MA352: COMPLEX ANALYSIS
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Office Hours: $\left\{ \begin{array}{l} 10 - 11{:}30 \text{ Tues} \\ 6-8 \text{ PM on W + Th} \end{array} \right\}$

Complex Numbers $\rightsquigarrow$ the set $\mathbb{C}$ of numbers/objects of the form

$z = x + iy$, such that $x, y \in \mathbb{R}$

For the moment, i is just a place holder - it's called the imaginary number.

x: real part. y: imaginary part.

$\operatorname{Re}(z) = \operatorname{Re}(x+iy) = x \qquad \operatorname{Im}(z) = \operatorname{Im}(x+iy) = y$

Note $\operatorname{Re}(z), \operatorname{Im}(z) \in \mathbb{R}$

$\mathbb{C}$ is equivalently the set of pairs (x, y) s.t. $x, y \in \mathbb{R}$

We'll write $(x, y) = x + iy$.

Algebraic structure:

* Addition: $z_1, z_2 \in \mathbb{C}$, $z_i = (x_i, y_i)$

Define $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

In terms of "pair" description, this is exactly the same as in $\mathbb{R}^2$.

Properties
1. Associativity $\rightarrow (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \quad \forall z_i$
2. Commutativity $\rightarrow z_1 + z_2 = z_2 + z_1 \quad \forall z_i$
3. 0: $\exists\, 0 \in \mathbb{C}$ $(0 = 0 + i0)$ s.t. $\forall z \in \mathbb{C}$, $z + 0 = 0 + z = z$

$3^{1/2}$ Existence of additive inverse. $\forall z\ \exists (-z) \in \mathbb{C}$ s.t. $z + (-z) = 0$
In fact, $z = x + iy$ then $(-z) = (-x) + i(-y) = -x - iy$

★ Multiplication $z_1, z_2 \in \mathbb{C}$, $z_i = x_i + iy_i$

Define $\boxed{z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)}$

Why is this the right thing?

If $i = \sqrt{-1}$, i.e. $i^2 = -1$ And we say "foil" is right, then

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + ix_2y_1 + \underbrace{i^2 y_1 y_2}_{-y_1y_2}$$

$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

(4) Properties
- $z_1(z_2z_3) = (z_1z_2)z_3 \quad \forall z_i \in \mathbb{C}$ (asso.)
- (5) $z_1 z_2 = z_2 z_1, \ \forall z_i \in \mathbb{C}$ (comm.)
- (6) $\exists\, 1 \in \mathbb{C}$ $(1 = 1 + i0)$ s.t $\forall z \in \mathbb{C}$, $1z = z1 = z$

(7) Also, given $z = x + iy \neq 0$

Define $z^{-1} = \dfrac{x}{x^2 + y^2} + i\left(\dfrac{-y}{x^2+y^2}\right)$

$\forall z \in \mathbb{C}, z \neq 0$, $zz^{-1} = z^{-1}z = 1$

(8) $\forall z \in \mathbb{C}$, $0z = z0 = 0$

Aside (Division) $z \in \mathbb{C}$, $w \neq 0$, $\dfrac{z}{w} = zw^{-1}$

(9) $\forall z_1, z_2, z_3 \in \mathbb{C}$,

$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Proposition $\mathbb{C}$ is a field

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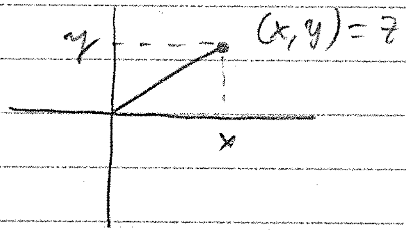
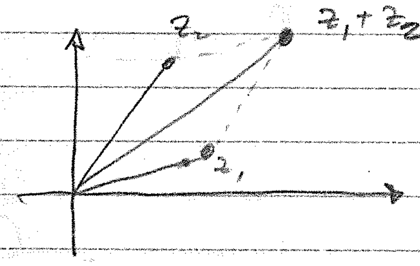
Some geometry

Given $z = x + iy = (x,y) \in \mathbb{C}$, we associate to it the point (x,y) in the plane.



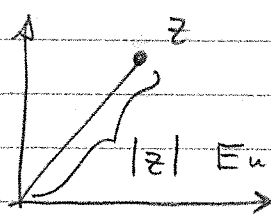
Geometrically, $\mathbb{C}$ is $\mathbb{R}^2$ is called the complex plane.

Of course, if $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ then



■ ~~We observe that~~, Given $z \in \mathbb{C}$, the modulus is the non negative, real number given by z is given by
$$|z| = \sqrt{x^2+y^2} \quad \text{where} \quad z = x+iy = (x,y).$$

■ If



$|z|$ Euclidean distance from 0 to z in $\mathbb{C}$

■ Notes
$$\boxed{|z|^2 = x^2 + y^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$$

Also
$$\mathrm{Re}(z) \le |\mathrm{Re}(z)| \le |z|$$
and
$$\mathrm{Im}(z) \le |\mathrm{Im}(z)| \le |z|$$

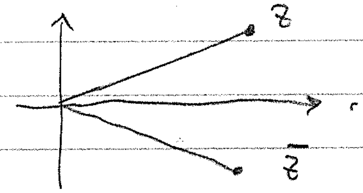
And $\forall z_1, z_2 \in \mathbb{C}$,
$$|z_1 + z_2| \le |z_1| + |z_2|$$
$$\big||z_1| - |z_2|\big| \le |z_1 \pm z_2|$$

Conjugates

Given $z = x + iy \in \mathbb{C}$, we define its conjugate to be

$$\bar{z} = x + i(-y) = x - iy \in \mathbb{C}$$

Geometrically



Properties

1. $\bar{\bar{z}} = z$
2. $|\bar{z}| = |z|$
3. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
4. $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$
5. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$
6. $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$
6. $\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$
7. $z\bar{z} = |z|^2$
8. For $z \neq 0$,

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

should be able to prove all statements

$\boxed{\text{pf (4)}}$ Given $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then $\bar{z}_1 = x_1 - iy_1$, $\bar{z}_2 = x_2 - iy_2$

$$z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

So $\overline{z_1 z_2} = (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1)$. On the other hand,

$$\bar{z}_1 \bar{z}_2 = (x_1x_2 - y_1y_2) + i(-x_1y_2 - x_2y_1) = \overline{z_1 z_2}$$ ✓

$\boxed{\text{Pf of (7)}}$

$$z\bar{z} = (x+iy)(x-iy) = x^2+y^2 = |z|^2$$

$\boxed{\text{Pf of (8)}}$

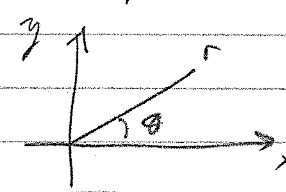
Observe that $\dfrac{z\bar{z}}{|z|^2} = \dfrac{1}{|z|^2}(z\bar{z}) = \dfrac{|z|^2}{|z|^2} = 1$

So by uniqueness of multiplicative inverse

$$z^{-1} = z^{-1}z\frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{|z|^2}.$$

$\boxed{\text{Exponential form}}$ Given $z = x+iy = (x,y)$, If $z \neq 0$, associated to (x,y) are polar coordinates.

$\exists\, \theta,\ r>0$, s.t. $x = r\cos\theta$
$y = r\sin\theta$



So $\boxed{z = r(\cos\theta + i\sin\theta) = r\cos\theta + ir\sin\theta}$

Here $r>0$ because $z\neq 0$ and θ is called the argument of z and written $\boxed{\theta = \arg(z)}$

Also, $\boxed{r = |z|}$

Note $\arg(z)$ is a "multi-valued" function...

Ex $1+i = \sqrt{2}\cos\dfrac{\pi}{4} + i\sqrt{2}\sin\dfrac{\pi}{4}$

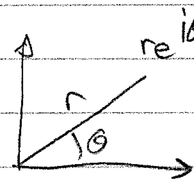
$$\arg(z) = \frac{\pi}{4} + k2\pi \quad k\in\mathbb{Z}$$

- We shall call $Arg(z)$ the value of $arg(z) \in (-\pi, \pi]$.
  It is called the "Principal Argument"

  $\hookrightarrow \quad arg(z) = Arg(z) + 2\pi n \quad , \quad n \in \mathbb{Z}$

- Given $z \in \mathbb{C}$, $z \neq 0$, $z = r\cos\theta + ir\sin\theta$, we write (formally / symbolically)

$$\boxed{re^{i\theta} = z}$$

- Geometrically, we have 

. $\boxed{\text{Note}}$ We don't know what $e^{i\theta}$ is, except formally

<u>define</u> $\boxed{e^{-i\theta} = e^{i(-\theta)}}$

- The relation $\boxed{e^{i\theta} = \cos\theta + i\sin\theta \text{ is called Euler's formula}}$

- Example (again)

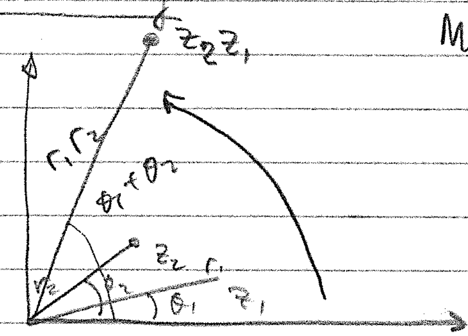
$$1+i = \sqrt{2}\cos\frac{\pi}{4} + i\sqrt{2}\sin\frac{\pi}{4} = \sqrt{2}e^{i\pi/4}$$

- Observe that for $z_1 = x_1 + iy_1 = r_1 e^{i\theta_1} = r_1\cos\theta_1 + ir_1\sin\theta_1$,
  $z_2 = x_2 + iy_2 = r_2 e^{i\theta_2} = r_2\cos\theta_2 + ir_2\sin\theta_2$

(A)
$$\left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = z_1 z_2 = \left[r_1\cos\theta_1 r_2\cos\theta_2 - r_1\sin\theta_1 r_2\sin\theta_2\right]$$
$$+ i\left[r_1\cos\theta_1 r_2\sin\theta_2 + r_2\cos\theta_2 r_1\sin\theta_1\right]$$
$$\| \qquad = r_1 r_2\left[\cos(\theta_1+\theta_2)\right] + i r_1 r_2\left[\sin(\theta_1+\theta_2)\right]$$
$$r_1 r_2 e^{i(\theta_1+\theta_2)} \qquad = r_1 r_2\cos(\theta_1+\theta_2) + i r_1 r_2\sin(\theta_1+\theta_2)$$

So $\boxed{(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}}$

Geometrically



Multiplying complex numbers means to multiply moduli & assign a sum of arguments.

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Also, $\dfrac{z_1}{z_2} = \dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \dfrac{r_1 e^{i\theta_1} e^{-i\theta_2}}{r_2 e^{i\theta_2} e^{-i\theta_2}} = \dfrac{r_1 e^{i(\theta_1-\theta_2)}}{r_2 e^{i0}} = \boxed{\dfrac{r_1 e^{i(\theta_1-\theta_2)}}{r_2}}$

$(z_2 \neq 0)$

So if $z_1 = r_1 e^{i\theta_1}$, $z_1 \neq 0$ then $\boxed{z_1^{-1} = \dfrac{1}{r_1} e^{-i\theta_1}}$

Prove $\boxed{\text{Given } z = re^{i\theta} \neq 0 \text{ \& } n = 0, \pm 1, \pm 2, \dots, \text{ then } z^n = r^n e^{in\theta}}$

Pf By induction. First we treat $n \in \mathbb{N}$ $(n = 1, 2, 3, \dots)$

Base case: $n = 1$. $z^n = z^1 = z = re^{i\theta} = r^1 e^{i\cdot 1\cdot\theta}$

Assume $z^n = r^n e^{i\theta n}$. Then

$$z^{n+1} = z^n \cdot z = (r^n e^{in\theta}) \cdot (re^{i\theta}) = r^n \cdot r \cdot e^{in\theta} e^{i\theta}$$
$$= r^{n+1} \cdot e^{i(n+1)\theta}$$

For $n = 0$, then $z^0 = 1 = r^0 e^0$

For $n \in \mathbb{Z} \setminus \{\mathbb{N} \cup \{0\}\}$, then

$$z^n = \left(\frac{1}{z}\right)^m \text{ where } m = -n \in \mathbb{N} = \left(\frac{1}{re^{i\theta}}\right)^m = \frac{1}{r^m} e^{-im\theta} = r^{-m} e^{-im\theta}$$
$$= r^n e^{in\theta}$$

Applications of this ...

If $z = e^{i\theta}$ $(|z|=1)$, then

$$\boxed{(\cos\theta + i\sin\theta)^n = z^n = 1^n e^{in\theta} = \cos n\theta + i\sin n\theta}$$

de Moivre's Formula

With this, one obtains many nice trig identities ...

- For $n=2$, $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$

$$(\cos^2\theta - \sin^2\theta) + 2i\sin\theta\cos\theta = \cos 2\theta + i\sin 2\theta$$

$$\Rightarrow \left\{ \begin{array}{l} \sin 2\theta = 2\sin\theta\cos\theta \\ \cos 2\theta = \cos^2\theta - \sin^2\theta \end{array} \right\}$$

<u>Example</u>

$(1+i)^{12}$. Here $z = (1+i) = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$

$$= \sqrt{2}\, e^{i\pi/4}$$

<u>So</u> $(1+i)^{12} = z^{12} = \left(\sqrt{2}\, e^{i\pi/4}\right)^{12} = (\sqrt{2})^{12}\left(e^{i\pi/4}\right)^{12}$

$$= 2^6 e^{3i\pi} = 64 e^{3i\pi}$$

$$= 64(\underbrace{\cos 3\pi}_{-1} + i\underbrace{\sin 3\pi}_{0}) = -64$$

- In other words, $(1+i)$ is a 12^{th} root of -64.

- Also, for $z = 1-i$, then $z = \sqrt{2}\, e^{-i\pi/4}$, then

$$z^{12} = 64 e^{-3i\pi} = 64\left(\cos(-3\pi) + i\sin(-3\pi)\right) = -64$$

$\boxed{\text{A couple of other remarks}}$ (Feb 8)

One consequence of the product formula is

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \qquad (\text{not Arg})$$

$$|z_2 z_1| = |z_2||z_1|$$

$\boxed{\text{ROOTS \& THINGS}}$

Given $z_0 = r_0 e^{i\theta_0}$, and $n \in \mathbb{N}$. Is it possible to find all the n^{th} roots of z_0?

Yes! Note, by def, z is an n^{th} root of z_0 if $z^n = z_0$.

let $z = re^{i\theta}$, then $z^n = r^n e^{in\theta} = z_0 = r_0 e^{i\theta_0}$.

So, $r^n = r_0$ $(r_1 > 0)$ or $\boxed{r = r_0^{1/n} = \sqrt[n]{r_0}}$

and

$$n\theta = \theta_0 + 2k\pi \qquad (k \in \mathbb{Z})$$

i.e.

$$\boxed{\theta = \frac{\theta_0}{n} + \frac{2k\pi}{n}}$$

So, all the distinct roots are of the form

$$\boxed{z = \sqrt[n]{r_0} \exp\left\{\frac{\theta_0}{n} + \frac{2k\pi}{n}\right\} \qquad k = 0, 1, 2, \ldots, n-1}$$

This gives all n distinct roots of z_0.

Observation

- All roots of z_0 live on a circle of $|z| = \sqrt[n]{r_0} = r_0^{1/n}$ ← radius
- All roots of z_0 are equally spaced around that circle $\left(\frac{2\pi}{n}\right)$

Example

(1) <u>Reality check</u>: $z_0 = 1$, $n = 2$

We seek complex numbers z s.t. $z^2 = 1$. We expect 2 roots.

$$r_0 = 1 \quad \text{so} \quad \sqrt{r_0} = 1 \qquad \theta_0 = 0$$

$$\theta_0 = 0 \qquad \text{so} \qquad \theta = \frac{\theta_0}{2} + \frac{2\pi}{2} \cdot k$$

So,

$$z = 1 \cdot \exp\{0 + \pi k\} \qquad k = 0, 1$$

So the roots are $\boxed{z = 1, \; z = -1}$

(2) <u>All 12^{th} roots of -64?</u>

$$z_0 = r_0 e^{i\theta_0} = (64) e^{i\pi}$$

$$z = \sqrt[12]{64} = \sqrt{2}$$

$$\theta = \frac{\theta_0}{n} + \frac{2\pi}{n} \cdot k = \frac{\pi}{12} + \frac{2\pi}{12} k$$

So all distinct roots are

$$z = \sqrt{2} \exp\left\{\frac{\pi}{12} + \frac{\pi}{6} k\right\} \qquad k = 0, 1, \ldots, 11$$

- <u>Note that</u> for $k = 1$, $z = \sqrt{2} \exp\{\pi/12 + \pi/6\} = \sqrt{2} e^{i\pi/4}$
$= (1+i)$

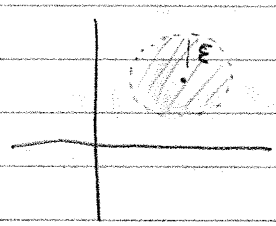
for $k = 10$, $z = \sqrt{2} \exp\left\{\pi/12 + \frac{10\pi}{6}\right\} = \sqrt{2} e^{i 7\pi/4} = \sqrt{2} e^{-i\pi/4}$
$= (1-i)$

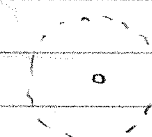
↳ draw...

Regions in the Complex Plane → a bit of point-set topology... – a study of closeness?

(*) Given $z_0 \in \mathbb{C}$, and $\varepsilon > 0$, the "ε neighborhood" of z_0 is the set of points

$$\boxed{B_\varepsilon(z_0) = \{ z \in \mathbb{C} : |z - z_0| < \varepsilon \}}$$





(*) The "deleted ε neighborhood" around z_0 is the set $\boxed{B_\varepsilon(z_0) \setminus \{z_0\}}$

or $B_\varepsilon(z_0) \setminus \{z_0\} = \{ z \in \mathbb{C} \mid 0 < |z - z_0| < \varepsilon \}$

<u>Sep 11, 2019</u> <u>Let</u> $S \subseteq \mathbb{C}$.

> z_0 is an <u>interior point</u> of S if some ε-neighborhood of z_0 is completely contained in S.
>
> i.e. $\exists\, B_\varepsilon(z_0)$ s.t. $B_\varepsilon(z_0) \subseteq S$.

The set of all interior points of S is called the <u>interior</u> of S, denoted

$$\boxed{\mathring{S} = \operatorname{int}(S)}$$

z_0 is an <u>exterior point</u> of S if $\exists\, B_\varepsilon(z_0)$ which does not intersect S.

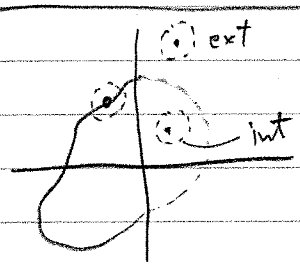
Equivalently, z_0 is an exterior pt of S if there exists $\varepsilon > 0$ s.t

$$B_\varepsilon(z_0) \subseteq S^c = \mathbb{C} \setminus S$$

i.e. an interior pt of the complement of S.

The set of exterior pts of S is called the <u>exterior</u> of S, denoted $\boxed{Ext(S)}$

If z_0 is neither an exterior pt nor an interior pt of S, z_0 is called a <u>boundary pt</u> of S. The set of boundary pts of S is called the <u>Boundary of S</u>, denoted by $\boxed{\partial S}$



Proposition

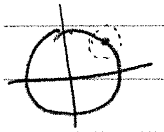
z_0 is a boundary pt of $S \Leftrightarrow \forall \varepsilon > 0$, $B_\varepsilon(z_0)$ contains at least one pt in S and at least one point in S^c

<u>Remark</u> ① $Int(S) \subseteq S$

② $Ext(S) \subseteq \mathbb{C} \setminus S$

③ $Ext(S) \cap S = \phi$

<u>Ex</u> Consider $S_1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \subseteq \mathbb{C}$ $\rightsquigarrow$ the unit circle...



① <u>Note</u> for $z_0 = e^{i\theta_0} \in S_1$, we have that $\underset{B_\varepsilon(z_0)}{\underset{\cap}{z = z_0 + \frac{\varepsilon}{2} e^{i\theta_0}}} \notin S$

$\Rightarrow \boxed{Int(S) = \phi}$

$$\boxed{\partial S_1 = S_1 \text{ , and } Ext(S_1) = \{ z \in \mathbb{C} \mid |z| \neq 1 \}}$$

More defn

- A set $\mathcal{O}$ is called open if it contains NONE of its boundary pts.

  that is $\partial\mathcal{O} \cap \mathcal{O} = \phi$

- A set C is closed if it contains ALL of its boundary pts.

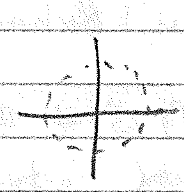
- The closure of a set S is the set $\boxed{cl(S) = \bar{S} = S \cup \partial S}$

Ex (1) $\boxed{\mathbb{C} \text{ is open and closed}}$

(2) $S = \{z \in \mathbb{C},\ Re(z) > 0,\ Im(z) \geq 0\}$ $\rightarrow$ $\boxed{\text{not open, not closed}}$

(3) $S = \{|z| = 1\}$ $\rightarrow$ $\boxed{\text{closed}}$, $\boxed{\text{not open}}$

(4) $B_1(0)$



Note $B_1(0) = S_1$ $\rightarrow$ open & not closed

Proposition : $\boxed{\text{Let } \mathcal{O} \subseteq \mathbb{C}.\ \mathcal{O} \text{ is open} \Leftrightarrow \forall z \in \mathcal{O},\ \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(z) \subseteq \mathcal{O}}$

Pf assume $\mathcal{O}$ is open, i.e. contains none of its boundary pts.

