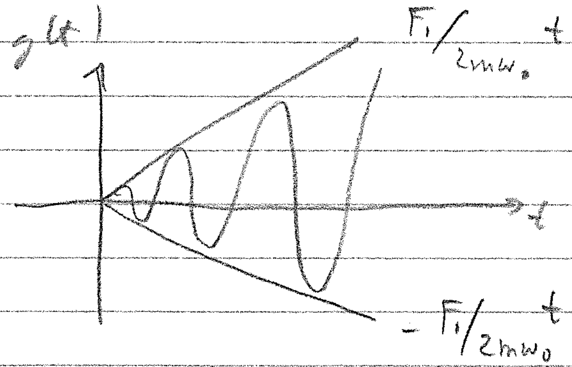
$$= -2m\omega_0\big(A\sin(\omega_0 t) - B\cos(\omega_0 t)\big) + \underbrace{t(k - m\omega_0^2)}_{0}\big(A\cos(\omega_0 t) + B\sin(\omega_0 t)\big)$$

$$= -2m\omega_0\big(A\sin(\omega_0 t) - B\cos(\omega_0 t)\big)$$

$$= F_0\cos(\omega_0 t)$$

So $\boxed{A = 0, \quad B = \frac{F_0}{2m\omega_0}}$

So, general solution: $\boxed{u(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) + \underbrace{\frac{F_0}{2m\omega_0}t\sin(\omega_0 t)}_{g(t)}}$




← All solutions $(C_1, C_2)$ are dominated by $u_p(t) = \frac{F_0}{2m\omega_0}$ si...

This osc grows without bound

→ why the bridge broke.

$\boxed{\text{Remarks}}$

↳ In reality, with $c \neq 0$, things still get bad

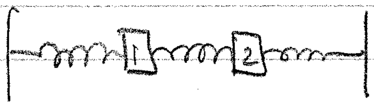
In reality, why did we assume $F(t) = F_0\cos(\omega t)$ where $\omega$ is anything?

↳ As it turns out, any reasonable function can be approximated by sines & cosines → so this method provides more general solutions

...r 30, 2018

## Systems of ODE

The equations ... is called a system of ODE

Ex 

$$m_1 u_1'' = k_2(u_2 - u_1) - k_1 u_1 + F_1(t)$$
$$m_2 u_2'' = k_2 u_1 - (k_2 + k_3) u_2 + F_2(t)$$

} $2^{nd}$ order sys of ODE

Simply
$$m_1 u_1'' = -(k_1 + k_2) u_1 + k_2 u_2 + F_1(t)$$
$$m_2 u_2'' = (k_2 u_1) - (k_2 + k_3) u_2 + F_2(t)$$

**Def** For $k = 1, 2, 3, \dots, n$. Consider $F_k : \mathbb{R}^n \mapsto \mathbb{R}$ form $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ by

$$F(\vec{x}) = \begin{pmatrix} F_1(\vec{x}) \\ \vdots \\ F_n(\vec{x}) \end{pmatrix}$$

The first order $n \times n$ system of ODE defined by $F$ is the system

$$\begin{cases} x_1' = F(x_1 \dots x_n) \\ \vdots \\ x_n' = F_n(x_1 \dots x_n) \end{cases}$$

Here, $x_1 \dots x_n$ are real-valued function of time, $t$

We write $\vec{x}(t) = (x_1(t) \dots x_n(t))^T$ This system to equiv to

$\dot{\vec{x}} = F(\vec{x}) \longrightarrow$ vector eqn

An IVP for $\dot{\vec{x}} = F(\vec{x})$ comes by specifying a time $t_0$ & $\vec{x}_0 \in \mathbb{R}^n$ seeking a once-continuously differentiable fn $\vec{x} : I \mapsto \mathbb{R}^n$ s.t. $\dot{\vec{x}} = F(\vec{x})$, $\vec{x}(t_0) = \vec{x}_0$

$\vec{x}$ is a solution to IVP if $\vec{x}(t_0) = \vec{x}_0$ and $\dot{\vec{x}}(t) = F(\vec{x}(t))$ $\forall t \in I$