($\Rightarrow$)

Let $z \in \mathcal{O}$. If $\forall \varepsilon > 0$, $B_\varepsilon(z) \not\subseteq \mathcal{O}$, i.e. $B_\varepsilon(z) \cap \mathbb{C} \setminus \mathcal{O} \neq \phi$

$\rightarrow$ z is a boundary point.

This cannot be true, for then $\mathcal{O}$ would contain a boundary point.

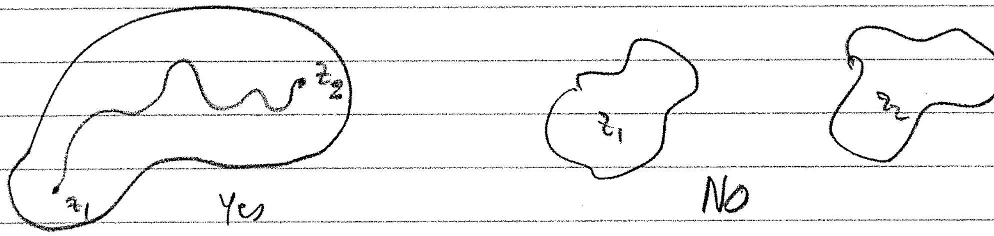
$\rightarrow$ $\exists \varepsilon > 0$ s.t. $B_\varepsilon(z) \subseteq \mathcal{O}$.

($\Leftarrow$) Assm. $\forall z \in \mathcal{O}$, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(z) \subseteq \mathcal{O}$. If $z \in \partial\mathcal{O}$, then $\forall \varepsilon > 0$, $B_\varepsilon(z)$ contains at least one point not in $\mathcal{O}$. But since $B_\varepsilon(z) \subseteq \mathcal{O}$, z cannot be in $\mathcal{O}$. So, $\partial\mathcal{O} \cap \mathcal{O}$, so $\mathcal{O}$ is open.

Defn

A set S is called (path) connected if $\forall z_1, z_2 \in S$, $\exists$ a continuous function $\gamma : [0,1] \longmapsto \mathbb{C}$ s.t.

$$\gamma(0) = z_1, \quad \gamma(1) = z_2 \ \underline{\text{and}} \ \gamma(t) \in S \ \forall \ t \in [0,1]$$

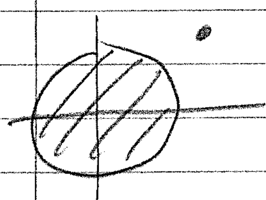


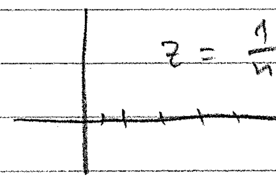
A set S is bounded (bdd) if $\exists R > 0$ s.t. $S \in B_R(0)$

A point z_0 is an <u>accumulation point</u> of a set S if $\forall \varepsilon > 0$, the deleted ε-neighborhood

$$B_\varepsilon(z_0) \setminus \{z_0\} \cap S \neq \emptyset,$$

<u>every</u> deleted neighborhood contains elements of S





0 is an accumulation point

Prop A set is closed iff it contains <u>all</u> of its accumulation pt

//

ANALYTIC FUNCTIONS

Sep 13, 2019

Defn A complex-valued function of one complex variable is a set $S \subseteq \mathbb{C}$ and a rule assigning to each $z \in S$ a (unique) number $w = f(z) \in \mathbb{C}$.

We denote such functions by $f : S \mapsto \mathbb{C}$, where S is called the domain of f and $S = dom(f) = Dom(f)$.

The set $R(f) = f(S) = \{ w \in \mathbb{C} : w = f(z), z \in S \}$ is called the range of f or the image of S under f.

Note ⚠ if only a rule is given for f, then you shall assume $S = dom(f)$ is the largest set of $\mathbb{C}$ for which the rule makes sense.

⚠ Part of the definition is the domain

(Pre-image) For a set $\Gamma \in \mathbb{C}$, $f^{-1}(\Gamma) = \{ z \in S \mid f(z) \in \Gamma \}$ is called the pre-image of Γ under f.

If $\Gamma = \{w_0\}$, we write $f^{-1}(w_0) = f^{-1}(\{w_0\})$

Ex $f(z) = z^2$. There we take $S = dom(f) = \mathbb{C}$

Note $z = (x, y) \Rightarrow z^2 = (x^2 - y^2 + 2ixy)$

In general, if $f(z) = f(x,y) = (u, v)$ $\quad u : S \to \mathbb{R}$, $v : S \mapsto \mathbb{R}$

$= u(x,y) + i(v(x,y))$

Here $u = u(x,y) = Re(f(x,y))$ & $v = Im(f(x,y))$

are called the real & img of f, respectively.

$\boxed{Ex}$ $f(z) = z^2$. Then $Re(f(z)) = x^2 - y^2$. $Im(f(z)) = 2xy$

Also, $f(z) = f(re^{i\theta}) = r^2 e^{i2\theta} = u(r,\theta) + iv(r,\theta)$ (polar)
$= r^2(\cos 2\theta + i \sin 2\theta)$

So $\left\{ \begin{array}{l} Re(f(z)) = r^2 \cos 2\theta = x^2 - y^2 \\ Im(f(z)) = r^2 \sin 2\theta = 2xy \end{array} \right\}$ identically

(*) If for $f(z) = f(x,y) = u(x,y) + iv(x,y)$, $v(x,y) \equiv 0$,
then ~~f(z) is~~ f is said to be real valued.

$\boxed{Ex}$ $\left. \begin{array}{l} f_1(z) = |z|^2, \\ f_2(z) = z + \bar{z} = 2Re(z) \end{array} \right\}$ both real valued.

Some special functions

* Polynomials: a fn $P: \mathbb{C} \mapsto \mathbb{C}$ of the form $P(z) = \sum_{n=0}^{m} a_n z^n$

for $a_i \in \mathbb{C}$ is called a polynomial function.

* Rational functions are fns of the form $R(z) = \frac{P(z)}{Q(z)}$ where

P, Q are polynomials. We take

$$\operatorname{dom}(R) = \{ z \in \mathbb{C} \mid Q(z) \neq 0 \}$$

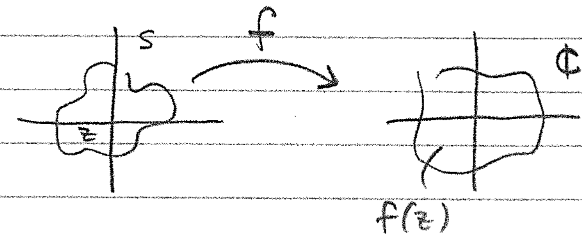
Visualizing functions

⊕ one way: We can graph real & imaginary parts

$$f(z) = u(x,y) + iv(x,y)$$

$$u : S \subset \mathbb{R}^2 \mapsto \mathbb{R}, \quad V : S \to \mathbb{R}.$$

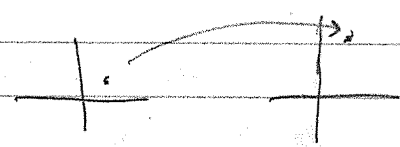
⊕ Another way: "mapping"



Shifts, Reflection, Rotations

$\boxed{\text{Ex}}$ $f(z) = z + i = x + i(y+1)$

↳ (shift)



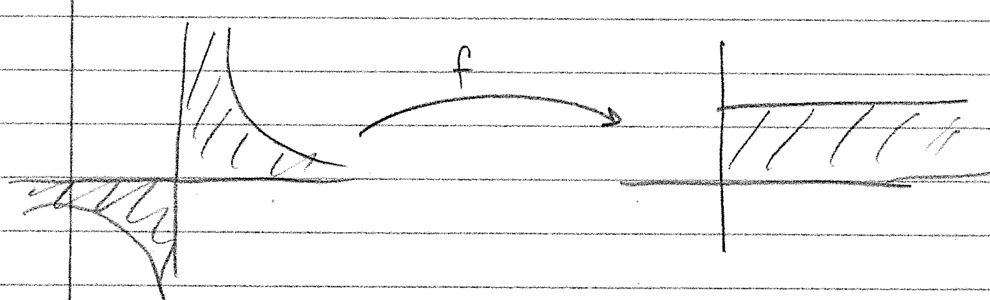
$f(z) = \bar{z}$ ~ reflection

$f(z) = ze^{i\theta}$ ⇝ rotation

$\boxed{\text{Ex}}$ Pre-images Consider $\Gamma = \{ z = x+iy : x \in \mathbb{R}, 0 < y < 2 \}$

for $f : \mathbb{C} \mapsto \mathbb{C}$, $f(z) = z^2$.

$f^{-1}(\Gamma) = ?$ well $0 < 2xy < 2 \Rightarrow 0 < xy < 1$



Sep 16, 2019

LIMITS

Defn Let f be a function ($f : \mathrm{Dom}(f) \subseteq \mathbb{C} \mapsto \mathbb{C}$) and let f be defined on some punctured neighborhood of z_0. We say that the limit of $f(z)$ is w_0 as z approaches z_0 and write

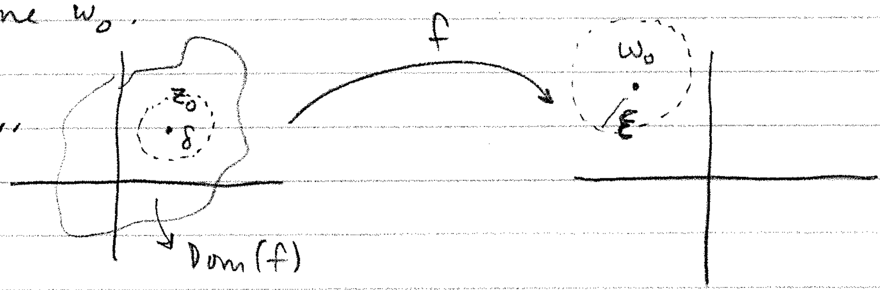
$$\lim_{z \to z_0} f(z) = w_0 \text{ if } \forall \varepsilon > 0, \exists \delta > 0 \text{ such that}$$

$$|f(z) - w_0| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta \text{ for}$$

$z \in \mathrm{Dom}(f)$.

{ We say that the limit of $f(z)$ as $z \to z_0$ exists, if this defn holds for some w_0.

Graphically...



Equivalently, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t

$$f\left(B_\delta(z_0) \setminus \{z_0\}\right) \subseteq B_\varepsilon(w_0)$$

$\boxed{\text{Example}}$ **Claim** $\displaystyle \lim_{z \to 1+i} \frac{i}{z} = \frac{i}{1+i} = \frac{1+i}{2}$

Scratchwork. $\varepsilon > 0$ given, Want $\left|f(z) - i/_{1+i}\right| = \left| 1/_z - 1/_{1+i} \right| < \varepsilon$

Note $\left| i/_z - i/_{1+i} \right| = \left| 1/_z - 1/_{1+i} \right| = \left| \dfrac{1+i-z}{z(1+i)} \right| = \left|\dfrac{1}{z}\right| \left|\dfrac{1}{1+i}\right| |z - (1+i)|$

$$= \frac{1}{|z|} \cdot \frac{1}{\sqrt{2}} |z - (1+i)|$$

Observe $\sqrt{2} = |1+i| = |(1+i) - z + z| \leq |(1+i) - z| + |z| < \delta + |z|$

So $\sqrt{2} - \delta \leq |z|$. If $\delta < \frac{\sqrt{2}}{2}$, then $\sqrt{2} - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} \leq |z|$ whenever

$|z - (1+i)| < \delta$.

So, whenever $|z - (1+i)| < \delta \leq \sqrt{2}/2$, then $\frac{1}{|z|} \leq \frac{2}{\sqrt{2}} = \sqrt{2}$, then

$$\underbrace{\frac{1}{|z|} \cdot \frac{1}{\sqrt{2}}}_{1} |z - (1+i)| < \delta$$

$\boxed{\text{Pf}}$ Let $\varepsilon > 0$. choose $\delta = \min(\varepsilon, \sqrt{2}/2)$. Then if

$0 < |z - (1+i)| < \delta$, then $\left|\frac{i}{z} - \frac{i}{1+i}\right| < |z - (1+i)| = \delta$

then $\frac{1}{|z|} < \sqrt{2}$ $\qquad\qquad \leq \varepsilon$

Thus $\lim_{z \to 1+i} \frac{i}{z} = \frac{i}{1+i}$ $\square$

$\boxed{\text{Proposition}}$ Limits are unique

Pf assume $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} f(z) = w_1$

Given $\varepsilon > 0$, choose δ_0, δ_1 such that

$|f(z) - w_0| < \varepsilon$ when $0 < |z - z_0| < \delta_0$
$|f(z) - w_1| < \varepsilon$ when $0 < |z - z_0| < \delta_1$

Consider $\delta = \min(\delta_0, \delta_1)$, we have, for some z s.t. $0 < |z - z_0| < \delta$, then

$|f(z) - w_0| < \varepsilon$ and $|f(z) - w_1| < \varepsilon$.

So, for this z, $|w_0 - w_1| = |f(z) - w_0 - f(z) + w_1|$

$$= |(f(z) - w_0) + (-f(z) + w_1)|$$

$$\leqslant |f(z) - w_0| + |f(z) - w_1|$$

$$< \varepsilon + \varepsilon = 2\varepsilon.$$

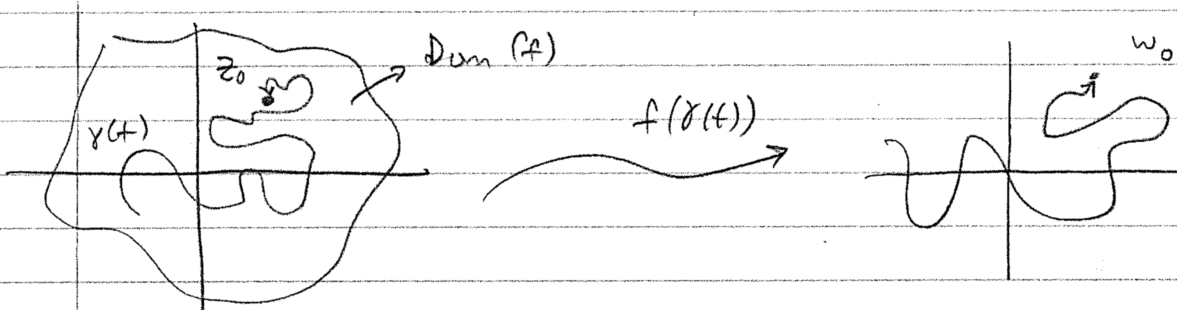
So, for any $\varepsilon > 0$, $|w_0 - w_1| < 2\varepsilon$. So, $w_0 = w_1$

$\square$

Proposition (a)

> If $\lim_{z \to z_0} f(z) = w_0$, then given any continuous function γ satisfying
>
> (1) $\gamma : [0,1] \to \mathbb{R}^2 \equiv \mathbb{C}$ is continuous
>
> (2) $\gamma(t) \neq z_0 \ \forall t > 0$, $\gamma(t) \in \mathrm{Dom}(f) \ \forall t > 0$
>
> (3) $\gamma(0) = z_0$
>
> then $\lim_{t \to 0^+} f(\gamma(t)) = w_0$.

Any path satisfying 1, 2, 3 is said to be <u>admissible</u> for f near z_0, or simply <u>admissible</u>.



$\boxed{\text{Corollary}}$ (a)

If given any two admissible paths γ_0 and γ, we have

$$\lim_{t \to 0^+} f(\gamma_0(t)) \neq \lim_{t \to 0^+} (f(\gamma_1(t)))$$

then $\lim_{z \to z_0} f(z)$ DNE

Example

PF: $\lim_{z \to 0} \frac{\bar{z}}{z}$ DNE

Consider $\gamma_0(t) = t + i0 = (t,0)$ an admissible path
$\gamma_1(t) = 0 + it = (0,t)$ an admissible path.

Note $f(\gamma_0(t)) = \frac{\overline{t+i0}}{t+i0} = \frac{t-i.0}{t+i.0} = 1$

So, $\lim_{t \to 0^+} f(\gamma(t)) = 1 = (1,0)$

Also, $f(\gamma_1(t)) = \frac{\overline{\gamma_1(t)}}{\gamma_1(t)} = \frac{\overline{0+i.t}}{0+i.t} = \frac{-it}{it} = -1 = (-1,0)$

So, $\lim_{t \to 0^+} f(\gamma(t)) = -1 = (-1,0) \neq (1,0) = 1 = \lim_{t \to 0^+} f(\gamma_0(t))$

Thus, by our corollary $\lim_{z \to z_0} f(z)$ DNE

$\square$

$\boxed{\text{Theorem}}$ $\rightsquigarrow$ Connecting to multi-var calc

Suppose that $f(z) = u(x,y) + iv(x,y)$ and $z_0 = x_0 + iy_0$.

Then $\lim\limits_{z \to z_0} f(z) = w_0 = a_0 + ib_0$

iff $\lim\limits_{(x,y) \to (x_0,y_0)} u(x,y) = u_0$ <u>and</u> $\lim\limits_{(x,y) \to (x_0,y_0)} v(x,y) = v_0$

Pf in book

Sep 18, 2019

Suppose that $\lim\limits_{z \to z_0} f(z) = w_0$; $\lim\limits_{z \to z_0} F(z) = W_0$, then

1. $\lim\limits_{z \to z_0} (f(z) + F(z)) = \cancel{\lim\limits_{z \to}}\, w_0 + W_0$
2. $\lim\limits_{z \to z_0} f(z) F(z) = w_0 W_0$
3. If $W_0 \neq 0$, then $\lim\limits_{z \to z_0} \dfrac{f(z)}{F(z)} = \dfrac{w_0}{W_0}$

Pf of 2 Let $z_0 = x_0 + iy_0$, $f(z) = u(x,y) + i v(x,y)$ and

$$F(z) = U(x,y) + iV(x,y). \text{ Then}$$

$$f(z)F(z) = (uU - vV) + i(uV + Uv)$$

Observe that since $\lim\limits_{z \to z_0} f(z) = w_0$ & $\lim\limits_{z \to z_0} F(z) = W_0$,

$\lim\limits_{(x,y) \to (x_0,y_0)} u(x,y) = u_0$; $\lim\limits_{(x,y) \to (x_0,y_0)} U(x,y) = U_0$

$\lim\limits_{(x,y) \to (x_0,y_0)} v(x,y) = v_0$; $\lim\limits_{(x,y) \to (x_0,y_0)} V(x,y) = V_0$

$\hookrightarrow$ by result from Monday

Appealing to the algebra of limits of fn: $\mathbb{R}^2 \mapsto \mathbb{R}$,

$$\lim_{(x,y)\to(x_0,y_0)} \Big(u(x,y)\,U(x,y) - v(x,y)\,V(x,y)\Big) = u_0 U_0 - v_0 V_0$$

$$= \operatorname{Re}(w_0 W_0) = \cancel{\operatorname{Re}(f(z_0)F(z_0))}$$

Similarly,

$$\lim_{(x,y)\to(x_0,y_0)} \Big(U(x,y)\,v(x,y) - V(x,y)\,u(x,y)\Big) = \cdots = \operatorname{Im}(w_0 W_0)$$

So, by thm from Monday, $\lim_{z\to z_0} f(z)F(z) = w_0 W_0$. $\square$

Two easy limits and a corollary

- **Fact** $\lim_{z\to z_0} z = z_0$

  Pf Given $\varepsilon > 0$, choose $\delta = \varepsilon$, then

  $|z - z_0| < \delta = \varepsilon$ whenever $0 < |z - z_0| < \delta$, so $\lim_{z\to z_0} z = z_0$. $\square$

- **Fact** $\lim_{z\to z_0} c = c$ for any $c \in \mathbb{C}$.

- By induction, we have that $\forall n \in \mathbb{N}$,

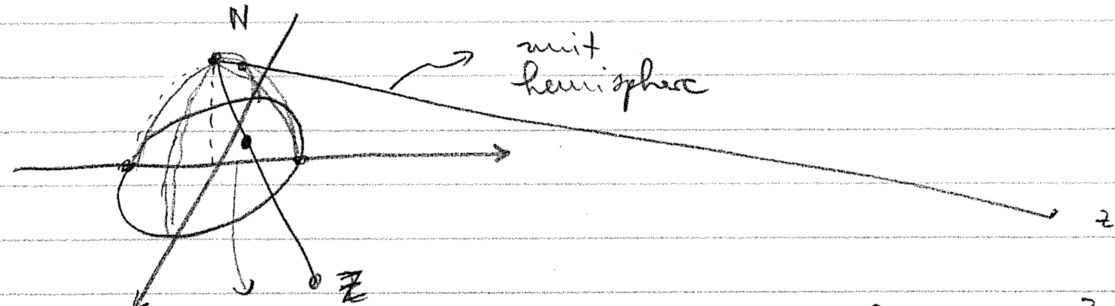
  $\lim_{z\to z_0} z^n = z_0^n$ (use thm to show $z^2 \ldots$)

- **Corollary** Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial and $z_0 \in \mathbb{C}$

  Then $\lim_{z\to z_0} p(z) = p(z_0)$ Pf use sum & product rule...

Riemann Sphere & Stereographic Projection

(how to see infinity ...)



interfect only at 1 point on sphere in $\mathbb{R}^3$
→ this is a 1 to 1 correspondence.

$\forall z \in \mathbb{C}, \exists r(z) \in S \setminus \{N\}$

S → unit sphere in $\mathbb{R}^3$

It is with this correspondence that we recognize points "near" ∞ in $\mathbb{C}$ as having $r(z)$ near N.

So ⇝ recognize $N =$ "$r(\infty)$" $=$ "∞".

Defn

> Given $\varepsilon > 0$, we call the set $B_\varepsilon(\infty) = \left\{ z \in \mathbb{C} : |z| > \frac{1}{\varepsilon} \right\}$ the ε-neighborhood of ∞

Defn

> Given $z_0 \in \mathbb{C}$ and f defined on a neighborhood of z_0. We say that the limit of $f(z)$ as $z \to z_0$ is ∞ and write
>
> $$\lim_{z \to z_0} f(z) = \infty \quad \text{if the following property holds}$$
>
> $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $f(z) \in B_\varepsilon(\infty)$ whenever $z \in \mathrm{Dom}(f)$ and $z \in \delta$-neighborhood of z_0, i.e.

Equivalently, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(z)| > \frac{1}{\varepsilon}$ whenever $0 < |z - z_0| < \delta$

We say
$$\lim_{z \to \infty} f(z) = w_0 \quad , \text{for } w_0 \in \mathbb{C} \quad \text{if}$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $f(z)$ lives in the ε-neighborhood of w_0 whenever z lives in the δ-neighborhood of ∞.

i.e. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(z) - w_0| < \varepsilon$ whenever $|z| > \frac{1}{\delta}$.

should include dom (f)

Sep 20, 2019

Defn We say that the limit of $f(z)$ as $z \to \infty$ is ∞ if $\forall$ ε neighborhood of ∞, $\exists$ δ-neighborhood of ∞ such that $f(z) \in B_\varepsilon(\infty)$ whenever $z \in B_\delta(\infty)$

{ Equivalently, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(z)| > \frac{1}{\varepsilon}$ whenever $|z| > \frac{1}{\delta}$. }

Thm Let $z_0, w_0 \in \mathbb{C}$, then $\displaystyle\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0$

Thm $\displaystyle\lim_{z \to \infty} f(z) = w_0 \iff \lim_{z \to 0} f(1/z) = w_0$

Thm $\displaystyle\lim_{z \to \infty} f(z) = \infty \iff \lim_{z \to 0} \frac{1}{f(1/z)} = 0$

$z^{-1} = (z^{-1}(\bar{z} w) = w$

$\boxed{\text{Pf of 3}}$ Suppose that $\lim_{z\to\infty} f(z) = \infty$.

Let $\varepsilon > 0$ be given. Then by assumption, $\exists\, \delta > 0$ s.t.

$|f(z)| > \frac{1}{\varepsilon}$ whenever $|z| \geq \frac{1}{\delta}$

Then $\frac{1}{f(z)} < \varepsilon$ whenever $|z| > \frac{1}{\delta}$. Notice that $|z| > \frac{1}{\delta}$ $(\Leftrightarrow)$ iff $|\omega| = \frac{1}{|z|} < \delta$. Then, for any $0 < |\omega| < \delta$, we have that

$$\left|\frac{1}{f(1/\omega)}\right| = \frac{1}{|f(z)|} < \varepsilon \text{ as long as } \omega = \frac{1}{z}, \ z = \frac{1}{\omega}, \text{ i.e.}$$

$\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $\left|\frac{1}{f(1/z)}\right| < \varepsilon$ when $|z| < \delta$.

So, $\lim_{z\to 0} \frac{1}{f(1/z)} = 0$ //

. The converse is gotten by reversing all things.

CONTINUITY

full, not punctured,

Defn Let f be defined on a neighborhood of z_0. We say that f is continuous at z_0 if the following things hold:

(1) $\lim_{z\to z_0} f(z)$ exists

(2) ~~lim~~ $f(z_0)$ exists

(3) $\lim_{z\to z_0} f(z) = f(z_0)$

$\boxed{\text{Ex}}$ + $f(z) = z^2 + c$ is continuous at $z_0 \in \mathbb{C}$, $\forall z_0$.

+ All polynomials are continuous everywhere.

+ All rational functions are continuous at points z_0 at which the denominator is non-zero.

—— // ——

Recall → We say that f is continuous on a set $S \subseteq \mathbb{C}$ if f is continuous at each $c_0 \in S$.

Thm Suppose that f is continuous at c_0 & g is continuous at $f(z_0) = w_0$. Then

$g \circ f(z_0)$ is continuous at z_0

Pf

Let $\varepsilon > 0$ be given. Given that g is cont at $w_0 = f(z_0)$. So $\exists \gamma > 0$ s.t. $|g(w) - g(w_0)| < \varepsilon$ whenever $|w - w_0| < \gamma$.

Given this γ. By the continuity of f at z_0, $\exists \delta > 0$ s.t $|f(z) - f(z_0)| < \gamma$ whenever $|z - z_0| < \delta$.

So, for $|z - z_0| < \delta$ we have that $|f(z) - f(z_0)| < \gamma$ and so

$$|g(f(z)) - g(f(z_0))| < \varepsilon.$$

☺

—— // ——

Thm Suppose that f is cont @ z_0 and $|f(z_0)| \neq 0$. $\exists \delta > 0$ s.t. $f(z) \neq 0$ $\forall z \in B_\delta(z_0)$

Pf Choose $\varepsilon = |f(z_0)/2| > 0$. By continuity of f @ z_0, $\exists \delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon = \frac{1}{2}|f(z_0)|$ $\forall z$ s.t $|z - z_0| < \delta$

Then, $\forall z$ s.t $|z - z_0| < \delta$ $(z \in B_\delta(z_0))$, we have that

$$|f(z_0)| = |f(z_0) + f(z) - f(z)| \leq |f(z) - f(z_0)| + |f(z)|$$
$$\leq \frac{|f(z_0)|}{2} + |f(z)|$$

So $\forall z \in B_\delta(z_0)$, we have $\left|\frac{f(z_0)}{2}\right| \leq f(z)$ ▨

Thm Let R be a closed & bounded subset of the complex plane. Let f be continuous on R. Then $\exists M \geq 0$ s.t

$|f(z)| \leq M$ $\forall z \in R$ and $\exists z_0 \in R$ @ which $|f(z_0)| = M$

Ex

$R = \overline{B_3(0)}$ → closed ball of radius 3.

$f(z) = z^2$.

Note that $R = \{0\} \cup \{re^{i\theta} ; 0 < r \leq 3 , \theta \in \mathbb{R}\}$

$\forall z$ s.t $f(z) = f(re^{i\theta}) = r^2 e^{i2\theta} \Rightarrow |f(z)| = r^2 \leq 9 = M$.

and note that if $z = 3i$ has $|z|^2 = 9$

DIFFERENTIABILITY

ep 23, 2019

Defn Let f be defined on a neighborhood of z_0. The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z \to z_0}$$

and it's defined whenever this limit exists. When this limit exists, we say f is differentiable at z_0

By expressing the difference $\Delta z = z - z_0$. We can write this limit as

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \text{ where}$$

$$\Delta w = f(z_0 + \Delta z) - f(z_0).$$

Ex $f(z) = z^2$. Claim. $f(z) = z^2$ is differentiable @ all $z \in \mathbb{C}$

Pf For fixed z_0, $\displaystyle\lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{2z\Delta z + \Delta z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} 2z + \Delta z$$

$$= 2z.$$

So $f'(z) = 2z$.

$\square$

We write $\boxed{\dfrac{df}{dz} = f'(z) = D_z f}$

Some fns that are differentiable everywhere

1) $f(c) = C \leftarrow$ constant ; $D_z f = 0$

2) $D_z z^n = n z^{n-1}$, $n \geq 0$

3) For $n \leq -1$, and $f(z)$ ~~diff~~ $= z^n = \dfrac{1}{z^{-n}}$ is diff' when $z \neq 0$ and

$$f'(z) = n z^{n-1}$$

4) ...

Bad examples Let $f(z) = \bar{z}$. (very continuous ...)

1) $$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{w \to 0} \frac{\bar{w}}{w} \leftarrow \text{DNE}$$

So f is differentiable nowhere.

2) $f(z) = |z|^2$ ...

$$\lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)\overline{(z+\Delta z)} - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z\overline{\Delta z} + \Delta z\,\bar{z} + \Delta z\,\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} z \cdot \frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z}$$

$$= \lim_{\Delta z \to 0} \bar{z} + z \cdot \frac{\overline{\Delta z}}{\Delta z} = \bar{z} + z \cdot \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} \quad \text{if } z \neq 0 \rightarrow \boxed{\text{DNE}}$$

but if $z = 0$ then $\lim_{\Delta z \to 0} = 0$

So $f(z) = |z|^2$ is not differentiable at any $z \neq 0$.

At $z = 0$, $\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \overline{\Delta z} = 0$

So $f(z) = |z|^2$ is differentiable at a single point $z = 0$, And $f(0) = 0$.

Aside: Real-valued fn diff at one point... $f(x) = x^2 \begin{cases} -1, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{cases}$

In terms of $\mathbb{R}^2$, $f(z) = f(x,y) = (x^2 + y^2, 0)$, $f(z) = |z|^2$

In the sense that $f: \mathbb{R}^2 \mapsto \mathbb{R}^2$, this fn is really nice. In fact, it is diff in a real sense everywhere, but complex diffable only at $z = 0$.

Suspect that complex differentiability is stringent / difficult to have

Proposition $\boxed{\text{If } f \text{ is differentiable at } z_0, \text{ it is cont. @ } z_0}$

Pf
$$\lim_{z \to z_0} (f(z) - f(z_0)) = \lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{(z - z_0)} \cdot (z - z_0) \right]$$

$$= \underbrace{\lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z \to z_0} \right]} \lim_{z \to z_0} (z - z_0)$$

$$= f'(z_0) \cdot \underbrace{\lim_{z \to z_0} (z - z_0)}_{0} = 0$$

Thus, $\lim_{z \to z_0} f(z) = f(z_0)$ ✓. So f cont. @ z_0. □

Proposition: Let f, g be diff. @ z_0, then

$f+g$, cf, $(c \in \mathbb{C})$, $f \cdot g$, are differentiable at z_0, with

$$D_z (f+g)(z_0) = f'(z_0) + g'(z_0)$$
$$D_z \, cf(z_0) = c f'(z_0)$$
$$D_z \, fg(z_0) = f'(z_0) g(z_0) + f(z_0) g'(z_0)$$

If, additionally, $g(z_0) \neq 0$, then $\frac{f}{g}$ diff' @ z_0, and

$$D_z \frac{f}{g}(z_0) = \frac{f'(z_0) g(z_0) - f(z_0) g'(z_0)}{g^2(z_0)}$$

<u>Pf of product</u>

① $$\lim_{\Delta z \to 0} \frac{\left((f(z_0+\Delta z)) g(z_0+\Delta z) \right) - f(z_0) g(z_0))}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[\underbrace{\left(f(z_0+\Delta z) - f(z_0) \right)}_{\Delta f} \left(g(z_0+\Delta z) \right) + f(z_0) g(z_0+\Delta z) - f(z_0) g(z_0) \right]$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[\Delta f \, \underbrace{g(z_0+\Delta z)}_{\text{exists by cont}} + f(z_0) . \Delta g \right] \quad (f, g \text{ diff @ } z_0 \text{ hence continuous})$$

$$= g(z_0) . f'(z_0) + f(z_0) g'(z_0)$$

//

Proposition Let f diff @ z_0 & g diff @ $w_0 = f(z_0)$. Then
$F(z) = g \circ f(z) = g(f(z))$ is diff @ z_0 and
$F'(z) = g'(f(z_0))f'(z_0)$ ...

Scratch...

$$\lim_{\Delta z \to 0} \frac{F(z_0 + \Delta z) - F(z_0)}{\Delta z}$$

$$= \lim_{z \to z_0} \frac{g \circ f(z) - g \circ f(z_0)}{z - z_0}$$

$$\overset{?}{=} \lim_{z \to z_0} \frac{1}{z - z_0}\left[\frac{g \circ f(z) - g \circ f(z_0)}{\underbrace{f(z) - f(z_0)}} \cdot f(z) - f(z_0)\right]$$

can I do this? probably NOT, because $f(z)$ can be $= f(z_0)$, ...

Sep 25, 2019

Pf On a neighborhood of w_0, defined on (can be defined on a larger domain than this)

$\phi : N \mapsto \mathbb{C}$ by

$$\phi(w) = \begin{cases} \dfrac{g(w) - g(w_0)}{w - w_0} - g'(w_0) & w \neq w_0 \\ 0 & w = w_0 \end{cases}$$

Observe that because g is diff', $\lim_{w \to w_0} \phi(w) = 0$. From this, it follows that ϕ is continuous on its domain. Also, for $w \in N$,

$$(w - w_0)\,\phi(w) = (g(w) - g(w_0)) - g'(w_0)(w - w_0) \qquad (*)$$

Thus, given the continuity of f at z_0, we can choose $\delta > 0$ s.t for $z \in B_\delta(z_0)$, we have $\overset{f(z)=}{w} \in N = B_\varepsilon(w_0)$, because

$$|f(z) - f(z_0)| = |w - w_0| < \varepsilon \text{ whenever } |z - z_0| < \delta .$$

So, consider $\forall$ such $z \in B_\delta(z_0)$, we have that $\Phi(f(z))$ makes sense.
Also, for these values of $z \neq z_0$

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{g(f(z)) - g(\overset{f(z_0)}{w_0})}{z - z_0}$$

$$= \frac{(w - w_0)\Phi(w) + g'(w_0)(w - w_0)}{z - z_0} \quad \text{by } (\star)$$

$$= \frac{(f(z) - f(z_0))\Phi(w) + g'(w_0)(f(z) - f(z_0))}{z - z_0}$$

So,

$$\lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{\overset{\text{diff}}{(f(z) - f(z_0))}\overset{\text{cont}}{\Phi(f(z))} + \overset{\text{constant}}{g'(f(z_0))}(f(z) - f(z_0))}{z - z_0} \text{ diff}$$

$$= f'(z_0)\underbrace{\Phi(f(z_0))}_{0} + g'(f(z_0))f'(z_0)$$

$$= g'(f(z_0))f'(z_0).$$

$\square$

$\boxed{Ex}$ Consider: $F(z) = (z + iz^2)^{200}$

Let $F(z) = g \circ f(z)$ where $g(w) = w^{200}$, $f(z) = z + iz^2$.

We know $f'(z) = 1 + 2iz$ and $g'(w) = 200w^{199}$. So, by ~~the~~ proposition,

F is diff everywhere and

$$F'(z) = g'(f(z))f'(z) = 200(z + iz^2)^{199} \cdot (1 + 2iz)$$

Cauchy - Riemann Equation

Suppose $f(z) = u(x,y) + i v(x,y)$ diff @ $z_0 = (x_0, y_0)$

This means that the (complex) limit.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \quad (\text{if exists})$$

Consider the path $\gamma_1(t) = (x_0 + t) + i y_0 = (x_0 + t, y_0)$

Observe this is an admissible path for $\frac{f(z) - f(z_0)}{z - z_0}$ near $z_0 = (x_0, y_0)$

By our proposition A, $f'(z_0) = \lim_{t \to 0^+} \frac{f(\gamma_1(t)) - f(z_0)}{\gamma_1(t) - z_0}$

$$= \lim_{t \searrow 0} \frac{u(x_0+t, y_0) + i v(x_0+t, y_0) - u(x_0, y_0) - i v(x_0, y_0)}{(x_0 + t) + i y_0 - x_0 - i y_0}$$

$$= \lim_{t \searrow 0} \frac{[u(x_0+t, y_0) - u(x_0, y_0)] + i [v(x_0+t, y_0) - v(x_0, y_0)]}{t}$$

$$= \lim_{t \searrow 0} \frac{u(\cdot + t) - u(\cdots)}{t} + i \lim_{t \searrow 0} \frac{v(\cdots + t) - v(\cdots)}{t}$$

(1) $$= \partial_x u(x_0, y_0) + i \partial_x v(x_0, y_0).$$

So the differentiability of f at z_0 guarantees that u, v have partial derivatives in x, y at (x_0, y_0), and

$$f'(z_0) = \partial_x u(x_0, y_0) + i \partial_x v(x_0, y_0)$$

Consider another path $\gamma_2(t) = (x_0, y_0 + t) \rightsquigarrow$ admissible

$$f'(z_0) = \lim_{t \searrow 0} \frac{u(x_0, y_0+t) + i v(x_0, y_0+t) - u(x_0, y_0) - i v(x_0, y_0)}{x_0 + i(y_0 + t) - x_0 - i y_0} = \cdots =$$

$$= \lim_{t \to 0} \frac{(u)}{it} + \frac{i(v)}{it} \rightsquigarrow \text{must exist...}$$

$$= \partial_y v(x_0, y_0) - i \partial_y u(x_0, y_0) \rightsquigarrow$$

(2) $$= \partial_y v(x_0, y_0) + i(-\partial_y u(x_0, y_0)).$$

From (1) and (2), $\partial_x u(x_0, y_0) + i \partial_x v(x_0, y_0) = \partial_y v(x_0, y_0) + i(-\partial_y u(x_0,$

$$\Leftrightarrow \quad \partial_x u = \partial_y v \text{ and } \partial_x v = -\partial_y u$$

Theorem: (Cauchy + worked or extended by Riemann)

Let $f(z) = u(x,y) + i v(x,y)$ be diff @ $z_0 = (x_0, y_0)$
Then the first order partial derivative of u & v exists @ (x_0, y_0) and

$$\partial_x u(x_0, y_0) = \partial_y v(x_0, y_0) \quad \& \quad \partial_x v(x_0, y_0) = -\partial_y u(x_0, y_0)$$

These are called the Cauchy - Riemann Eqn

Sep 27, 2019

Is the converse true? $\boxed{\text{NO}}$

non-ex

$$f(z) = \begin{cases} \dfrac{\cancel{z}^2}{z} & z \neq 0 \\ 0 & z = 0 \end{cases} \rightsquigarrow \text{in HW}$$

This fn's real and imaginary parts satisfy Cauchy-R eqn @ $z = 0$, yet f' exists nowhere.

Thm Let $f(z) = u(x,y) + iv(x,y)$ be defined on a neighborhood of

$$z_0 = x_0 + iy_0.$$

Suppose (1) u, v have partial derivatives on a neighborhood of z_0
(2) All first order partial derivatives are <u>continuous</u> on this neighborhood of z_0 and the C-R equations hold, i.e.

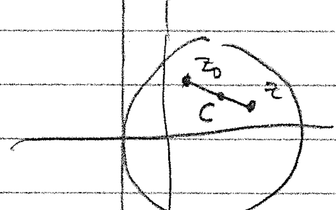
$$u_x(x_0,y_0) = v_y(x_0,y_0) \quad \& \quad u_y(x_0,y_0) = -v_x(x_0,y_0)$$

Then f is differentiable @ z_0 and

$$f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0)$$

<u>Pf</u> The assumption that $u: \mathbb{R}^2 \mapsto \mathbb{R}$ and $v: \mathbb{R}^2 \mapsto \mathbb{R}$ have continuous partials on a neighborhood of (x_0, y_0) guarantees that ~~in fact~~ u, v are diff on this neighborhood, $N \left(= B_\delta(z_0)\right)$ in the sense of functions mapping $\mathbb{R}^2 \to \mathbb{R}$

As a consequence, for each $z \in N$ of z_0, $\exists\, c_z$ living on the line segment between z and z_0 such that



$$u(x,y) - u(x_0,y_0) = \underset{\substack{\uparrow \\ \text{(Jacobian)}}}{D} u(c_z) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

by the mean value theorem in $\mathbb{R}^2$. And $\exists\, \tilde{c}_z$ s.t.

$$v(x,y) - v(x_0,y_0) = Dv(\tilde{c}_z) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

Now, this gives $u(x,y) - u(x_0,y_0) = Du(z_0)\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$

$$+ \left[Du(c_z) - Du(z_0) \right] \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

And, $Du = (u_x, u_y)$, $Dv(v_x, v_y)$. So we have

$$u(x,y) - u(x_0, y_0) = u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y$$
$$+ [u_x(C_z) - u_x(x_0,y_0)]\Delta x + [u_y(C_z) - u_y(x_0,y_0)]\Delta y$$

Similarly,

$$v(x,y) - v(x_0,y_0) = v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y$$
$$+ [v_x(\tilde{C_z}) - v_x(x_0,y_0)]\Delta x + [v_y(\tilde{C_z}) - v_y(x_0,y_0)]\Delta y$$

Write

$$\varepsilon_{u,x}(z) = u_x(C_z) - u_x(x_0,y_0)$$
$$\varepsilon_{u,y}(z) = u_y(C_z) - u_y(x_0,y_0)$$
$$\varepsilon_{v,x}(z) = v_x(C_z) - v_x(x_0,y_0)$$
$$\varepsilon_{v,y}(z) = v_y(C_z) - v_y(x_0,y_0)$$

So

$$\frac{\Delta w}{\Delta z} = \frac{f(z)-f(z_0)}{\Delta z} = \frac{(u(x,y) - u(x_0,y_0)) + i(v(x,y) - v(x_0,y_0))}{\Delta x + i\Delta y}$$

$$= \frac{u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + i(v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y)}{\Delta x + i\Delta y} \cdot \frac{\Delta x - i\Delta y}{\Delta x - i\Delta y}$$

$$+ \frac{\varepsilon_{ux}\Delta x + \varepsilon_{uy}\Delta y + \varepsilon_{vx}\Delta x + \varepsilon_{vy}\Delta y}{\Delta x + i\Delta y} \cdot \cancel{\frac{\Delta x - i\Delta y}{\Delta x - i\Delta y}}$$

$= \dots = \dots =$ By Cauchy - Riemann...

@(x_0,y_0)

$$= (u_x + iv_x) + \frac{(\varepsilon_{ux} + \varepsilon_{vx})\Delta x + (\varepsilon_{uy} + \varepsilon_{vy})\Delta y}{\Delta z}$$

$$= \left(u_x(x_0,y_0) + \varepsilon_x \frac{\Delta x}{\Delta z}\right) + i(v_x(x_0,y_0)) + \varepsilon_y \frac{\Delta y}{\Delta z}$$

When $z \to z_0$, $\tilde{C}_z$, C_z get squeezed ... so by continuity

$$\boxed{\lim_{z\to z_0} \frac{\Delta f}{\Delta z} = u_x(x_0,y_0) + iv_x(x_0,y_0)}$$

ANALYTIC FUNCTIONS

Oct 2, 2019

Defn A function f is analytic at a point $z \in \mathbb{C}$ if it is diff on some neighborhood of z_0 at every point in $B_\delta(z_0)$ for some $\delta > 0$

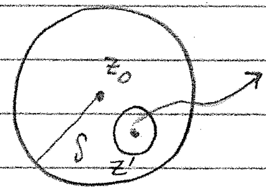
f is said to be analytic on an open set $\mathcal{O}$ if it is analytic at each $z \in \mathcal{O}$

If f is analytic on a set S, we say it is analytic on an open set $\mathcal{O} \supseteq S$.