Utility Fact: All $n^{th}$-order eqn can be reduced to $(n \times n)$

Consider $y^{(n)} = G(t, y^{(n-1)} \dots y)$, a general $n^{th}$ order EQ

Define $x_1 = y^{(0)} \dots x_n = y^{(n-1)}$

So $\dot{x}_1 = x_2$, ... $\dot{x}_{n-1} = x_n$, $\dot{x}_n = y^{(n)} = G(t, y^{(n-1)} \dots y)$

So $\dot{x}_n = G(t, x_n, x_{n-1} \dots x_1)$

So: $y$ solves the ODE $\Leftrightarrow$ $\vec{x}(t) = (x_1 \dots x_n)^T = (y^{(0)}, \dots y^{(n-1)})^T$ solves the $n \times n$ system: $\dot{\vec{x}} = F(\vec{x}, t) = (x_2 \dots x_n, G(t, x_n \dots x_1))^T$

Ex $my'' + cy' + ky = f(t)$

So $y'' = -\frac{c}{m} y' - \frac{k}{m} y + f(t)/m$

Let $x_1(t) = y(t)$
$x_2(t) = y'(t) = x_1'(t)$

So
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{c}{m} x_2 - \frac{k}{m} x_1 + f(t)/m \end{cases}$$

So $\vec{F}(t, \vec{x}) = \begin{pmatrix} x_2 \\ -\frac{c}{m} x_2 - \frac{k}{m} x_1 \dots \end{pmatrix}$

Want to solve

$$\dot{\vec{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{c}{m} x_2 - \frac{k}{m} x_1 + \frac{f}{m} \end{pmatrix}$$

Let $f = c = 0$, $m = k = 1$ $\rightarrow$ $y'' + y = 0$ Let $y = \sin t$

So $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_2 - x_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Let's verify that $\vec{x}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

So $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

**Theorem** (Picard–Lindelöf theorem)

Consider the IVP $\dot{\vec{x}} = F(t, \vec{x})$ where $\vec{x}(t_0) = \vec{x}_0$, where $\vec{F}(t, \vec{x}) = (F_1(t, \vec{x}), \ldots F_n(t, \vec{x}))^T$

If $(D_{\vec{x}} F)(t, \vec{x})$ $\leftarrow$ Jacobian matrix in $\vec{x}$ derivative is continuous for $t \in I$ continuous $t$

equiv

~~if~~ if $\frac{\partial F_i}{\partial x^j}$ $\forall i, j$ are continuous on $I \times R$ where $t \in I$, $R \ni \vec{x}_0$, $R \subseteq \mathbb{R}^n$, then the IVP has a unique solution

$\vec{x}(t) : J \mapsto \mathbb{R}^n$, where $J \subseteq I$, $\vec{x}$ is also continuously differentiable on $J$.

— t

$\boxed{\text{Corollary P-L in 1D}}$

Consider IVP $\begin{cases} y' = G(t,y) \\ y(t_0) = y_0 \end{cases}$

there, our system is $1\times 1$. and $F(t,x) = G(t,x) = F_1$

$$\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x} G(t,x) = \frac{\partial}{\partial y} G(t,y) \longrightarrow 1$$

Dec 3, 2018

An $n\times n$ linear system is an eqn of the form

(★) $\dot{\vec{x}} = P(t)\vec{x} + \vec{g}(t)$ $P(t)$ is $n\times n$ matrix for each $t$

$\vec{g} : I \mapsto \mathbb{R}^n$

This system is homogeneous if $\vec{g}(t) = \vec{0}$.

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} P_{11}(t) & P_{12}(t) & \cdots & P_{1n}(t) \\ \vdots & & & \\ P_{n1}(t) & \cdots & & P_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}$$

$\boxed{\text{Proposition (Superposition)}}$

If $\vec{x}_1$ & $\vec{x}_2$ solve (★) & $\vec{g}(t) = \vec{0}$, then $\alpha\vec{x}_1 + \beta\vec{x}_2$ solves (★)

Ex $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x_1(t) + x_2(t) \\ 4x_1(t) + x_2(t) \end{pmatrix}$.

$x^{(1)}(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$, $x^{(2)}(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$

$\begin{pmatrix} 3e^{3t} \\ 6e^{3t} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$ ✓ So $\alpha\begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} + \beta\begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$ solves

**Def** The set of vector valued function $\vec{x} : I \mapsto \mathbb{R}^n$ for which $\vec{x}$ has one continuous derivative is denoted $C^1(I, \mathbb{R}^n)$

Ex $\vec{x}^{(1)}(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} \in C^1(\mathbb{R}, \mathbb{R}^2)$

**Fact**

$C^1(I, \mathbb{R}^n)$ is an $\infty$-dim vector space.

**Theorem** Consider $\dot{\vec{x}} = P(t)\vec{x}$ (1/2) assume entries of $P$ are continuous functions of $t$. The set of solutions to (1) is an $n$-dim subspace of $C^1(I, \mathbb{R}^n)$

Given a basis $\{\vec{x}^{(1)}, \vec{x}^{(2)}, \dots \vec{x}^{(n)}\}$ a solution space.

General solution is $\boxed{\vec{x}(t) = \sum_{i=1}^{n} c_i \vec{x}^{(i)}(t)}$

In practice, If you have a collection $\{\vec{x}^1, \vec{x}^2, \dots \vec{x}^{(n)}\}$ solve the system,

$\dot{\vec{x}} = P(t)\vec{x}$. If they are linearly indep, then they form a basis from which we get the general solution.

A tool for checking linear independence is the Wronskian. Given $\{\vec{x}^1, \dots \vec{x}^{(n)}\}$, we define

$$W(t, \vec{x}^{(1)}, \vec{x}^{(2)}(t) \dots \vec{x}^{(n)}(t)) = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{x}^{(1)}(t) & \dots & \vec{x}^{(n)}(t) \\ \downarrow & & \downarrow \end{pmatrix}$$

and its determinant $\omega(t, \vec{x}^{(1)} \dots \vec{x}^{(n)}) = \det W$

**Proposition** If $\omega(t, \vec{x}^{(1)} \dots \vec{x}^{(n)}) \neq 0$ for some $t$, then $\{\vec{x}^{(1)}(t) \dots \vec{x}^{(n)}(t)\}$ is linearly independent

Consider $\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\vec{x}^{(1)}(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$,

$\vec{x}^{(2)}(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$

Ask, does this pair form a basis?

$$W = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} \to \det(W) = e^{2t}(-2-2) = -4e^{2t} \neq 0 \quad \forall t$$

$\to \vec{x}^{(1)}, \vec{x}^{(2)}$ form a basis...

(Ex) $P(t) = A$, $A$ is a constant $n \times n$ matrix

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = A\vec{x} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

__Crucial observation__. Suppose that $\vec{v} \in \mathbb{R}^n$ is an eigenvector for $A$ with ass. eigenval $\lambda$.

Consider $\vec{x}(t) = \phi(t)\vec{v}$ with $\Phi : I \to \mathbb{R}$

Can, for some choice of $\Phi$, $\vec{x} = \Phi\vec{v}$ is a solution?

$$\dot{\vec{x}} = \frac{d}{dt}\Phi(t)\vec{v} = \vec{v}\,\dot{\Phi}(t)$$

__Also__ $A\vec{x} = A\,\Phi(t)\vec{v} = \Phi(t)A\vec{v} = \Phi(t)\lambda\vec{v}$

__We see__ $\vec{x}(t)$ is a solution $\Leftrightarrow$ $\dot{\Phi}\vec{v} = \dot{\vec{x}} = A\vec{x} = \Phi(t)\lambda\vec{v}$

$$\Leftrightarrow \boxed{\dot{\Phi} = \lambda\Phi}$$

Solution is

$$\boxed{\Phi(t) = e^{\lambda t}}$$

**Fact** If $\vec{v} \in \mathbb{R}^n$ is an eigenvector of $A$ with eigenval $\lambda$, then $\vec{x}(t) = e^{\lambda t}\vec{v}$ solves the ODE $\dot{\vec{x}} = A\vec{x}$

$\boxed{\text{HW9: } 36, 37, 38, 39, 40, 41}$

Recall if $\vec{v}$: eigenvec of $A$ with eigval $\lambda$, $\vec{x} = e^{\lambda t}\vec{v}$ solves $\dot{\vec{x}} = A\vec{x}$

if $A$ is $n\times n$, has $n$ distinct eigenvalues, then $A$ has $n$-linearly indep pairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2) \dots (\lambda_n, \vec{v}_n)$

and $\{\vec{x}^{(n)} = e^{\lambda_n t}\vec{v}_n\}$ form a basis for solution space to $\dot{\vec{x}} = A\vec{x}$

Eg $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \to \det(A - \lambda I) = 0 \to (1-\lambda)(4-\lambda) - 6 = 0$

$\lambda^2 - 5\lambda - 2 = 0$

$\lambda = \lambda_1, \quad \lambda = \lambda_2$

$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \to \lambda = 5, -1$

$\lambda = 5 \Rightarrow \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}\vec{v} = \vec{0} \to \vec{x} = \begin{pmatrix} 1 \\ +2 \end{pmatrix}$

$\lambda = -1 \Rightarrow \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}\vec{v} = \vec{0} \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

So $\boxed{\vec{x} = Ae^{5t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + Be^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$ is general solution...