Analytic $\equiv$ Holomorphic

A fn f is said to be entire if f is analytic on $\mathbb{C}$

If $z_0 \in \mathbb{C}$ is such that f is analytic @ every point in a nbh centered at z_0, but NOT at z_0 (analytic on $B_\delta(z_0) \setminus \{z_0\}$) we say z_0 is a SINGULAR point for f



analytic here ...

Ex

1. Polynomials $\rightsquigarrow$ entire
2. $f(z) = \dfrac{1}{z}$ $\rightsquigarrow$ analytic on $\mathbb{C} \setminus \{0\}$
3. $f(z) = z \cdot \mathrm{Im}(z)$ $\rightsquigarrow$ diff only at 0, but not analytic anywhere ...
4. $f(x+iy) = x^2 + iy^2$ $\rightsquigarrow$ diff only $x = y$, but not analytic anywhere

Proposition Suppose f, g are analytic on open set $\mathcal{O}$, then $f \pm g$, fg are also analytic on $\mathcal{O}$. If $g(z) \neq 0 \ \forall z \in \mathcal{O}$ then

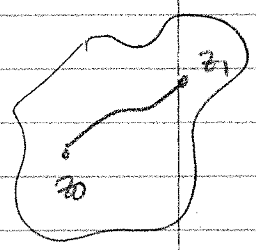
$\frac{f}{g}$ is also analytic on $\mathcal{O}$

Conclusion

$\hookrightarrow$ The set of analytic functions on an open set $\mathcal{O}$ forms a ring (commutative) $Hol(\mathcal{O})$

Proposition Sup. D is a domain (open, nonempty, path connected) and f is analytic on D. If $f'(z) = 0$ $\forall z \in D$, then f is constant on D

Pf Given $z_0, z_1 \in D$, $\exists$ a C' path $\gamma : [0,1] \mapsto D$ s.t. $\gamma(0) = z_0$, $\gamma(1) = z_1$, and γ' is continuous



By the C-R eqn... $f = u + iv$ for u, v diff from $\mathbb{R}^2$ to $\mathbb{R}$

Consider $h(t) = \mathrm{Re}(f \circ \gamma)(t) = u(\gamma(t))$

By our observation & multivariate chain rule, $h(t)$ is cont on $[0,1]$ and diff. on $(0,1)$ and

$$h'(t) = u_x(\gamma(t))\gamma_1'(t) + u_y(\gamma(t))\gamma_2'(t)$$

$$\text{with } \gamma(t) = (\gamma_1(t), \gamma_2(t)) \quad \forall t \in (0,1)$$

By MVT, $\exists\, c \in (0,1)$ s.t.
$$\begin{aligned} h(1) - h(0) &= h'(c) \cdot (1-0) \\ &= h'(c) \\ &= u_x(\gamma(c))\gamma_1'(c) \\ &\quad + u_y(\gamma(c))\gamma_2'(c) \end{aligned}$$

$$= u_x(\gamma(c)).\gamma_1'(c) - v_x(\gamma(c))\gamma_2'(c)$$

But $f'(z) = u_x + iv_x = 0$, so

$\exists c \in (0,1)$ s.t. $h(1) - h(0) = 0 \Leftrightarrow h(1) = h(0)$. So,

$Re(f(z_0)) = Re(f(\gamma(0))) = h(0) = h(1) = Re(f(\gamma(1))) = Re(f(z_1))$

Similarly, we can show $Im(f(z_0)) = Im(f(z_1))$

Therefore $f(z_0) = f(z_1)$ $\forall z_0, z_1 \in D$. Thus f is constant.

~~Proposition If f & $\bar{f}$ are analytic~~

Theorem (C-R for analytic fn) (★)(★)

Let f be a fn defined on an open set $O \subseteq \mathbb{C}$, then f is analytic on O **iff** for $f = u + iv$

1. u, v have first-order partial derivatives on all of O
2. u_x, u_y, v_x, v_y are cont on all of O.
3. C-R eqns are satisfied i.e. $u_x = v_y$, $u_y = -v_x$ on all of O

Application Sup f & $\bar{f}$ are both analytic on D then f is constant

Pf Sup $f = u + iv$, ~~then~~ $\bar{f} = U + iV$ where $u = U$, $v = -V$
If f analytic, then
$$u_x = v_y,\ u_y = -v_x \text{ on all of } D$$
If $\bar{f}$ analytic, then, $U_x = V_y$, $U_y = -V_x$ on all of D $\Rightarrow$

$u_x = U_x = V_y = -v_y = -u_x \Rightarrow u_x = 0$ on D.

Similarly $v_x = 0$ on D

So $f' = u_x + i v_x = 0$ on all of $D \Rightarrow f$ constant on D. ☒

App.

> If $|f(z)| = c \ \forall z \in D$, D is a domain, then f constant on D and f is analytic

Pf If $c = 0$, then o.k.
If $c \neq 0$, then

$\bar{f}(z) f(z) = |f(z)|^2 = c^2 > 0$. In particular,

$f(z) \neq 0 \ \forall z \in D$, and so $\bar{f}(z) = \dfrac{c^2}{f(z)}$, this says

$\bar{f}(z)$ is analytic on D. And so by App 1, f is constant

HARMONIC FUNCTIONS

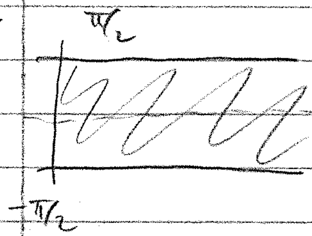
A function u is said to be harmonic on a set Θ if

$$\Delta u = u_{xx} + u_{yy} \equiv 0 \text{ on } \Theta.$$

This eqn is called Laplace' Eqn. Appears in theory of heat..., electrostatics, magnetostatics, etc in mathematical physics.

Oct 4, 2019

$\boxed{\text{Ex}}$ $T(x,y) = e^{-x}\cos y$ on $\overline{D} = \{(x,y) \in \mathbb{C},\ x \geq 0,\ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\}$



Is T a harmonic fn in D

$$\left.\begin{aligned} T_{xx} &= e^{-x}\cos y \\ T_{yy} &= -e^{-x}\cos y \end{aligned}\right\} \to \Delta T = 0 \ ✓$$

Coming from PDE, pose BVP. Find T such that

$$\begin{cases} \Delta T = 0 & \text{in } D \\ T(x, \mp\frac{\pi}{2}) = 0 & \forall\, x \geq 0 \\ T(0,y) = \cos y & -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \end{cases}$$

<u>Connection to analytic function</u>.

<u>Thm</u> $\boxed{\text{if } f(z) = u(x,y) + i v(x,y) \text{ is analytic in a domain } D \text{ Then } u, v \text{ are harmonic in } D}$

We won't prove this, but we can prove the following...

Thm $\boxed{\text{If } f(z) = u(x,y) + i v(x,y) \text{ is analytic in } D \text{ and } u, v \text{ are twice differentiable in } D, \text{ Then } u, v \text{ are harmonic}}$

with continuous partials

<u>Pf</u> By C-R, $u_x = v_y$
$u_y = -v_x$

equality of mixed partials...

So $u_{xx} = v_{yx}$
$u_{yy} = -v_{xy} = -v_{yx} = -u_{xx} \Rightarrow \boxed{\Delta u = 0}$

Similarly, $\boxed{\Delta v = 0}$ So u, v are harmonic. $\square$

Ex $f(z) = e^{-z} = e^{-(x+iy)} = e^{-x}(\cos y - i\sin y)$

$$= \underbrace{e^{-x}\cos y}_{u} + i\,(\underbrace{-e^{-x}\sin y}_{v})$$

Here, as we will see, $f(z)$ is entire, and so u, v are harmonic.

Ex $f(z) = \frac{1}{z}$, $D = \mathbb{C} \setminus \{0\}$

We know that in fact f is analytic on D.

By the theorem, we have that $f(z) = \underbrace{\frac{x}{x^2+y^2}}_{u} + i\,\underbrace{\frac{-y}{x^2+y^2}}_{v}$

and u, v are harmonic.

Defn Given a harmonic fn u on D and another harmonic fn v on D. If u, v satisfy C-R eqn, then we say v is a harmonic conjugate of u. (not symmetric)

Thm. A fn $f(z) = u + iv$ on a domain D is analytic iff v is a harmonic conjugate of u

Pf If f is analytic, then u, v are harmonic. But C-R thm also says u, v satisfy C-R

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

So v is a harmonic conjugate of u.

Conversely, if v is harm. conj of u, then C-R hold everywhere in D & by thm, $f = u + iv$ is analytic in D

ELEMENTARY FUNCTIONS

Defn Exponential Function

$\exp : \mathbb{C} \mapsto \mathbb{C}$ by $\exp(z) = e^x e^{iy} = e^x(\cos y + i \sin y)\ \forall z = x+iy \in \mathbb{C}$

Properties ① when $z = x + i0 \in \mathbb{R}$, then $\exp(z) = e^x$

exp is an _____ extention of the e^x seen in calc.

② $\boxed{\forall z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2 \in \mathbb{C},\quad e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}}$

pf
$$e^{z_1+z_2} = e^{(x_1+x_2)+i(y_1+y_2)} = e^{x_1+x_2}(\cos(y_1+y_2) + i\sin(y_1+y_2))$$
$$= \cdots$$
$$= e^{z_1} e^{z_2}.$$

③ $e^z \neq 0\ \forall z \in \mathbb{C}$.

pf if $e^z = 0$ then $e^0 = e^{z_1 - z_1} = e^{(0-z_1)} \cdot e^{z_1} = 0$
But $e^0 = 1 \Rightarrow$ contradiction...

④ $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$

⑤ $|e^z| = e^x$

⑥ $\arg(e^z) = y + 2\pi n,\ n \in \mathbb{Z}$

⑦ e^z is periodic with period $(2\pi i)$

⑧ e^z is entire with $D = \mathbb{C}$, $\dfrac{d}{dz} e^z = e^z$

Pf $\quad e^z = e^x(\cos y + i\sin y) \qquad u(x,y) = e^x \cos y$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad v(x,y) = +e^x \sin y$.

Can show $\quad u_x = v_y$

$\qquad\qquad u_y = -v_x$. These are continuous on $\mathbb{C} = \mathbb{R}^2$.

So, $\quad (e^z)' = u_x + iv_x = e^x\cos y + ie^x \sin y = e^z$ by C-R equs. with

and of course

$\qquad e^z$ is analytic...

Defn $\quad$ The complex log $\quad$ To construct the log, we begin by finding an inverse relationship with e^z. For $z \neq 0$, can we find $w \in \mathbb{C}$ s.t. $e^w = z$?

If $w = u + iv$, $z = re^{i\theta}$, $r > 0$, $\theta = \arg(z)$. Then, we have that

$$e^{u+iv} = re^{i\theta} \Rightarrow \begin{array}{l} e^{u} = r = |z| \\ e^{iv} = e^{i\theta} \end{array}$$

So,

$$\boxed{\begin{array}{l} u = \ln(r) = \ln|z| \\ v = \theta + 2\pi n, \quad n \in \mathbb{Z} \end{array}}$$

So, given $z = re^{i\theta} \neq 0$, $\boxed{\log(z) = \ln(r) + i(\theta + 2\pi n) \quad n \in \mathbb{Z}}$

multivalued function.

Then

$$e^{\log(z)} = z$$

Note $\quad \log(z) = \ln(|z|) + i(\underbrace{\theta}_{\arg(z)} + 2\pi n)$

Principal value of log...

Given $z = re^{i\Theta}$ where $|z| = r$, and $\Theta = \operatorname{Arg}(z) \in (-\pi, \pi]$, we defin

$$\boxed{\operatorname{Log}(z) = \ln(r) + i\,\Theta = \ln|z| + i\,\operatorname{Arg}(z)}$$

Ex For $z = x + i0 \in \mathbb{R}\setminus\{0\}$, $\operatorname{Log}(z) = \log(x)$

- If $x>0$, $\Theta = 0$, so $\log(x) = \ln|z| + i(\overset{\to 0}{\Theta} + 2\pi n)$ $n \in \mathbb{Z}$
  $= \ln x + (i\,2\pi n)$ $n \in \mathbb{Z}$

  there $\operatorname{Log}(x) = \ln(x)$

- If $x<0$, $z = x = |x|e^{-i\pi} = |x|e^{-3i\pi}$

  There, $\log(x) = \ln|x| + i(-\pi + 2\pi n)$ $n \in \mathbb{Z}$

  $= \ln(-x) + i(\pi + 2\pi n)$ $n \in \mathbb{Z}$.

  Also,
  $$\boxed{\operatorname{Log}(x) = \ln|x| + i\pi = \ln(-x) + i\pi}$$

Note $\operatorname{Log}(-1) = \ln(1) + i\pi = i\pi$

We had, for any $z \neq 0$, $e^{\log(z)} = z$.
Notice that

$$\log(e^z) = \log(e^x e^{iy}) = \ln|e^x| + i(y + 2\pi n) \quad n \in \mathbb{Z}$$

$$= (x + iy) + 2\pi i n, \quad n \in \mathbb{Z}$$

So $\boxed{\log(e^z) = z + (2\pi i)n, \ n \in \mathbb{Z}}$

For $z = x + iy$, where $-\pi < y \leq \pi$, we get that

$$\operatorname{Log}(e^z) = z$$

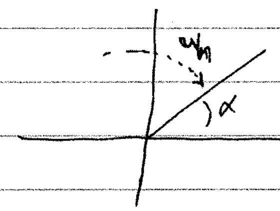
Branches & analyticity Given $\alpha \in \mathbb{R}$, define the α-branch of log by

$\log_\alpha(z) = \ln|z| + i\,\theta_\alpha$ where θ_α is the arg of $z \neq 0$, which lives between α and $\alpha + 2\pi$.

$\Rightarrow$ This fixes a single-valued function... $\mathrm{Log}_\alpha : \mathbb{C}\setminus\{0\} \longrightarrow \mathbb{C}$.

This function is NOT continuous... • Consider sequence.... $w_n = e^{i\alpha + i/n}$

Then $\log_\alpha(w_n) = i\left(\alpha + \frac{1}{n}\right) \to i\alpha$



• Next, consider... $w_n = e^{i(\alpha - \frac{1}{n})}$

Then $\log_\alpha(w_n) = i\left(\alpha - 1/n + 2\pi\right) \longrightarrow i(\alpha + 2\pi)$

These give two paths and different limits along those paths, so not continuous

However, if we restrict $\mathrm{Dom}(\log_\alpha) = \{z \neq 0 \mid \arg(z) \neq \alpha\}$, then $\log_\alpha$ on D is continuous and as you showed in thm, it is analytic

Sept 9, 2019

We saw $\mathrm{Log}(z) = \ln|z| + i\arg(z)$

Restriction to single valued log $\rightsquigarrow$ the α^{th}-branch of log

This is the function $\log_\alpha : \mathbb{C}\setminus\{0\} \longrightarrow \mathbb{C}$

defined by $\log_\alpha(z) = \ln|z| + i\theta_\alpha$ where θ_α = unique value of the argument of z which $\alpha < \theta_\alpha \leq \alpha + 2\pi$

Notice $\boxed{\log_\alpha(\mathbb{C}\setminus\{0\}) = \text{Range}(\log_\alpha) = \{(x,y) : x\in\mathbb{R},\ y\in(\alpha,\alpha+2\pi]\}}$

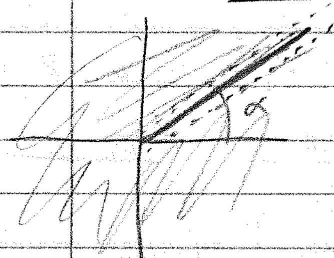
$\boxed{\text{Ex}}$

$\underline{\alpha=0}$, $\log_\alpha(1+i) = \ln|1+i| + i\theta_0 \qquad 0<\theta_0\le 2\pi$

$$= \ln\sqrt{2} + i\frac{\pi}{4}$$

$$\log_\alpha(1) = \ln(1) + i\theta_0 \qquad 0<\theta_0\le 2\pi$$
$$= \ln(1) + 2\pi i$$
$$= (2\pi i)$$

The <u>restriction</u> of $\log_\alpha$ to the domain $\mathcal{D}_\alpha = \{z=re^{i\theta} : r>0,\ \theta\neq\alpha\}$



$$= \{z\neq 0 \mid \arg(z)\neq\alpha\}.$$

On this D_α, $\log_\alpha$ is continuous and <u>in fact</u> analytic.
Here, by calculation in HW

$$\frac{d}{dz}\log_\alpha(z) = \frac{1}{z} \quad \forall z\in\mathcal{D}_\alpha$$

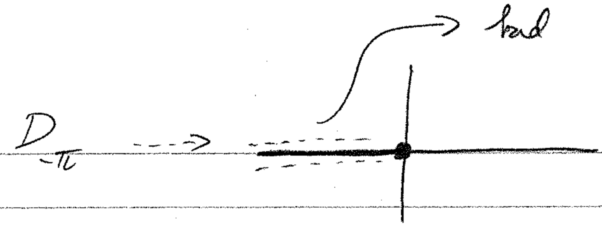
Note

> For general multi-valued functions, a <u>branch cut</u> is associated with a curve at which the restricted map fails to be analytic. A point shared by all branch cuts is a <u>branch point</u>.
>
> The "branch" of a function is generally that defined on $\mathbb{C}\setminus$ branch cut.

Note $\left\{\begin{array}{l}\text{Branch cut for } \log_\alpha \text{ is the ray of angle } \alpha. \\ \text{The branch point for the logarithm is } z=0.\end{array}\right\}$

Defn

> The principal branch of log is the function $\text{Log} = \log_{-\pi}$
>
> $\text{Log} : \mathbb{C} \longrightarrow \{x+iy,\ x\in\mathbb{R},\ -\pi<y\le\pi\}$. Restricting to
> $D = D_{-\pi} = \{re^{i\theta} \mid r>0,\ -\pi<\theta\le\pi\} = \mathbb{C}\setminus\{\mathbb{R}^{-}\}$
>
> non positive



We have for $\log$, $\log_\alpha$, $\operatorname{Log}(z)$

$$\boxed{e^{\log(z)} = z}$$

If $z = x + iy$ s.t $\alpha < y \le \alpha + 2\pi$ then

$$\boxed{\log_\alpha (e^z) = z}$$

$$\boxed{\log_\alpha : D_\alpha \longrightarrow \{x+iy : x \in \mathbb{R},\ \alpha < y < \alpha + 2\pi\} \text{ is a 2-sided inverse of } \exp.}$$

Warning Log properties (for complex log) don't work the way you expect.

Ex $\log_{-\pi}(i^3) = \log_{-\pi}(-i) = \log_{-\pi}\left(e^{-i\pi/2}\right) = 0 + -\frac{\pi}{2} i$

$3\log_{-\pi}(i) = 3 \cdot \frac{\pi}{2} i = \frac{3\pi}{2} i \ne -\frac{\pi}{2} i$

So

$$3\log_{-\pi}(i) \ne \log_{-\pi}(i^3)$$

It is true, however, that for multi-valued log, some properties work

Ex $\log(z_1 z_2) = \log(z_1) + \log(z_2) \quad \forall z_1, z_2 \ne 0$

COMPLEX POWERS

We want to define z^c when $z \neq 0$, $c \in \mathbb{C}$.

Motivate: Assume that $c = n \in \mathbb{Z}$, then

$$z^c = z^n = (re^{i\theta})^n = r^n e^{in\theta} = e^{n\ln(r)} e^{in\theta}$$

$$= e^{n(\ln r + i\theta)}$$

$$= e^{n \log z} = e^{c \log(z)}$$

And so... when $c \in \mathbb{Z}$, this is a single-valued function of z.

$$z^c = e^{c\log(z)}$$

Let's define for $c \in \mathbb{C}$, $z \neq 0$. $z^c = e^{c\log(z)} \to$ a multi-valued fn

For the α-branch of log ... $\log_\alpha$.

$$z_\alpha^c = e^{c\log_\alpha(z)} = e^{c\{\ln|z| + i\theta_\alpha\}} = e^{c\ln|z| + ic\theta_\alpha}$$

$\hookrightarrow$ this is single-valued of z.

Restricted to $\mathcal{D}_\alpha$, $z^c = z_\alpha^c$ is analytic on $\mathcal{D}_\alpha$ by the chain rule (and domains work) and

$$\frac{d}{dz} z^c = \frac{d}{dz}\left[e^{c\log_\alpha(z)}\right]$$

$$= e^{c\log_\alpha(z)} \cdot \frac{d}{dz}\left[c\log_\alpha(z)\right] \quad c/z$$

$$= c\,\frac{e^{c\log_\alpha z}}{e^{\log_\alpha z}} = ce^{(c-1)\log_\alpha(z)}$$

So $\boxed{\frac{d}{dz} z^c = cz^{c-1}}$

$$= cz^{c-1}$$

When, in this course, "P.V." is put in front of sth, it is meant to be that constructed by Log. P.V. = Principal Value

i.e. P.V. $\log(z) = \mathrm{Log}(z)$

$$\text{P.V. } z^c = \exp\{c\,\mathrm{Log}(z)\} = \exp\{c\log_{-\pi}(z)\}$$

Analytic in $D = \mathbb{C} \setminus \{\text{non positive reals}\}$

$\boxed{\text{Ex}}$

$$\text{P.V. } (1+i)^i = \exp\{(i)\,\mathrm{Log}(i+1)\}$$

$$= \exp\{(i)[\ln\sqrt{2} + i\tfrac{\pi}{4}]\}$$

$$= \exp\{-\tfrac{\pi}{4} + i\ln\sqrt{2}\}$$

$$= e^{-\pi/4}\{\cos(\ln\sqrt{2}) + i\sin(\ln\sqrt{2})\}$$

$$\text{P.V. } i^{1+i} = \exp\{(i+1)\,\mathrm{Log}(i)\}$$

$$= \exp\{(i+1)(+\tfrac{\pi}{2}i)\}$$

$$= \exp\{-\tfrac{\pi}{2} + i\tfrac{\pi}{2}\}$$

$$= (e^{-\pi/2})\{\cos\pi/2 + i\sin\pi/2\}$$

$$= ie^{-\pi/2}$$

Ex $i^c = \exp(c\,\mathrm{Log}(i))$

$$= \exp\left(c\tfrac{\pi}{2}i\right) = \exp\left((u+iv)\tfrac{\pi}{2}i\right)$$

$$= \exp^{-v\pi/2}\exp^{i(u\pi/2)}$$ → not imaginary generally.

$$\boxed{\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \quad ; \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}} \rightsquigarrow \text{ENTIRE.}$$

properties are as expected.

$$\frac{d}{dz} \sin(z) = \cos z \quad ; \quad \frac{d}{dz} \cos z = -\sin z$$

Oct 11, 2019

Consider an interval $I = [a, b]$. A function $z : I \mapsto \mathbb{C}$ is called continuous if

$z(t) = x(t) + iy(t)$ and $x(t), y(t)$ are continuous, real-valued fns on I. In this case, we call z continuous on I and write

$z \in C^0(I; \mathbb{C})$ = set of continuous functions from I to $\mathbb{C}$.

If $x(t), y(t)$ are differentiable on $[a, b]$ and $x'(t), y'(t)$ are continuous then, a member of $z \in C^0(I, \mathbb{C})$ is said to be once continuously differentiable on I and we write
$z \in C^1(I; \mathbb{C})$.

In this case, $z'(t) = x'(t) + iy'(t) \quad \forall \, t \in I$

Ex $\quad z(t) = e^{it}$. $\quad I = [0, 2\pi]$

$= \cos(t) + i\sin(t) \rightsquigarrow \cos\theta, \sin\theta$ are continuously diff' on I

$z'(t) = -\sin(t) + i\cos(t) = ie^{it}$

Remark: These functions $z : I \to \mathbb{C}$ are complex-valued functions of a real var. So diff'ty is not complex diff'ty.

Of course, $z \in C^0(I; \mathbb{C})$ traces a curve in $\mathbb{C}$. Such a curve, the set of points

$$\mathcal{C} = \{ z(t) = (x(t), y(t)) : t \in I \}$$

is a subset of the complex plane. This set is necessarily bounded whenever I is a bounded interval

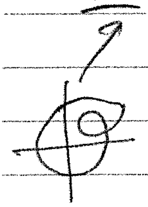
Given a set C, a function z, s.t. $\mathcal{C} = \{ z(t), t \in I \}$ is said to be a <u>parameterization</u> of the curve $\mathcal{C}$ (parametric representation)

<u>Warning</u> → Sets C can have multiple parameterization...

$$e^{it},\ e^{2\pi i t} \quad \text{parameterize the same set (unit circle)}$$

<u>So</u>, we shall generally think of curve $\mathcal{C}$ coming with parameterizations.

NO



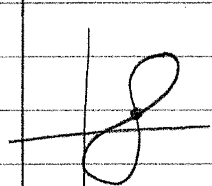
$\equiv$ curve

We say that $\mathcal{C}$ is an <u>arc</u>. We say it is <u>SIMPLE</u> if its param has

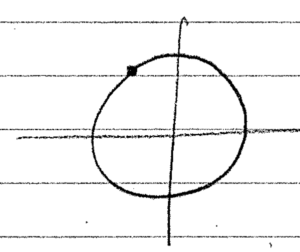
$$z(t_1) \neq z(t_2) \quad \forall\ t_1 \neq t_2$$

→ Curve does not self intersect

We say that $\mathcal{C}$ is a simple closed curve if $z(a) = z(b)$ yet $z(t_1) \neq z(t_2)\ \forall\ t_1 \neq t_2 \in (a, b)$



→ ~~simple~~ closed



simple, closed

→ <u>we can define orientation</u> { A simple closed curve is said to be positively oriented if z traces in CCW fashion }

Given a curve $\mathcal{C}$ with param $z \in C^1(I, \mathbb{C})$ (cont. diff'), we define the length of $\mathcal{C}$ to be

$$\boxed{L(\mathcal{C}) = \int_a^b |z'(t)|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\;dt}$$

Consider the unit circle ... $\mathcal{C} = S^1 \to$ simple closed curve w/ param.

$$z(t) = e^{it}, \quad t \in [0, 2\pi]$$

$$L = \int_0^{2\pi} |z'(t)|\,dt = \int_0^{2\pi} |ie^{it}|\,dt = 2\pi$$

Oct 14, 2019

<u>Summarize</u> Curves = sets + parameterization (+ associated direction)

Proposition The arc length $\mathcal{L}(\mathcal{C})$ is invariant under parameterization.

<u>Pf</u> Assume $z \in C^1([a,b])$ and $\tilde{z} \in C^1([\alpha,\beta])$ are both parameterization of curve $\mathcal{C}$. We shall also assume that these are injective maps and z' and $\tilde{z}'$ are nonzero everywhere...

~~once continuously diff bijection~~ $\phi(\alpha) = a, \ \phi(\beta) = b \ldots$

By these assumptions, $\exists\ \phi: [\alpha,\beta] \longmapsto [a,b]$ and that
$\tilde{z}(\tau) = z[\phi(\tau)] \quad \forall\ \tau \in [\alpha,\beta]$ and $\phi'(\tau) > 0$.

Letting $f(t) = \sqrt{x'(t)^2 + y'(t)^2}$, then noting that

$$\tilde{z} = z(\phi(\tau))$$

$$\tilde{z}' = \frac{d}{d\tau}\left[\tilde{x}(\tau) + i\tilde{y}(\tau)\right] = \frac{d}{d\tau}\left[x(\phi(\tau)) + iy(\phi(\tau))\right]$$

(chain rule) $\longrightarrow \quad = z'(\phi(\tau))\,\phi'(\tau)$

so $$\int_\alpha^\beta |\tilde{Z}'(\tau)|\,d\tau = \int_{\alpha}^\beta \sqrt{\tilde{X}'(t)^2 + \tilde{Y}'(t)^2}\,d\tau$$

$$= \int_\alpha^\beta \underbrace{|\phi'(\tau)|}_{\geq 0} \sqrt{X'(\phi(t))^2 + Y'^2(\phi(t))}\,d\tau$$

$$= \int_\alpha^\beta |Z(\phi(\tau))|\,\phi'(\tau)\,d\tau$$

change of vars $\rightsquigarrow$ $$= \int_{\phi(\alpha)}^{\phi(\beta)} |Z(t)|\,dt = \int_a^b |Z(t)|\,dt$$

So, ~~Flts~~ makes sense to write

$$\boxed{\mathcal{L}(C) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt}$$

Given a curve or arc C and a parametrization $Z \in C^1$, we say the curve is "smooth" if $Z'(t) \neq 0$

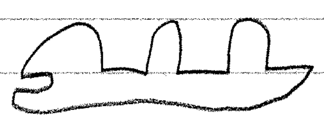
↳ alternatively... ~~a~~ $\boxed{\text{non-degenerate}}$ curve...

$\boxed{\text{Fact}}$

we can re-parametrize C by an arclength parameter

$\sqrt{x'^2 + y'^2} = 1$ in the case of a non-degenerate parametrization

A $\boxed{\text{CONTOUR}}$ is a path/curve C with parametrization $Z \in C^0([a,b],\mathbb{C})$ where Z is differentiable at all but finite number of points in $[a,b]$. Everywhere else it is continuously diff and non degenerate.

Ex 

smooth arcs pieced together...

Ex C = upper half of unit circle + line from -1 to 1

$$z(t) = \begin{cases} e^{i\pi t} & t \in [0,1] \\ t-2+0i & 1 \le t \le 3 \end{cases}$$

Note $z \in C^0([0,3])$

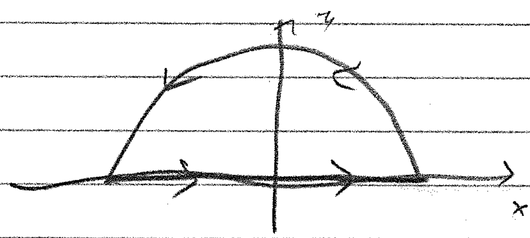
Note z fails to be diff @ $t=1$ but everywhere else...

$$z'(t) = \begin{cases} i\pi e^{i\pi t} & t \in [0,1) \\ 1 & t \in (1,3] \end{cases}$$

→ non degenerate
simple closed, not simple
CCW → (+) oriented

$\boxed{\text{JORDAN CURVE THEOREM}}$

The points on a simple closed contour are the boundary points of two domains, (1) a bounded region, called the interior, and (2) an unbounded region, called the exterior. These domains don't intersect.



$\mathbb{C}$-valued integrals... Given $z \in C^0([a,b], \mathbb{C})$

we define: $$\boxed{\int_a^b z(t)\,dt = \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt}$$

Ex $z(t) = (1+it)^2$

$$\int_0^1 z(t)\,dt = \int_0^1 (1-t^2)\,dt + i\int_0^1 2t\,dt = \dots = \frac{2}{3} + i$$

$\boxed{\text{FTC?}}$ Yes!

Proposition Suppose $w \in C^1([a,b], \mathbb{C})$ & $w'(t) = z(t)$ $\forall t$.

Then $$w(b) - w(a) = \int_a^b z(t)\,dt$$

MVT? given $z \in C^0([a,b], \mathbb{C})$, does $\exists\, c \in [a,b]$ s.t.

$$\int_a^b z(t)\,dt = z(c)(b-a) \;?$$

Nope! $z(t) = e^{2\pi i t}$, $t \in [0,1]$

$$\int_0^1 z(t)\,dt = \frac{1}{2\pi i} e^{2\pi i t}\Big|_0^1 = 0 \overset{?}{=} \underbrace{z(c)}_{e^{2\pi i c}}[1-0]$$

$\rightarrow$ no such c, since $e^{2\pi i c} \neq 0$ $\forall c$.

$\boxed{\text{MVT} \rightarrow \text{does not hold}}$.

CONTOUR INTEGRALS

D Supp C is a contour with param $z \in C^0([a,b], \mathbb{C})$, and $f : \mathcal{O} \mapsto \mathbb{C}$ where $C \subseteq \mathcal{O}$. ($\mathcal{O} \subseteq \mathbb{C}$)

We define the contour int of f along C (direction matters) is

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

(This makes sense because z' exists everywhere except a finite # pts, which don't contribute to integral...

Fact $\int_C f(z)\,dz$ is independent of parameterization (direction matters)
↳ same proof as before...

Ex **Contour** $z(t) = 2e^{it}$ $t \in [0, 2\pi]$
↳ circle of radius 2, CCW

① $f(z) = z \ldots$

$$\int_C f(z)\,dz = \int_0^{2\pi} 2e^{it}(2ie^{it})\,dt = \int_0^{2\pi} 4ie^{i2t}\,dt = \ldots = 0$$

② Find $\int_C \bar{z}\,dz$

Oct 16, 2014 **More examples**

Consider $\mathcal{C}: z(t) = 2e^{it}$ $0 \le t \le 2\pi$

$$I = \int_C \bar{z}\,dz = \int_0^{2\pi} 2e^{-it}\, 2 \cdot i e^{it}\,dt = 4i(2\pi) = 8\pi i$$

Note $\forall z \in \mathcal{C}$, $z\bar{z} = 2e^{it}\,2e^{-it} = 4$

So $\bar{z} = \frac{4}{z}$, so

$$\int_C \bar{z}\,dz = \int_C \frac{4}{z}\,dz \quad 4\int_C \frac{1}{z}\,dz = 8\pi i$$

$$\Rightarrow \boxed{2\pi i = \int_C \frac{1}{z}\,dz}$$

7/12-30

Okay... consider path $C: z(t) = Re^{it} \quad t \in [0, 2\pi],\ R>0$

$$\boxed{\int_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{R} e^{-it} Rie^{it}\,dt = 2\pi i}$$

<u>Some properties</u>. Spose C is a contour and f, g are piecewise continuous on C. Then, for any $z_0 \in C$

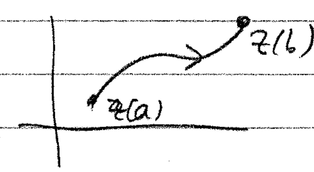
(1) $$\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz$$

(2) $$\int_C f(z) + g(z)\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$$

(3) <u>Reversing orientation</u>: Spose C is a contour with param

$z(t) = C^1([a,b]; \mathbb{C})$, then

Define $-C$ as that given by



$\tilde{Z}(t) = z(-t) \quad -b \le t \le -a \ldots$

So $$\int_{-C} f(z)\,dz = \int_{-b}^{-a} f(\tilde{z})\,\tilde{z}'(t)\,dt = \qquad s = -t$$

$$= \int_{-b}^{-a} f(z(-t))\, z'(-t)\,(-1)\,dt = -\int f(z)\,dz$$

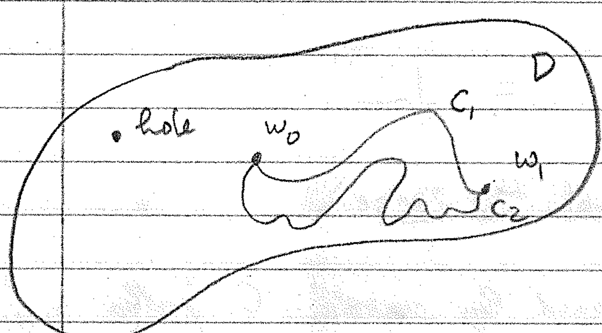
So $$\boxed{\int_{-C} f(z)\,dz = -\int_C f(z)\,dz}$$

Given two points w_0 and w_1 and any fn f which is piecewise continuous on an open set containing w_0, w_1. In what sense does

$$\int_C f(z)\,dz$$

depend on the path C (a contour from w_0 to w_1 staying inside the open set)?

What does this have to do with f?

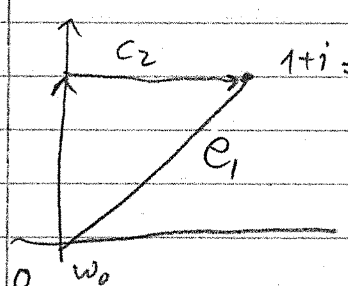


When is it the case that

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz \;?$$

__Example when NOT path independent__

$f(z) = y - x - 3ix^2$ and $w_0 = 0$, $w_1 = 1+i$



__Step 1__ parametrize C_1:

$$z_1(t) = (1+i)t \qquad t \in [0,1]$$
$$z_1'(t) = (1+i) \qquad =: (f_1 t)$$

So $$\int_{C_1} f(z)\,dz = \int_0^1 \left((t-t) - 3it^2\right)(1+i)\,dt = -i(i+1) = \boxed{-i(1+i)}$$

__Step 2__ parametrize C_2:

$$z_2(t) = \begin{cases} it & t \in [0,1] \\ (t-1)+i & t \in [1,2] \end{cases}$$

So $\displaystyle\int_{C_1} f(z)\,dz = \int_0^1 \cdots + \int_1^2 \cdots$

$$= \int_0^1 t\,(i)\,dt + \int_1^2 \left(1-(t-1)-3i(t-1)^2\right)\cdot 1\,dt$$

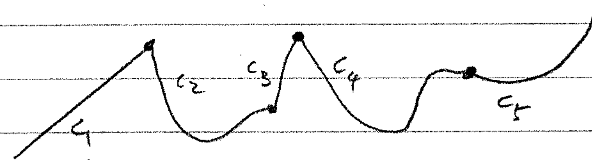
$$= \frac{i}{2} + \int_1^2 \left[1-(t-1)-3i(t-1)^2\right]dt$$

$$= \frac{i}{2} + 2 + \int_1^2 -t\,dt - 3i\int_1^2 (t-1)^2\,dt$$

$$= \cdots = \boxed{\frac{1-i}{2}}$$

We note that they are not the same.

Let C be the contour formed by ~~smooth~~ C' paths, $C_1, C_2, \ldots C_n$



Then the following is true:

$$\boxed{\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz}$$

Let C be a C' path from $w_0 \to w_1$ with param $z \in C'([a,b],\mathbb{C})$

Claim $\boxed{\displaystyle\int_C z\,dz = \frac{w_1^2 - w_0^2}{2}}$ ⇝ path independence for the special fn...

Pf (?)

Pf $\displaystyle\int_C z\,dz = \int_a^b z(t)\,z'(t)\,dt = \,?$

Since $z \in C^1$, $\displaystyle\frac{d}{dt}\left[\frac{(z(t))^2}{2}\right] = z(t)\,z'(t) \quad \forall\, t \in (a,b)$

So $\displaystyle\int_C z\,dz = \int_a^b \frac{d}{dt}\left(\frac{z(t)^2}{2}\right)dt = \frac{z(b)^2 - z(a)^2}{2} = \frac{w_1^2 - w_0^2}{2}$

FTC.

Note We assumed C^1 path. But what about contours?

Prop

$$\boxed{\text{If } C \text{ contour from } w_0 \text{ to } w_1, \text{ then } \int_C z\,dz = \frac{w_1^2 - w_0^2}{2}}$$

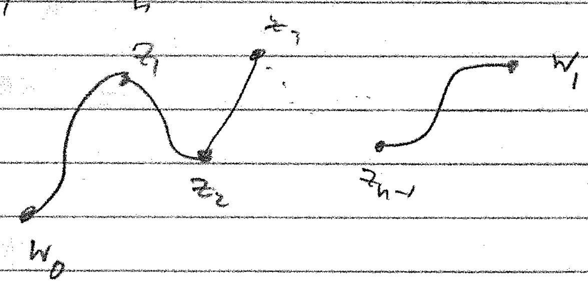
Pf Decompose C to C^1 paths $C_1 \dots C_n$

$C_1 \quad w_0 \to z_1$

$C_2 \quad z_1 \to z_2$

$\vdots$

$C_n \quad z_{n-1} \to w_1$



Then $\displaystyle\int_C z\,dz = \sum_{k=1}^{n} \int_{C_k} z\,dz = \cancel{\sum_{k=1}^{n}} \frac{-w_0^2 + z_1^2}{2} + \frac{-z_1^2 + z_2^2}{2} + \dots + \frac{-z_{n-1}^2 + w_1^2}{2}$

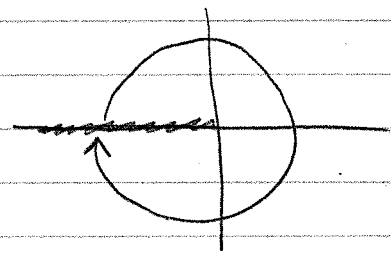
So $\displaystyle\int_C z\,dz = \frac{w_1^2 - w_0^2}{2}.$

$\square$

Oct 18, 2019

CONTOUR INTEGRALS – BRANCH CUTS

It's not a problem to integrate in which a contour includes a branch cut. Suppose we want to integrate a function involve Log on $|z|=1$, start at -1, close curve to -1



Ex $\displaystyle\int_{C_R} z^{a-1}\,dz$, $a\in\mathbb{R}$, $C_R=\{|z|=R\}$

$z(t)=Re^{it}$ $\quad t\in[-\pi,\pi]$

$$\int_{C_1} z^{a-1}dz = \int_{C_R}\exp\{(a-1)\operatorname{Log} z\}\,dz$$

$$= \int_{-\pi}^{\pi}\exp\{(a-1)\operatorname{Log}(Re^{it})\}\, iRe^{it}\,dt$$

$$= iR\int_{-\pi}^{\pi}\exp\{(a-1)\{\ln R + it\}\}\,e^{it}\,dt$$

$$= iR\int_{-\pi}^{\pi} e^{(a-1)\ln R}\cdot e^{i(a-1)t}\,e^{it}\,dt$$

$$= iR\int_{-\pi}^{\pi}(R^{a-1})\,e^{iat}\,dt$$

$$= iR^{a}\int_{-\pi}^{\pi} e^{iat}\,dt$$

$$= \begin{cases} iR^{a}\,\dfrac{e^{iat}}{ia}\Big|_{-\pi}^{\pi} & a\neq 0 \\ iR^{a}\,t\Big|_{-\pi}^{\pi} & a=0 \end{cases} = \begin{cases} \dfrac{R^{a}}{a}\left(e^{ia\pi}-e^{-ia\pi}\right) & a\neq 0 \\ (2\pi i) \end{cases}$$

$$= \begin{cases} \frac{R^a}{a}(2i)\sin(a\pi) & a \neq 0 \\ 2\pi i & a = 0 \end{cases}$$

So
$$\begin{cases} 0 \text{ when } & a \neq 0,\ a \in \mathbb{Z} \\ 2\pi i \text{ when } & a = 0 \\ \frac{2i R^a}{a}\sin(a\pi) & a \in \mathbb{R}\setminus\mathbb{Z}. \end{cases}$$

Note
$$\int_{C_R} \frac{1}{z}\,dz = \int_{C_R} z^{0-1}\,dz = 2\pi i$$

$$\boxed{\text{MODULI} \;\, z \;\, \text{CONTOUR}} \quad (\text{estimating})$$

Lemma: Let $w \in C^0([a,b], \mathbb{C})$, then

then $$\left|\int_a^b w(t)\,dt\right| \leq \int_a^b |w(t)|\,dt$$

(triangle inequality...)