Check Wronskian $W = \begin{pmatrix} e^{5t} & e^{-t} \\ 2e^{5t} & -e^{-t} \end{pmatrix}$ $\quad \det(W) = e^{4t}(-1-2) = -3e^{4t} \neq 0$

$\boxed{\text{What if } A \text{ has complex } \lambda\text{?}}$ Suppose $A = 2\times 2$. If $\lambda = \alpha + i\beta$ is an eigenvalue, then $\lambda_2 = \alpha - i\beta$ is also an eigenvalue for $A$.

$\Rightarrow$ $\boxed{\vec{v}_1 = \vec{a} + i\vec{b}, \quad \vec{v}_2 = \vec{a} - i\vec{b}}$ where $\vec{a}, \vec{b}$ are real valued vectors... $\in \mathbb{R}^2$

$\to$ $\boxed{\vec{v}(t) = \left(c_1 e^{(\alpha+i\beta)t}(\vec{a}+i\vec{b}) + c_2 e^{(\alpha-i\beta)t}(\vec{a}-i\vec{b})\right)}$

SCHO: "TOOO NON TRIVIAL FOR ME!"

A solution is gotten by $c_1 = c_2 = \frac{1}{2}$

$$\vec{x}(t) = \frac{e^{\alpha t}}{2}\left(2\cos(\beta t)\,\vec{a} + i2i\sin(\beta t)\,\vec{b}\right) = e^{\alpha t}\left(\cos(\beta t)\,\vec{a} - \sin(\beta t)\ldots\right.$$

$$\boxed{\vec{x}(t) = e^{\alpha t}\cos(\beta t)\,\vec{a} - e^{\alpha t}\sin(\beta t)\,\vec{b}} \qquad \left(c_1 = c_2 = \frac{1}{2}\right)$$

Choosing $c_1 = \frac{-i}{2}$, $c_2 = \frac{i}{2}$ gives

$$\boxed{\vec{x}(t) = e^{\alpha t}\sin\beta t\,\vec{a} + e^{\alpha t}\cos\beta t\,\vec{b}}$$

General solution

$$\boxed{\vec{x}(t) = c_1 e^{\alpha t}\left(\cos\beta t\,\vec{a} - \sin\beta t\,\vec{b}\right) + c_2 e^{\alpha t}\left(\sin(\beta t)\,\vec{a} + \cos\beta t\,\vec{b}\right)}$$

Theorem — consider linear system: $\dot{\vec{x}} = A\vec{x}$. ~~If A has distinct real $\lambda_1, \lambda_2$ w~~

Let $A$ be $2\times 2$ real matrix. If $A$ has distinct real $\lambda_1, \lambda_2$ with $\vec{v}_1, \vec{v}_2$, then GS = $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

If $A$ has complex $\lambda$, which are necessarily distinct $(\alpha \pm i\beta)$ with $\vec{v} = \vec{a} \pm i\vec{b}$, then GS is of form:

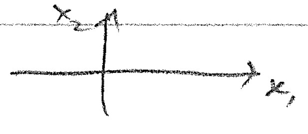
$$\vec{x}(t) = c_1 e^{\alpha t}\left(\cos\beta t\,\vec{a} - \sin\beta t\,\vec{b}\right) + c_2 e^{\alpha t}\left(\sin(\beta t)\,\vec{a} + \cos(\beta t)\,\vec{b}\right)$$

Dec 7, 2018 — **Geometry of autonomous $2\times 2$ systems**

Def: An $n\times n$ sys is called autonomous if it's of the form $\dot{\vec{x}} = F(\vec{x})$ where $F: \mathbb{R}^n \to \mathbb{R}^n$ → time does not appear explicit