Proof Let $r_0 = \left|\int_a^b w\,dt\right|$. If $r_0 = 0$, the statement is obvious (int of non negative fn is non negative)

Suppose $r_0 > 0$. In this case, $\exists\, \theta_0 \in \mathbb{R}$ s.t

$$\int_a^b w(t)\,dt = r_0 e^{i\theta_0}. \text{ So } r_0 = e^{-i\theta_0}\int_a^b w(t)\,dt = \int_a^b w(t) e^{-i\theta_0}\,dt$$
$$= \mathrm{Re}\left(\int_a^b e^{-i\theta_0} w(t)\,dt\right) = \int_a^b \mathrm{Re}\left(e^{-i\theta_0} w(t)\right) dt \quad \overset{\Uparrow}{\mathbb{R}}$$

But $\mathrm{Re}(e^{-i\theta}w(t)) \le \left|\mathrm{Re}(e^{-i\theta_0}w(t))\right|$

$$\le \left|e^{-i\theta_0}w(t)\right| = |w(t)| \quad \forall t \in [a,b]$$

So $\left|\int_a^b w(t)\,dt\right| = r_0 \le \int_a^b |w(t)|\,dt \ldots$ $\square$

Thm Let C be a contour, and let $f : \mathrm{Dom}(f) \to \mathbb{C}$ be piecewise continuous on C. If
$|f(z)| \le M \quad \forall z \in C$ then

$$\left|\int_C f(z)\,dz\right| \le M\,\ell(C)$$ length of C

Pf from lemma, this follows.

Let $z : [a,b] \mapsto \mathbb{C}$ be param, then

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))\, z'(t)\,dt\right|$$

$$\le \int_a^b |f(z(t))\, z'(t)|\,dt \quad \rightsquigarrow \text{lemma}$$

$$\le \int_a^b |f(z(t))|\,|z'(t)|\,dt$$

$$\le M \underbrace{\int_a^b |z'(t)|\,dt}$$

$$= M\,\ell(C)$$

$|e^z| \leq ?$

$|e^{e^{it}}| \leq ?$

(67)

Ex Use lemma to ~~say~~ estimate $e^{(\cos t + i\sin t)}$ $e^{\cos t}\, e^{i\sin t}$

$$\int_C e^z\,dz \qquad C: z(t) = 1 + e^{it} \qquad t \in [0, \pi]$$

$$\int_C e^z\,dz = \int_0^{\pi} e^{(1+e^{it})}\, i e^{it}\,dt$$

$$= \int_0^{\pi} e^1 \left(e^{\sin t}\right) e^{i\cos t}\; i e^{it}\,dt$$

$$= \int_0^{\pi} e\left(e^{\cos t}\right)\left\{\cos(\sin t) + i\sin(\cos t)\right\} i e^{it}\,dt$$

$$|e^{1+e^{it}}| \leq \cancel{\sqrt{\mathrm{Re}^2 + \mathrm{Im}^2}} = |e^{1+\cos t}| \leq e^2.$$

$$\text{So} \quad \left|\int_C e^z\,dz\right| \leq e^2(\pi) \approx 23.2$$

Note
$$\int_C e^z\,dz = \int_0^{\pi} e^{z(t)} z'(t)\,dt = e^{z(t)}\Big|_0^{\pi} = e^0 - e^2 = 1 - e^2$$

$$\text{So} \left|\int_C e^z dz\right| \leq e^2 - 1 \leq e^2 \cdot \pi$$

INDEPENDENCE OF PATH

Thm Suppose f is continuous on a domain D, TFAE

1. $\exists$ diff' fn (analytic fn) F on D st $F' = f$ (existence of antiderivative)
2. Given any 2 points $z_1, z_2 \in D$ and any contour $C \subseteq D$ going from $z_1 \to z_2$,
$$\int_C f(z)\,dz = F(z_2) - F(z_1)$$
(independence of path)

(3) Given any simple closed contour $\mathcal{C} \subseteq D$,

$$\int_{\mathcal{C}} f(z)\,dz = 0.$$

Oct 23, 2019

Thm

Let f be continuous on domain D. TFAE

(1) $f(z)$ has an antiderivative $F(z)$ throughout D.

(2) Given any $z_1, z_2 \in D$ and contours $C_1, C_2 \subseteq D$ both going from z_1 to z_2

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

In other words, the integral is independent of contour

(3) Given any close contour $\mathcal{C} \subseteq D$

$$\int_{\mathcal{C}} f(z)\,dz = 0$$

In the case that one (and hence every) condition is satisfied, we have:

For any $z_1, z_2 \in D$ and contour C from $z_1 \to z_2$ (all $\in D$)

$$\int_C f(z)\,dz = F(z_2) - F(z_1)$$

where F's existence is guaranteed by (1)

Ex Consider $f(z) = z$, $D = \mathbb{C}$. We saw that for any $z_1, z_2 \in \mathbb{C}$ and contour C from z_1 to z_2

$$\int_C f(z)\,dz = \frac{z_2^2}{2} - \frac{z_1^2}{2}$$

So property (2) is true.

(Of course, easy to see $F(z) = \frac{z^2}{2}$ is anti-deri of $f(z) = z$ throughout D, so (1) is true)

<u>Ex</u> $f(z) = \frac{1}{z}$. Taking this to be continuous on $D = \mathbb{C} \setminus \{0\}$,

observe that the unit contour $C : z(t) = e^{it}$, $t \in [0, 2\pi]$ gives

$$\int_C \frac{1}{z}\, dz = 2\pi i .$$

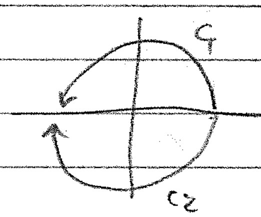
Here C = unit circle contour is a simple closed curve lies entirely in $D = \mathbb{C} \setminus \{0\}$.

$\hookrightarrow$ (3) is not true $\Leftrightarrow$ (1) & (2) not true.

Based on this, $f(z) = \frac{1}{z}$ has no antiderivative defined on all of $\mathbb{C} \setminus \{0\}$.

Also, $\exists\ z_1, z_2 \neq 0$ and two paths $C_1, C_2 \subseteq \mathbb{C} \setminus \{0\}$ going from z_1 to z_2 such that

$$\int_{C_1} \frac{1}{z}\, dz \neq \int_{C_2} \frac{1}{z}\, dz$$



$C_1 - C_2$ = unit contour $= C$

<u>So</u> $$\int_C \frac{1}{z}\, dz = 2\pi i = \int_{C_1 - C_2} \frac{1}{z}\, dz = \int_{C_1} \frac{1}{z}\, dz + \int_{-C_2} \frac{1}{z}\, dz$$

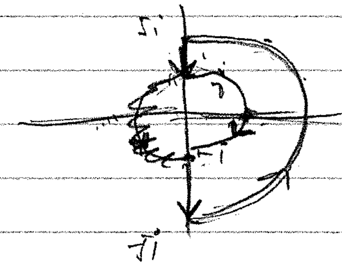
$$\boxed{2\pi i = \int_{C_1} \frac{1}{z}\, dz - \int_{C_2} \frac{1}{z}\, dz}$$

<u>Ex</u> Consider $f(z) = \frac{1}{z}$ with $D = \mathbb{C} \setminus \{x, 0 : x \leq 0\}$

Notice that $F(z) = \mathrm{Log}(z) = \log_{-\pi}(z)$ is an antiderivative on D. So we should expect (theorem) that all integrals over closed contours in D are <u>zero</u>.

$C : C_1 + C_2 + C_3 + C_4$

$C_1 : z_1(t) = 5e^{it} \quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$



$C_2 : z_2(t) = 0 + i(5-t) \quad t \in [0, 4]$

$C_3 : z_3(t) = e^{-it} \quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$C_4 : z_4(t) = 0 + i(t+1) \quad t \in [0, 4]$

$\ln 1 + (i\theta_{-\pi})$

Not allowed

$\int_{C_1} \frac{1}{z} dz = \text{Log}(z_f) - \text{Log}(z_i) = \text{Log}\left(\frac{z_f}{z_i}\right) = \text{Log}(-1) = +i\pi$

$\int_{C_2} \frac{1}{z} dz = \text{Log}\left(\frac{z_f}{z_i}\right) = \text{Log}\left(\frac{i}{5i}\right) = \text{Log}\left(\frac{1}{5}\right) = -\ln 5 + \cancel{i\theta} \quad \theta_{-\pi} = 0$

$\int_{C_3} \frac{1}{z} dz = \text{Log}\left(\frac{-i}{i}\right) = \cancel{\text{Log}(-1)} = -i\pi$

$\text{Log}(-i) - \text{Log}(i)$

$\int_{C_4} \frac{1}{z} dz = \text{Log}\left(\left(\frac{1}{5}\right)^{-1}\right) = \ln 5 + \cancel{i\theta}\ 0$

$\int_C \frac{1}{z} dz = (\ln 5 - \ln 5) = 0.$

Do this explicitly... use $\int_C f(z) dz = \int_{t_i}^{t_f} f(z(t))\, z'(t)\, dt$

Also, note, this C is path from $z_1 = 5$ to $z_2 = 5$

By thm $\int_C \frac{1}{z} dz = 0.$

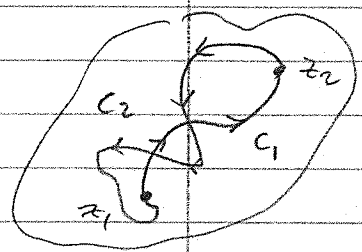
$\boxed{\text{Ex}}$

$f(z) = \frac{1}{z^2}$, $D = \mathbb{C} \setminus \{0\}$. We note that $F(z) = \frac{-1}{z}$ is an antiderivative of $f(z) = \frac{1}{z^2}$ valid throughout $D = \mathbb{C} \setminus \{0\}$.

$\Rightarrow$ By thm, any closed path $C \subseteq \mathbb{C}\setminus\{0\}$ has $\int_C \frac{1}{z}\,dz = 0$.

$\boxed{\text{Pf of Thm}}$

$\boxed{2 \Leftrightarrow 3}$ Suppose (2) is valid & let C be a closed curve (contour) in D. Then C contains 2 points z_1, z_2 and we can divide C into 2 pieces $C_1 + C_2$ where C_1 goes $z_1 \to z_2$ and C_2 goes from $z_2 \to z_1$.



Now, by reversing the direction of C_2, $-C_2$ goes from $z_1 \to z_2$. So, C_1 & $-C_2$ both go from $z_1 \Rightarrow z_2$ & stay inside D. Thus,

$$\int_C f\,dz = \int_{C_1+C_2} f\,dz = \int_{C_1} f\,dz + \int_{C_2} f\,dz$$

$$= \int_{C_1} f\,dz - \int_{-C_2} f\,dz$$

By (2), $\int_{C_1} f\,dz = \int_{C_2} f\,dz \Rightarrow \int_C f\,dz = 0$. $\Rightarrow$ $\boxed{(2) \Rightarrow (3)}$

Pf $\boxed{(3) \Rightarrow (2)}$ Let z_0, z_1 be in D, Let $C_1, C_2 \subset D$ be contours going from z_0 to z_1.

Observe that $C := C_1 - C_2$ is a closed contour in D. So, by property 3

$$0 = \int_C f = \int_{C_1 - C_2} f = \int_{C_1} f + \int_{-C_2} f = \int_{C_1} f - \int_{C_2} f.$$

Show $\boxed{(1) \Leftrightarrow (2)}$

$(1 \to 2)$ Let z_0, z_1 be in D and still let C be a contour from $z_0 \to z_1$.
So, $C: z(t) \in C^1([a,b], \mathbb{C})$ piecewise diff', $z(a) = z_0$ & $z(b) = z_1$

As F is an antiderivative of f, for all $t \in [a,b]$ for which $z'(t)$ exists the chain rule gives $\frac{d}{dt} F(z(t)) = F'(z(t))\, z'(t) = f(z(t))\, z'(t)$

So $\displaystyle\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{a_k}^{b_k} f(z(t))\, z'(t)\,dt$ where a_k, b_k are points at which z fails to be diff!

$$a = a_1 < b_1 = a_2 < b_2 \ldots b$$

$$= \sum_{k=1}^{n} \int_{a_k}^{b_k} \frac{d}{dt} F(z(t))\,dt$$

$$\overset{FTC}{=} \sum_{k=1}^{n} F(z(b_k)) - F(z(a_k)) = F(z(\underbrace{b_n}_{b})) - F(z(\underbrace{a_1}_{a}))$$

$$= F(z_1) - F(z_0)$$

So, given any 2 contours $C_1, C_2 \in D$ from z_0 to z_1

$$\int_{C_1} f = F(z_1) - F(z_0) = \int_{C_2} f \qquad (1 \to 2) \checkmark$$

$(2 \to 1)$ We need to construct an antiderivative F. Let $z_0 \in D$ and define $F : D \mapsto \mathbb{C}$ by

$$F(z) = \int_{C_z} f(w)\,dw \quad \text{where } C_z \text{ is a contour from } z_0 \to z_1.$$

<u>Note</u> Since D is a domain, it is path connected, and so for each z, a path C_z exists. By (2), this is not dependent on the choice of contour C_z.

i.e. F is well-defined.

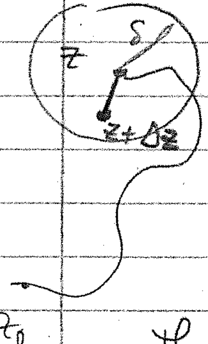
<u>To show</u> $F(z)$ diff' and its derivative is f.

Let $z \in D$ and choose $\varepsilon > 0$. Given the continuity of f, let δ be chosen so that

(1) $\qquad \left| f(w) - f(z) \right| < \frac{\varepsilon}{2} \quad \forall \quad |w - z| < \delta$

(2) $\qquad B_\delta(z) \subseteq D \qquad$ (D is open)

Given a $\Delta z \in \mathbb{C}$ s.t $\Delta z < \delta$, consider path $C_{z,\Delta z}$ defined by
$w(t) = z + t\Delta z \quad , \quad t \in [0,1]$



Note $\quad C_z$ (fixed from $z_0 \to z$) is a path from $z_0 \to z$
and so,

$C_z + C_{z,\Delta z}$ is a contour in D from $z_0 \to z + \Delta z$.

Then,

$$\frac{1}{\Delta z}\Big(F(z+\Delta z) - F(z)\Big)$$

$$= \frac{1}{\Delta z}\left(\int_{C_z + C_{z+\Delta z}} f(w)\,dw - \int_{C_z} f(w)\,dw\right)$$

$$= \frac{1}{\Delta z}\int_{C_{z+\Delta z}} f(w)\,dw = \frac{1}{\Delta z}\int_0^1 f(z+t\Delta z)\cdot(z+t\Delta z)'\,dt$$

$$= \frac{1}{\cancel{\Delta z}}\int_0^1 f(z+t\Delta z)\cdot \cancel{\Delta z}\,dt$$

$$= \int_0^1 f(z+t\Delta z)\,dt = \int_0^1 f(z+t\Delta z)\,dt$$

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So, for $|\Delta z| < \delta$, $\quad \left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right|$

$$= \left|\int_0^1 f(z+t\Delta z)\,dt - f(z)\right|$$

$$= \left|\int_0^1 f(z+t\Delta z) - f(z)\,dt\right|$$

by Lemma, $\left| \int_0^1 (f(z+t\Delta z) - f(z))\,dt \right|$

$$\leq \int_0^1 \left| f(z+t\Delta z) - f(z) \right| dt$$

$$\leq \int_0^1 (\varepsilon/2)\,dt \quad \text{by choice of } \delta$$

$$\leq \varepsilon/2 < \varepsilon.$$

We have shown: given $z \in D$ and $\varepsilon > 0$, $\exists \delta > 0$ for which

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| < \varepsilon \quad \text{whenever } |\Delta z| < \delta$$

So, F is diff' at z and $F'(z) = f(z)$ $\blacksquare$

CAUCHY - GOURSAT THEOREM

Suppose that C is a simple closed contour & f is <u>analytic</u> on the interior of C and all points of C, then

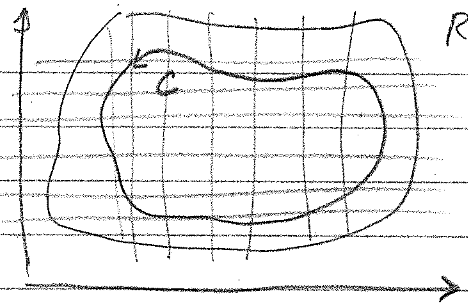
$$\int_C f(z)\,dz = 0$$

Cauchy proved this assuming f' continuous → easy (vector calc)
Goursat removed this condition.

<u>Pf</u> Lemma

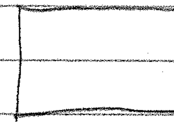
Let f be analytic on a region R containing C and its interior. For every $\varepsilon > 0$, C + C's interior can be covered by a finite number of squares and partial squares σ_j, $j = 1, 2, \ldots, n$ and, in each σ_j $\exists$ z_j for which

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \varepsilon \quad \forall\, z_j \in \sigma_j \setminus \{z_j\}$$

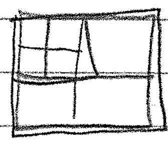


__Sketch of proof__

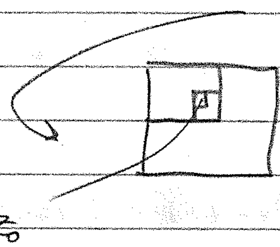
Suppose it cannot be done, i.e. $\exists$ a bad square



$\rightarrow$ __cut__



At each level, we produce a bad square $\sigma^{(j)}$ for which the conclusion is false. This gives a sequence of compact squares $\sigma^{(j)}$ which are nested..



__Cantor intersection thm__

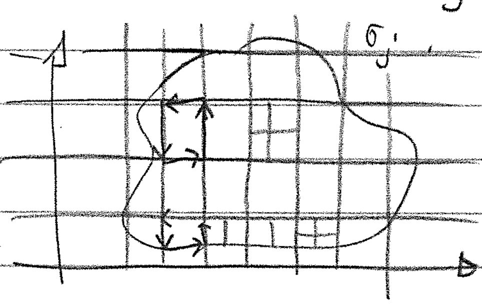
$\bigcap_{j=1}^{\infty} \sigma^{(j)} \neq \phi$. Let z_0 be such an element $\in \sigma^{(j)} \; \forall j$.

By construction, $\exists \varepsilon > 0$ s.t.

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| \geq \varepsilon \quad \forall \; z \neq z_0, \; z \in \sigma^{(j)} \; \forall j = 1 \ldots$$

Since the collection of $\sigma^{(j)}$ contracts around z_0, this shows that f is not diff' @ z_0. $\Rightarrow$ contradiction. $\square$

__Pf of thm__ Let $\varepsilon > 0$, split the region of $C \cup$ its interior into n squares $\cup$ partial squares for which the conclusion of the lemma holds. Let these squares be denoted by $\sigma^{(j)}$, associated points z_j, side length s_j, and C_j the ~~positively~~ positively oriented boundary of σ_j.



Assuming C is positively oriented...

$$\int_C f(z)\,dz = \sum_{j=1}^{n} \int_{C_j} f(z)\,dz$$ (cancellations between adjacent σ_j)

On each σ_j, define $\delta_j : \sigma_j \to \mathbb{C}$ defined by

$$\delta(j)(z) = \begin{cases} 0 & z = z_j \\ \dfrac{f(z) - f(z_j)}{z - z_j} - f'(z_j) & z \neq z_j \end{cases}$$

Note

(1) $|\delta_j(z_j)| < \varepsilon \quad \forall z \in \sigma_j$.

(2) $\displaystyle\lim_{z \to z_j} \delta_j(z) = \lim_{z \to z_j} \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) = 0$

So

$\delta_j(z)$ is continuous $\forall j = 1, 2, \ldots, n$.

So, on σ_j,

$$f(z) = f(z_j) + f'(z_j)(z - z_j) + \delta_j(z)(z - z_j)$$

$$= f(z_j) + f'(z_j)(z) - f'(z_j) z_j + \delta_j(z)(z - z_j)$$

$\forall z \in \sigma_j$

So, for each $j = 1, 2, \ldots, n$

$$\int_{C_j} f(z)\,dz = \int_{C_j} \underbrace{\left(f(z_j) - f'(z_j) z_j\right)}_{\text{constant}} dz + \int_{C_j} \underbrace{f'(z_j)}_{\text{constant}} z\,dz + \int_{C_j} \delta_j(z)(z - z_j)\,dz$$

$$\left(\int_{C_j} dz = 0, \int_{C_j} z\,dz = 0\right)$$

$$= 0 + 0 + \int_{C_j} \delta_j(z)(z - z_j)\,dz$$

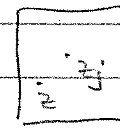
So $$\int_{C_j} f(z)\,dz = \int_{C_j} \delta_j(z)(z - z_j)\,dz$$

Oct 30, 2019

.... $$\int_C f(z)\,dz = \sum_{j=1}^{n} \int_{C_j} \delta_j(z)(z - z_j)\,dz$$ $C_j = \partial\sigma_j$, ccw

By Δ-inequality,

$$\left| \int_C f(z)\,dz \right| \leq \sum_{j=1}^{n} \left| \int_{C_j} \delta_j(z)(z-z_j)\,dz \right|$$



$|z - z_j| \leq \sqrt{2}\, S_j$

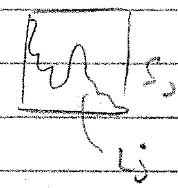
$\forall z \in \delta_j$

$|\delta_j(z)| < \varepsilon$

$$\leq \sum_{j=1}^{n} \int_{C_j} |\delta_j(z)| \cdot |z - z_j|\,dz$$

$$< \sum_{j=1}^{n} \int_{C_j} \varepsilon \cdot \sqrt{2}\, S_j\,dz$$

$$= \sum_{j=1}^{n} (\varepsilon\sqrt{2})\, \mathcal{L}(C_j)$$



If σ_j square, then $\mathcal{L}(C_j) = 4S_j$.

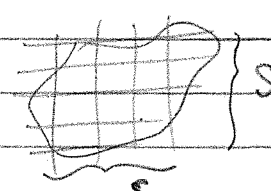
If σ_j is a partial square $\mathcal{L}(C_j) \leq 4S_j + L_j \rightsquigarrow$ portion of C_j in σ_j

So

$$\left| \int_C f(z)\,dz \right| < \sum_{j=1}^{n} \varepsilon(\sqrt{2})(S_j)\left[4S_j + L_j \right]$$ has zero if σ_j is a partial square.

$$< \sum_{j=1}^{n} \varepsilon\left(4\sqrt{2}\, S_j^2 + \sqrt{2}\, S_j L_j\right)$$

$$< \sum_{j=1}^{n} \varepsilon\sqrt{2}\left(4S_j^2 + S_j\, C\right)$$ $\mathcal{L}(C) > L_j$



$$= \varepsilon\sqrt{2}\left(4S^2 + S\mathcal{L}(C)\right)$$

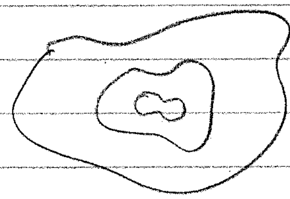
We have shown that $\forall \varepsilon > 0$, $\left| \int_C f(z)\,dz \right| < \varepsilon\sqrt{2}\left(4S^2 + S\mathcal{L}(C)\right)$

So $\int_C f(z)\,dz = 0$. (Cauchy-Goursat)

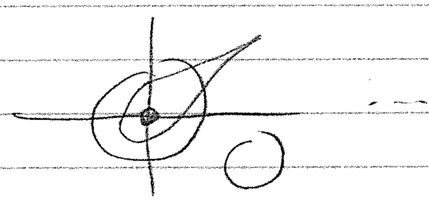
$\boxtimes$

Defn A domain D is called simply connected if every simple closed contour $C \subseteq D$ contains only points of D in its interior.

i.e. Every simple closed contour is contractable to a point



Defn A multiply-connected domain D is a domain which is not simply connected

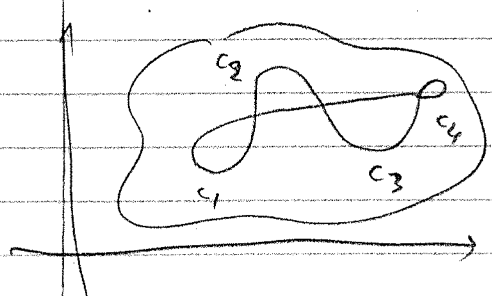


Thm Cauchy - Goursat for simply-connected domain

Let D be a simply connected domain. f analytic in D. $\forall$ closed contour $C \subset D$,

$$\int_C f(z)\,dz = 0$$

Pf-ish Let C be a closed contour in D with finite number of self intersections ...



Given that C only has n intersections
$\rightarrow$ can split C into finite number m of simple closed curves $C_j = 1 \ldots m$

Also, given D is simple connected, the C_j and interior of each C_j lies in D.

By previous thm, $\int_{C_j} f(z)\,dz = 0 \;\forall j = 1, \ldots, m \Rightarrow \int_C f(z)\,dz = \int_{\Sigma C_j} f(z)\,dz = 0$

Corollary If f is analytic on a simply connected domain D then f has an antiderivative F everywhere D.

(TFAE)

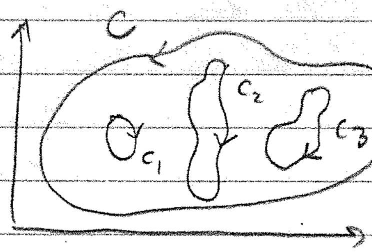
Multiply-connected Regions

Thm Suppose (a) C is a simple closed curve (ccw)

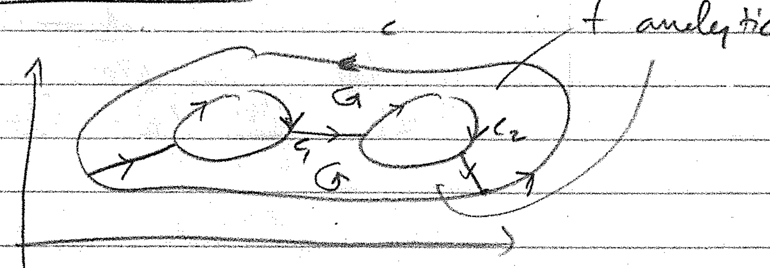
(b) C_j, $j = 1, 2, \ldots, n$ are simple closed curves disjoint and all lie in the interior to C (cw)

If f is analytic on C, C_j $\forall j$ and the region between C, C_j (in C and outside C_j) then

$$\int_C f(z)\,dz + \sum_{j=1}^{n} \int_{C_j} f(z)\,dz = 0$$



Pf by picture

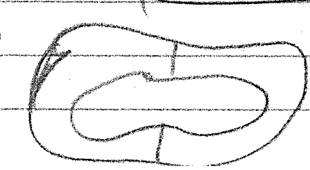


$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz = 0 \to \int_C f(z)\,dz + \sum_{j=1}^{n} \int_{C_j} f(z)\,dz = 0.$$

Corollary Let C_1 & C_2 be simple closed curves. C_1 lies in the interior of C_2. Both oriented ccw.
If f is analytic on the region between C_1, C_2 then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Pf



done by thm $\int_{C_2} f\,dz + \int_{-C_1} f\,dz = 0 \Rightarrow \int_{C_2} f = \int_{C_1} f$

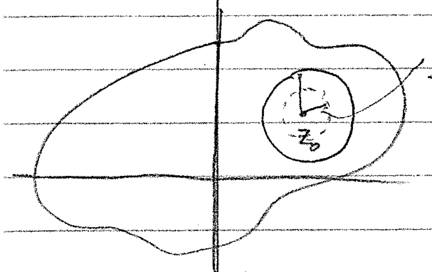
Nov 1, 2019

Cauchy's Integral Formula

Thm Let C be a (+) or. simple closed contour and let f be analytic on C and its interior. If z_0 lies interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$$

Pf z_0 to be interior to C. Let $\delta < 1$ be small enough so that $|z-z_0| < \delta$ places $z \in \mathrm{Int}(C)$



Since the quotient $\frac{f(z)}{z-z_0}$ is analytic on the region exterior to $B_\delta(z_0)$ and interior to C

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_\rho} \frac{f(z)}{z-z_0}\,dz$$ where $\rho < \delta$, C_ρ is the (+) or. circle @ z_0, radius ρ.

$$\varepsilon = \left(\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz - f(z_0) \right)$$

$$= \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{z-z_0}\,dz - \frac{f(z_0)}{2\pi i}\int_{C_\rho} \frac{1}{z-z_0}\,dz$$

$$\varepsilon = \frac{1}{2\pi i}\left\{ \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right\}$$

Given that $f(z)$ is continuous @ z_0, $\forall \varepsilon' > 0\ \exists \rho^{\varepsilon} > 0$ s.t. $|f(z)-f(z_0)| < \varepsilon$ whenever $|(z-z_0)| < \cancel{\delta}\ \varepsilon_\rho < \delta$

~~δ ...~~ Since $|z-z_0| = \rho < \varepsilon_\rho$ on C_ρ,

$$\left| \frac{f(z)-f(z_0)}{z-z_0} \right| = \frac{1}{\rho}|f(z)-f(z_0)| < \frac{\varepsilon}{\rho} \text{ on } C_\rho.$$

So $\quad |E| \le \frac{1}{|2\pi i|} \frac{\varepsilon}{\rho} \mathcal{L}(C_\rho) = \frac{1}{2\pi} \frac{\varepsilon}{\rho}(2\pi\rho) = \varepsilon.$

So given any $\varepsilon > 0$, $|E| \le \varepsilon$. This says

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz = f(z_0)$$

∎

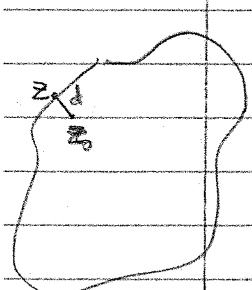
Thm

CAUCHY'S INTEGRAL FORMULA FOR DERIVATIVES

Thm: Let C be (+) or. simple closed contour - let ~~z_0 lie interior~~ f be analytic on the interior of C and on C

If $z_0 \in \text{Int}\, C$ then $\quad f'(z_0) = \frac{1}{2\pi i}\int \frac{f(z)}{(z-z_0)^2}\,dz$

Pf Let $M = \max|f(z)|$, $z \in C$. Given $z_0 \in \text{Int}(C)$, let



$d = \min_{z \in C} |z - z_0| > 0$. Suppose that $h = \Delta z$ is such that

$|h| = |\Delta z| < d$.

using Cauchy's Int formula, $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$

B/c $|h| < d$, $z_0 + h \in \text{Int}(C)$

So $\quad f(z_0+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-(z_0+h)}\,dz.$

Observe that $\quad E = \frac{f(z_0+h) - f(z_0)}{h} - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz.$

$$= \frac{1}{2\pi i}\frac{1}{h}\int_C \frac{f(z)}{z-(z_0+h)} - \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$$

$$= \frac{1}{2\pi i}\frac{1}{h}\int_C f(z)\left(\frac{1}{z-(z_0+h)} - \frac{1}{z-z_0}\right)dz - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$$

$$= \frac{1}{2\pi i}\frac{1}{\cancel{h}}\int_C f(z)\cdot\frac{\cancel{h}}{(z-(z_0+h))(z-z_0)}\,dz - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$$

$$= \frac{1}{2\pi i}\int_C f(z)\left\{\frac{1}{(z-(z_0+h))(z-z_0)} - \frac{1}{(z-z_0)^2}\right\}dz.$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\cdot\frac{h}{z-(z_0+h)}\,dz$$

for all $z \in \mathrm{Int}(C)$, $d \leq |z-z_0|$, so $\frac{1}{|z-z_0|^2} \leq d^2$.

Also, $d < |z-z_0| = |z-z_0-h+h|$

$\leq |z-(z_0+h)| + |h|$

~~$\leq |z-(z_0+h)|+\delta$~~

So $0 < \cancel{|z-z_0-h|} \leq |z-z_0-h|$

$d-|h|$

continued

Jun 4, 2019

Thm

$$\boxed{\begin{array}{l}\text{Let } C \text{ be (+) simple closed curve \& } f \text{ analytic on } C \text{ and interior of } C. \text{ If } z_0 \text{ is interior to } C \text{ then}\\ \displaystyle f'(z_0) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz\end{array}}$$

Pf

Let $\mathcal{O}$ be the interior of C. $M = \max_{z\in C}|f(z)|$, $d = \min_{z \in C}|z-z_0| > 0$ and we showed that for

$|h| < d$, $z_0 + h \in \mathcal{O}$

Defining

$$E(h) = \frac{f(z_0+h)-f(z_0)}{h} - \frac{1}{2\pi i}\int \frac{f(z)}{(z-z_0)^2}\,dz.$$

For all $|h|<d$, we showed

$$E(h) = \frac{1}{2\pi i}\int_C \frac{f(z)\,h}{(z-z_0)^2\,(z-(z_0+h))}\,dz$$

For $z \in C$, $|z-z_0|^2 \geq d^2 \Leftrightarrow \dfrac{1}{|z-z_0|^2} \leq \dfrac{1}{d^2}$

Also, $0 \leq d-|h| \leq |z-z_0-h| \quad \forall\ |h|<d$.

So $\forall z \in C$, any $|h|<d$,

$$\left|\frac{f(z)h}{(z-z_0)^2(z-(z_0+h))}\right| \leq \frac{|f(z)|\,|h|}{|z-z_0|^2\,|z-(z_0+h)|^2} \leq \frac{M|h|}{d^2\cdot(d-|h|)}$$

So any $|h|<d$, $\quad |E(h)| \leq \left|\dfrac{1}{2\pi i}\right| \dfrac{M|h|}{d^2(d-|h|)}\cdot L(C)$

$$= \frac{M|h|L(C)}{2\pi\, d^2(d-|h|)}$$

Let $\varepsilon>0$ be given and choose

$$\delta = \min\left\{\frac{d}{2},\ \frac{\pi d^3}{M L(C)}\varepsilon\right\}.$$ Then for $|h|<\delta \leq \frac{d}{2} < d$,

$$\frac{1}{d-|h|} \leq \frac{1}{d/2}$$

So $|E(h)| \leq \dfrac{M L(C)|h|}{2\pi d^2\, d/2}$

$$= \frac{M L(C)|h|}{\pi d^3} < \frac{M L(C)}{\pi d^3}\cdot\frac{\pi d^3\varepsilon}{M L(C)} = \varepsilon.$$

So $\boxed{\displaystyle\lim_{h\to 0}\frac{f(z_0+h)+f(z_0)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz}$

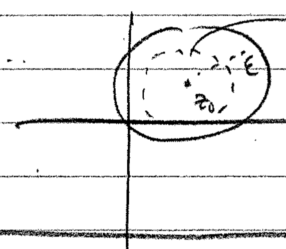
Thm Let C be (+) or. simple closed contour, f analytic on C & its interior. Then $\forall$ z_0 interior to C, ~~deriv~~ and $n \in \mathbb{N}$, f is n-times diff'able @ n at z_0 and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

<u>Consequences</u>

Thm If f is analytic @ z_0 then f has derivatives of all orders which are also analytic at z_0.

Pf by picture ...



$\to C_{\varepsilon/2} \to$ apply preceding Thm to $C = C_{\varepsilon/2}$ ✓

Corollary If D is a domain & f analytic on D then f has analytic of all orders and each deriv is analytic on D.

$\to$ f is infinitely diff'able on D.

Thm Let $f(z) = u(x,y) + iv(x,y)$ be analytic at $z_0 = (x_0, \ldots$ then

U and V have continuous partial derivatives of all orders at z_0. Further, if $f = u + iv$ is analytic on D, then u, v are ∞-diff in D, i.e.

$$u, v \in C^\infty(D)$$

Pf Cauchy-Riemann.

Thm (Hörmander's Thm) If u is harmonic in a domain D then u is smooth $\Leftrightarrow$ $u \in C^\infty(D)$

Pf By Sec 104, u has harmonic conjugate $v \to f = u + iv \ldots$ everything follows ...

Morera's Thm (converse to simply connected version of Cauchy-Goursat)

> Let f be cont. on D. If $\forall$ simple closed curves $C \subseteq D$,
> $$\int_C f(z)\,dz = 0,$$
> then
> f is analytic on D.

Pf By TFAE, f has F throughout D. But F analytic because $F' = f \Rightarrow F$'s derivative is analytic throughout D as well $\rightarrow f$ analytic throughout D. ☑

CAUCHY'S INEQUALITY

> Let f be analytic on and inside (+) circle C_R w/ center z_0. Let $M_R = \max_{z \in C_R} |f(z)|$, then $\forall\, n \in \mathbb{N}$
> $$|f^{(n)}(z_0)| \le \frac{n!\, M_R}{R^n}$$

Pf
$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \right|$$

$$\le \frac{n!}{\cancel{2\pi}} \frac{M_R}{R^{n+1}} \cdot \underbrace{(\cancel{2\pi} R)}_{L(C_R)}$$

$$= \frac{n!\, M_R}{R^n}. \checkmark$$

Nov 8, 2011

Theorem   Liouville's Thm

$\boxed{\text{If } f \text{ is bounded + entire, } f \text{ is constant}}$

Let $M \geq 0$ for which $|f(z)| \leq M \ \forall z \in \mathbb{C}$. Given any $z_0 \in \mathbb{C}$, f is analytic on every neighborhood of z_0, and so, $\forall R > 0$

$$|f'(z_0)| \leq \frac{1!M_R}{R} \quad \text{where} \quad M_R = \max_{z \in C_R(z_0)} |f(z)| \leq M.$$

So, for any $z_0 \in \mathbb{C}$, $R > 0$,

$|f'(z_0)| \leq \frac{M}{R}$. This shows $f'(z_0) = 0 \ \forall z_0 \in \mathbb{C}$. So, f is constant b/c $\mathbb{C}$ is a domain.

$\square$

Thm: $\boxed{\text{The fundamental thm of algebra}}$

If $P(z)$ is a non constant polynomial, i.e.

$$P(z) = a_0 + a_1 z^1 + \dots + a_n z^n, \quad a_n \neq 0, n = \deg(P)$$

then $\exists z_0 \in \mathbb{C}$ at which $P(z_0) = 0$

Pf $w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \dots + \frac{a_{n-1}}{z}$ and note that

$$\boxed{P(z) = (w + a_n) z^n}$$

We observe that z^k for $k \in \{1, 2, 3, \dots\}$ has

$$\frac{1}{z^k} \to 0 \text{ as } z \to \infty.$$

So, given $\varepsilon = \frac{|a_n|}{2}$, $\exists R > 0$ for which

$$|w| < \frac{|a_n|}{2} \quad \forall |z| > R$$

So, for $|z| > R$, $|w + a_n| \geq \big||w| - |a_n|\big| = |a_n| - |w| \geq \frac{|a_n|}{2}$

So $\left|\frac{1}{P(z)}\right| = \frac{1}{|w+a_n||z^n|} \leq \frac{2}{|a_n|}\frac{1}{|z^n|} \leq \frac{2}{|a_n|}\frac{1}{R^n}$ when $|z| > R$

Suppose that $P(z) \neq 0$ $\forall z \in \mathbb{C}$. Since $P(z)$ never vanishes,
$f(z) = \frac{1}{P(z)}$ is entire.

Since, in particular, $\frac{1}{P(z)}$ continuous, it is ~~continuous~~ bounded on all closed & bounded set

So $\exists M > 0$ s.t. $\left|\frac{1}{P(z)}\right| \leq M$ $\forall z$ s.t. $|z| \leq R$.

By what we've just shown, we have

$$\left|\frac{1}{P(z)}\right| \leq \text{Max}\left\{M, \frac{2}{|a_n|R^n}\right\} \rightarrow \text{bounded \& entire.}$$

Liouville thm $\Rightarrow$ $\frac{1}{P(z)}$ is constant, (contradiction) $\square$

Corollary If $P(z)$ has degree n, $\exists$ $c \in \mathbb{C}$ and $z_1, z_2, \ldots, z_n \in \mathbb{C}$ s.t.

$$P(z) = c(z - z_1)\cdots(z - z_n)$$

MAXIMUM MODULUS PRINCIPLE

Idea: An analytic function on a region $R = D \cup \partial D$ where D is a domain (open set)

Then $f(z)$ is maximized on ∂D.

Lemma Suppose that an analytic fn f has $|f(z)|$ maximized @ z_0 in some nbhd $B_\varepsilon(z_0)$ for some $\varepsilon > 0$. Then $f(z)$ is constant on $B_\varepsilon(z_0)$

Pf Take $0<\rho<\varepsilon$ and by invoking Cauchy's Int formula

$$f(z_0) = \int_{C_\rho} \frac{f(z)}{z-z_0}\,dz \cdot \frac{1}{2\pi i}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+\rho e^{it})}{(z_0+\rho e^{it}-z_0)}\, i\rho e^{it}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{it})\,dt$$

So,

$$|f(z_0)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0+\rho e^{it})\,dt\right|$$

maximized

$$\le \frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(z_0+\rho e^{it})|}_{\le |f(z_0)|}\,dt$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,dt = |f(z_0)|$$

So,
$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{it})|\,dt$$

So
$$0 = \frac{1}{2\pi}\int_0^{2\pi}\Big(\underbrace{|f(z_0)| - |f(z_0+\rho e^{it})|}_{\ge 0}\Big)\,dt$$

So
$$|f(z_0)| = |f(z_0+\rho e^{it})| \quad \forall\, t\in[0,2\pi]$$
$$\forall\, \rho<\varepsilon.$$

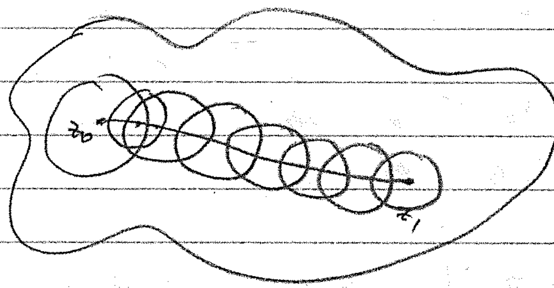
So, since this is true $\forall\, \rho<\varepsilon$, $|f(z_0)| = |f(z)|$
$\forall\, z\in B_\varepsilon(z_0)$

$\square$

Thm: Maximum modulus

Let f be analytic & non-constant on a domain D (open connected), then $|f(z)|$ cannot be maximized in D

Pf (Suppose it is ~~[illegible]~~ maximized at $z_0 \in D$. Let $z_1 \in D$ be arbitrary



Done!

POWER SERIES

Nov 11, 2019

Taylor Series ... Laurent Series

Defn Consider a sequence $\{z_n\} = (z_0, z_1, z_2, \dots)$ of complex write $\{z_n\} \subseteq \mathbb{C}$. We say that the sequence **converges** if $\exists\, z \in \mathbb{C}$ for which the following holds

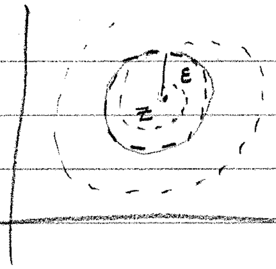
$\forall\, \varepsilon > 0$, $\exists\, N = N_\varepsilon \in \mathbb{N}$ s.t.

s.t.
$$|z_n - z| < \varepsilon \quad \forall\, n \geq N$$

In this case, we also say $\{z_n\}$ converges to z and call z the limit of the sequence.

$$\lim z_n \equiv \lim_{n \to \infty} z_n = z$$

Picture



Thm: Let $z_n = x_n + iy_n$ be a sequence, then $z_n \to z = x+iy$ $\iff$ $x_n \to x$, $y_n \to y$ in the sense of real numbers

Thm Cauchy: A sequence $\{z_n\}$ is called a Cauchy sequence if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t.

$$|z_n - z_m| < \varepsilon \quad \forall n, m \geq N$$

Thm: A sequence $\{z_n\}$ is convergent iff it's Cauchy

Series

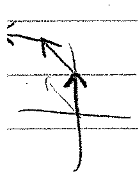
Consider a sequence $\{z_n\}_{n=0}^{\infty}$ and the series formed with the sequential elements as its terms

$$\sum_{k=0}^{\infty} z_k = z_0 + z_1 + z_2 + \dots$$

where, a priori, we don't assume they add to anything...