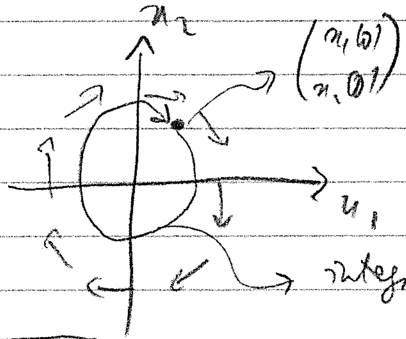
Example $\dot{\vec{x}} = A\vec{x}$ ($A$ : $n\times n$ matrix)

☆ The phase plane for a $2\times 2$ autonomous sys is the $(x_1, x_2)$ plane




$F(\vec{x}) = F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ defines a vector fie

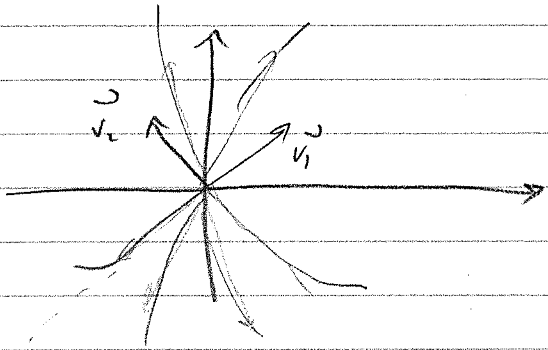
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} = F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$




Solution is $\vec{x}(t) : I \mapsto \mathbb{R}^2$

integral curve / solution curve

$\boxed{\text{Some cases}}$ $A = 2\times 2$ with $\lambda_1, \lambda_2 > 0$



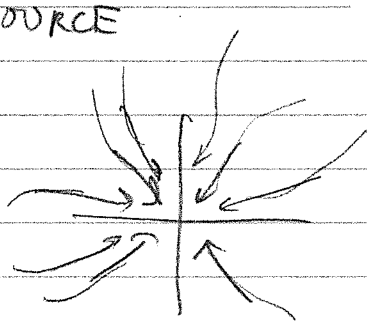

$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

Note any solution starting near but not at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ wishes away from $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \to$ SOURCE

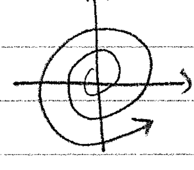
$\to$ unstable equilibrium.

$\boxed{\text{Some cases}}$ $A = 2\times 2$ matrix, $\lambda_1, \lambda_2 < 0$

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a sink

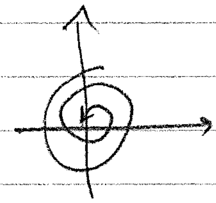


$\boxed{\text{Complex}}$ $\alpha > 0$



spiral source

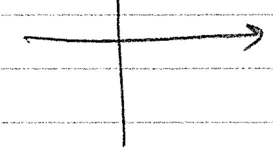
$\alpha < 0$



spiral sink

$\alpha = 0$

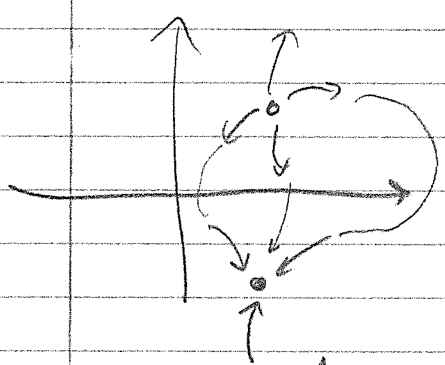
($\lambda_1, \lambda_2$ purely imaginary)



$\to (0,0)$ is a center

$\boxed{\text{Non-linear realm}}$ $\dot{\tilde{x}} = F(\tilde{x})$. **Def** A point $\begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$ s.t.

$$F\begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ is called an equilibrium point



Any solution starting at an eq. point will be constant. (stay there)

To determine the behavior of solution near eq. point.

At an eq. point, $\tilde{x}_0 \in \mathbb{R}^2$.

$$F(\tilde{x}) \cong DF(\tilde{x}_0)(\tilde{x} - \tilde{x}_0) + \underbrace{F(\tilde{x}_0)}_{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

So eq pts says $DF(\tilde{x}_0)(\tilde{x} - \tilde{x}_0) = F(\tilde{x})$