Given a series $\sum_{k=0}^{\infty} z_k = z_0 + z_1 + \dots$ for each $N \in \mathbb{N}$, define the N^{th}-partial sum as

$$S_N = \sum_{k=0}^{N} z_k$$



Defn: The series $\sum_{k=0}^{\infty} z_k$ converges if $\{S_N\}$ is a convergent sequence, i.e.

$$S = \lim_{N\to\infty} S_N \text{ exists}$$

In this case, we call S the sum of the series and write

$$\sum_{k=0}^{\infty} z_k = S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{k=0}^{N} z_k$$

Now, this has meaning as a complex number.

We also say $\sum z_k$ converges to S.

Thm $\boxed{\text{Given } z_n = x_n + iy_n, \text{ then } \sum z_n \text{ converges to } x+iy \iff \sum x_n \to x \text{ and } \sum y_n \to y}$

Thm $\boxed{\text{If } \sum z_n \text{ converges, then } \lim_{n\to\infty} z_n = 0 \quad \text{(converse false)}}$

Pf Let $\varepsilon > 0$. Given that $\sum z_n$ converges, $\{S_N\}$ converges so, $\{S_N\}$ is Cauchy, so $\exists M \in \mathbb{N}$ s.t.

s.t. $|S_n - S_m| < \varepsilon \quad \forall n, m \geq M$

Setting $N = M+1$, for $n \geq m+1$

$$|z_n - 0| = |z_n| = \left| \sum_{k=0}^{n} z_k - \sum_{k=0}^{n-1} z_k \right|$$

$$= |S_n - S_{n-1}| < \varepsilon \text{ b/c } n-1 \geq M$$

$\boxed{Ex}$ ⓐ $\sum_{k=0}^{\infty} e^{ik\theta}$ for any fixed θ — no (contra (+) of thm)

ⓑ $\sum_{k=0}^{\infty} \frac{e^{ik\theta}}{2^k}$. Try $S_N - S_M = \cancel{\frac{e^{in\theta}}{2^n} - \frac{e^{im\theta}}{2^m}}$

$$= \left| \frac{e^{i(n+1)\theta}}{2^{N+1}} + \frac{e^{i(n+2)\theta}}{2^{n+2}} + \cdots + \frac{e^{iM\theta}}{2^M} \right| \leq \frac{1}{2^{N+1}} + \frac{1}{2^{N+2}} + \cdots \frac{1}{2^M}$$

$$= \frac{1}{2^N}\left(\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^{M-N}}\right) \le \frac{1}{2^N}\cdot 1 = \frac{1}{2^N} < \varepsilon$$

provided N is sufficiently large.

Pf Let $\varepsilon > 0$, choose N_1 s.t. $\left(\frac{1}{2}\right)^{N} < \varepsilon$. Then for $n, m \ge N_1$, $|s_n - s_m| < \varepsilon$ □

A series $\sum z_k$ is said to be absolutely convergent if $\sum |z_k|$ is convergent as a series of real, non-neg real no

Thm If $\sum z_n$ is absolutely convergent then $\sum z_n$ is convergent

Sketch of pf
$$|s_N - s_M| = \left|\sum_{k=N+1}^{M} z_k\right| \overset{\Delta\text{-ineq}}{\le} \sum_{k=N+1}^{M} |z_k|$$

with this ineq, the Cauchiness of $\sum |z_k|$ implies the Cauchiness of $\sum z_k$ □

Nov 13, 2019

Suppose we're investigating the convergence of a series

$$\sum_{k=0}^{\infty} z_k.$$

If S is to be the sum of the series, we define the remainder

$$\rho_N = S - s_N = S - \sum_{n=0}^{N} z_n.$$

Thm the series $\sum_{n=0}^{\infty} z_n$ conv. to S $\Leftrightarrow$ $\lim_{N\to\infty} \rho_N = 0$

This is useful if you have a candidate S in mind — somehow a way to usefully express S_N.

$\boxed{Ex}$ (Geometric series). Let $z \in \mathbb{C}$, $z \neq 1$.

Then $\sum_{n=0}^{N} z^n = S_N$ for the "geometric series" $\sum_{n=0}^{\infty} z^n$.

Observe

$$(1-z)S_N = (1-z)(1+z+z^2+\cdots+z^N) = 1-z^{N+1}$$

So $\boxed{S_N = \dfrac{1-z^{N+1}}{1-z}}$

Thm: For any $z \in \mathbb{C}$ s.t. $|z|<1$, $\sum_{n=0}^{\infty} z^n$ converges and its sum is $\dfrac{1}{1-z}$

Pf For each $N \in \mathbb{N}$, $\rho_N = \dfrac{1}{1-z} - \sum_{n=0}^{N} z^n = \dfrac{1}{1-z} - \dfrac{1-z^{N+1}}{1-z}$

So $\rho_N = \dfrac{z^{N+1}}{1-z}$.

Since $|z|<1$, $\lim_{N\to\infty} z^{N+1} = 0$, So $\lim_{N\to\infty} \rho_N = \lim_{N\to\infty} \dfrac{z^{N+1}}{1-z} = 0$.

So, by previous thm that $\sum_{n=0}^{\infty} z^n = \dfrac{1}{1-z}$. ∎

Taylor's Thm Let $f(z)$ be analytic in a disk $B_{R_0}(z_0)$ with center z_0 and radius R_0. Then, for any $z \in B_{R_0}(z_0)$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

Remark: (1) In particular, the series $\sum_{n=0}^{\infty} \dfrac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ converges

[In fact, for each $r<R_0$, the series converges uniformly on $B_r(z_0)$]

(2) The sum is f.

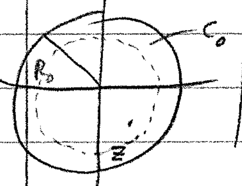
(3) For real functions $h:\mathbb{R}\to\mathbb{R}$. If h is diff'able on an open set containing x_0, it might not be twice differentiable, i.e. $h^{(2)}(x)$ might not exist...

(3b) For infinitely differentiable function $h\in C^{\infty}(\mathbb{R})$. Now the series makes sense, but it is here the case that h is representable by Taylor series...

$$\text{E.g.}\quad h(x)=\begin{cases} e^{-1/x^2} & x\neq 0\\ 0 & x=0\end{cases}$$

↳ infinite diff' but $\neq$ its Maclaurin series...

Pf. Without loss of generality, assume $z_0=0$ & consider $B_{R_0}(0)$ on which f is analytic... Let $z\in B_{R_0}(0)$. Let $|z|<R_0$ & define the positively oriented circle centered @ 0 & radius R_0, C_0.



Since $z\in \mathrm{Int}(C_0)$, Cauchy says $f(z)=\frac{1}{2\pi i}\int_{C_0}\frac{f(w)}{w-z}\,dw$.

Since $w\neq 0$, $$\frac{1}{w-z}=\frac{1}{w}\cdot\frac{1}{1-z/w}$$

Note $$\frac{1}{1-z/w}=\frac{1-(z/w)^{N+1}}{1-z/w}+\frac{(z/w)^{N+1}}{1-z/w}=\sum_{n=0}^{N}\left(\frac{z}{w}\right)^n+\frac{(z/w)^{N+1}}{1-z/w}$$

So $$\frac{1}{w-z}=\sum_{n=0}^{N}\frac{z^n}{w^{n+1}}+\frac{(z/w)^{N+1}}{w-z}$$

By Cauchy's deriv formula...

$$f^{(n)}(0)=\frac{n!}{2\pi i}\int_{C_0}\frac{f(w)}{(w-0)^{n+1}}\,dw.\ \text{So,}\ a_n=\frac{f^{(n)}(0)}{n!}=\frac{1}{2\pi i}\int_{C_0}\frac{f(w)}{w^{n+1}}\,dw$$

So, let
$$\rho_N=f(z)-\sum_{n=0}^{N}a_nz^n=\frac{1}{2\pi i}\int_{C_0}\frac{f(w)}{w-z}\,dw-\sum_{n=0}^{N}\frac{1}{2\pi i}\int_{C_0}\frac{f(w)}{w^{n+1}}\cdot z^n\,dw$$

$$= \frac{1}{2\pi i}\int_{C_0} f(w)\Big\{\underbrace{\frac{1}{w-z} - \sum_{n=0}^{N}\frac{z^n}{w^{n+1}}}\Big\}\,dw$$

$$= \frac{1}{2\pi i}\int_{C_0} f(w)\cdot\frac{(z/w)^{N+1}}{w-z}\,dw$$

Let $d = \min_{z\in C_0}|w-z|$. Then $|\rho_N| = \frac{1}{2\pi}\left|\int_{C_0}\frac{(z/w)^{N+1}}{w-z}f(w)\,dw\right|$

$M = \max_{z\in B_{R_0}(0)}|f(z)|$

$$\le \frac{1}{2\pi}\,\frac{|z/w|^{N+1}}{d}\,M\,\underbrace{\ell(C_0)}_{2\pi r_0}$$

$$= \frac{1}{\cancel{2\pi}}\,\frac{|z/w|^{N+1}}{d}\,M\,(\cancel{2\pi}r_0)$$

We have shown, given $z\in B_{R_0}(0)$ $\exists\ |z|<r_0<R_0$ for which

$$|\rho_N| \le M\left(|z|^{N+1}/r_0^{N+1}\right)\cdot\frac{r_0}{d} = \ldots \le \left(\frac{M|z|}{d}\right)\left(\frac{|z|}{r_0}\right)^N \quad \forall N\in\mathbb{N}$$

Since we've chosen $|z|<r_0<R_0$, $\frac{|z|}{r_0}<1$ and >0,

Given $\varepsilon>0$, $\exists N_0\in\mathbb{N}$ for which $\forall N\ge N_0$, $\left(\frac{|z|}{r_0}\right)^N < \frac{\varepsilon d}{M|z|}$

So, $\forall N\ge N_0$, $|\rho_N| \le \frac{M|z|}{d}\left(\frac{|z|}{r_0}\right)^N < \varepsilon$.

So $f(z) = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\sum_{n=0}^{N} a_n z^n$.

So

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n$$

$\boxed{\text{FE}}$

n-gillion group

cyclique group

Nov 15, 2019

Ex Consider $f(z) = e^z = e^x \cos y + i e^x \sin y \qquad (z = x+iy)$

We've shown that e^z is entire, so it is analytic on every nbh about every point. In particular, it is analytic on $B_R(0)$ $\forall R>0$.

Thus $e^z = \sum_{n=0}^{\infty} a_n (z-0)^n = \sum_{n=0}^{\infty} a_n z^n$

where $a_n = \left(\frac{d}{dz^n} e^z \Big|_{z=0}\right) \Big/ n! = \frac{1}{n!}$

So, $\boxed{e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \quad \forall z}$

~~Application...~~ $\forall x \in \mathbb{R}$, $e^x = e^{x+i\cdot 0} = \sum_{n=0}^{\infty} \frac{(x+i\cdot 0)^n}{n!} = \sum_{n=0}^{\infty} x^n/n!$

convergence of the real counterpart...

Ex e^z is analytic on $B_R(1+0\cdot i)$ $\forall R>0$

Expand around $z_0=1$, ... $e^z = \sum_{n=0}^{\infty} a_n (z-1)^n$

$$= \sum_{n=0}^{\infty} \frac{d^n/dz^n\, e^z|_{z=1}}{n!} = \sum_{n=0}^{\infty} \frac{e}{n!}$$

So, $e^z = \sum_{n=0}^{\infty} \frac{e(z-1)^n}{n!} = e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$

$e^{z-1} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$

Ex $f(z) = z^7 e^z$ is analytic on all $B_R(z_0)$, $R>0$,

$z^7 e^z = f(z) = \sum_{n=0}^{\infty} a_n (z-0)^n = \sum_{n=0}^{\infty} a_n z^n$

$a_n = \frac{d^n}{dz^n} \frac{(z^7 e^z)}{n!}\Big|_{z=0}$ ~ bad...

$$z^3 e^z = z^3 \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{n+3}}{n!} = \sum_{n=3}^{\infty} \frac{z^n}{(n-3)!}$$

Note $\quad f^{(10)}(0) = \frac{d^{10}}{dz^{10}}(z^3 e^z)\Big|_{z=0} = 10!\, a_{10} = \frac{10!}{(10-3)!} = 10\cdot 9\cdot 8$

$$= a_n \cdot n!$$

Ex $\quad f(z) = \frac{1+2z^2}{z^3+z^5} = \frac{1}{z^3} \underbrace{\frac{1+2z^2}{1+z^2}}_{g(z)}$

$$g(z) = 1 + \frac{z^2}{1+z^2} \Rightarrow f(z) = \frac{1}{z^3}\left(1+\frac{z^2}{1+z^2}\right) = \frac{1}{z^3} + \frac{1}{1+z^2}\cdot\frac{1}{z}$$

Note $\quad \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-z^2)^n \quad \forall\ |z|<1$

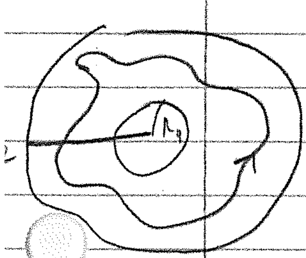
$$= \sum_{n=0}^{\infty} (-1)^n z^{2n} \ldots$$

Taylor series for $z \neq 0$, $|z|<1$,

$$f(z) = \frac{1}{z^3} + \frac{1}{z}\sum_{n=0}^{\infty} (-1)^n z^{2n} = \frac{1}{z^3} + \sum_{n=0}^{\infty} (-1)^n z^{2n-1}$$

$$= \frac{1}{z^3} + \frac{1}{z} - z + z^3 - z^5 + \ldots \leftarrow \boxed{\text{LAURENT SERIES}}$$

valid for $0<|z|<1$.



Thm (Laurent Series) Suppose that f is analytic on the region $R_1 < |z-z_0| < R_2$ where $R_1 \geq 0$ and let C be a simple closed curve, (+) oriented in this annular region. Then at each z such that $R_1 < |z-z_0| < R_2$, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

where $a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$, $\quad b_n = \frac{1}{2\pi i}\int \frac{f(z)}{(z-z_0)^{-n+1}}\,dz$

Quiz

Nov 18, 2019

Let z_n convergent, then $\lim_{n\to\infty} z_n = z$.

Proof 1 Then, for $\varepsilon = 1$, $\exists\, n_0$ s.t. $\forall\, n \geq n_0$, $|z_n - z| < 1$. Then for $n \geq n_0$,

$$\begin{aligned} |z_n| &= |z_n - (z - z)| \\ &\leq |z_n - z| + |z| \\ &\leq 1 + |z|. \end{aligned}$$

Let $M = \max\{1 + |z|, |z_1|, \ldots, |z_{n_0 - 1}|\}^{\to 0}$ then

Then $|z_n| \leq M \quad \forall\, n \in \mathbb{N}$. $\square$

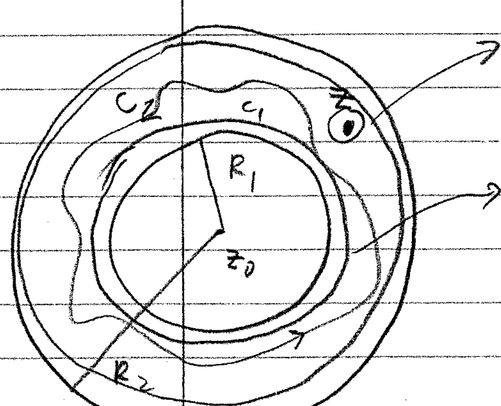
So, in all cases, $|z_n| \leq M$.

Proof 2 $\to$ use Calc 1, write $z = x + iy$, $\begin{cases} x_n \to x \\ y_n \to y \end{cases} \Rightarrow \ldots$

$$\to \quad \begin{aligned} |z_n| &= |x_n + i y_n| \\ &\leq |x_n| + |y_n| \\ &\leq M_1 + M_2 = M \end{aligned}$$

$\boxed{\text{Pf of Laurent's Thm}}$ W.L.O.G. assume $z_0 = 0$.

Picture



$C_2 : \{z \in \mathbb{C}, |z| = r_2 < R_2\}$

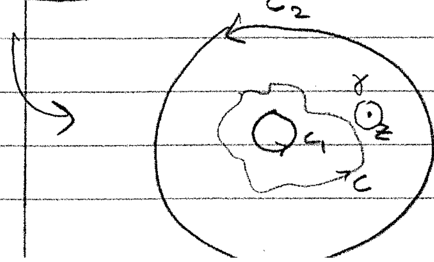
so that c_2 encloses z, C ...

$C_1 : \{z \in \mathbb{C}, |z| = r_1 > R_1\}$

Assume int C_2 contains C_1, z, C

int C_2 contains C_1

ext C_1 contains z, C



$\gamma \to$ curve around z, s.c.c $(+)$, interior to C_2, exterior to C_1

An appeal to Cauchy-Goursat for multiply connected domains shows that

$$\int_{C_2} \frac{f(s)}{s-z}\,ds - \int_{C_1} \frac{f(s)}{s-z}\,ds - \int_{\gamma} \frac{f(s)}{s-z}\,ds = 0$$

By CIF, $f(z) = \dfrac{1}{2\pi i}\displaystyle\int_{\gamma} \frac{f(s)}{s-z}\,ds$

$$= \frac{1}{2\pi i}\left\{ \int_{C_2} \frac{f(s)}{s-z}\,ds - \int_{C_1} \frac{f(s)}{s-z}\,ds \right\}$$

$$= \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\,ds + \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{z-s}\,ds$$

Note for 1st integral, $\dfrac{1}{s-z} = \dfrac{1}{s}\cdot\dfrac{1}{1-z/s}$

$$= \frac{1}{s}\left\{ \sum_{n=0}^{N-1} \left(\frac{z}{s}\right)^n + \frac{(z/s)^N}{1-z/s} \right\}$$

$$= \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + \left(\frac{z}{s}\right)^N \cdot \frac{1}{s-z}$$

2nd integral, $\dfrac{1}{z-s} = \dfrac{1}{z}\left(\dfrac{1}{1-s/z}\right) = \cdots$ $\quad s \leftrightarrow z$

$$= \sum_{n=0}^{N-1} \frac{s^n}{z^{n+1}} + \frac{1}{z-s}\left(\frac{s}{z}\right)^N$$

$$= \sum_{n=1}^{N} \frac{s^{n-1}}{z^n} + \frac{1}{z-s}\left(\frac{s}{z}\right)^N$$

$$= \sum_{n=1}^{N} \frac{1}{s^{-n+1}} z^{-n} + \frac{1}{z-s}\left(\frac{s}{z}\right)^N$$

So,

$$f(z) = \frac{1}{2\pi i}\int_{C_2} f(s)\left\{ \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + \left(\frac{z}{s}\right)^N \frac{1}{s-z} \right\} ds + \frac{1}{2\pi i}\int_{C_1} f(s)\left\{ \sum_{n=1}^{N} \frac{z^{-n}}{s^{-n+1}} + \frac{1}{z-s}\left(\frac{s}{z}\right)^N \right\} ds$$

So
$$f(z) = \sum_{n=0}^{N-1} \overbrace{\left\{\frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{s^{n+1}}\,ds\right\}}^{\alpha_n} z^n + \sum_{n=1}^{N}\overbrace{\left\{\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{s^{-n+1}}\,ds\right\}}^{\beta_n} z^{-n}$$
$$+ \rho_N + \sigma_N$$

where $$\rho_N = \frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{s-z}\left(\frac{z}{s}\right)^N ds$$

and $$\sigma_N = \frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{z-s}\left(\frac{s}{z}\right)^N ds$$

On $C_2 \Rightarrow \dfrac{1}{|s-z|} \le \dfrac{1}{r_2-r}$. On $C_1 \Rightarrow \dfrac{1}{|z-s|} \le \dfrac{1}{r-r_1}$

where $|z| = r$, $r_1 < r < r_2$.

Letting $M = \max\limits_{s\in C_1\cap C_2} |f(s)|$. By Δ-ineq, we have that

$$|\rho_N| = \left|\frac{1}{2\pi i}\right|\left|\int_{C_2}\frac{f(s)}{s-z}\left(\frac{z}{s}\right)^N ds\right| \le \frac{1}{\cancel{2\pi}} M \cdot \frac{1}{r_2-r}\cdot\left(\frac{r}{r_2}\right)^N \cancel{2\pi}\, r_2$$

$$= \frac{M}{1-r/r_2}\cdot\left(\frac{r}{r_2}\right)^N$$

Similarly,
$$|\sigma_N| \le \frac{M}{1-r_1/r}\cdot\left(\frac{r_1}{r}\right)^N$$

We see that $\sigma_N \to 0$, $\rho_N \to 0$ as $N\to\infty$.

$\therefore$ It follows (with ε's, N's)

$$f(z) = \sum_{n=0}^{\infty}\alpha_n z^n + \sum_{n=1}^{\infty}\beta_n z^{-n}$$

where $$\alpha_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{s^{n+1}}\,ds, \quad \beta_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{s^{-n+1}}\,ds$$

By analogy to Cauchy-Goursat for multiply-connected regions,

$$\alpha_n = \frac{1}{2\pi i}\int_C (\ ) \, ds = a_n$$

$$\beta_n = \ldots \int_C \quad = b_n \quad \forall n$$

Note, the formula $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \frac{1}{z^n}$

is equiv to

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \quad \text{where}$$

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(s)}{s^{n+1}}\, ds$$

$$a_n = c_n \quad n \geq 0 \ \ldots$$
$$c_n = b_{-n} \quad n < 0 \ \ldots$$

Nov 20, 2019

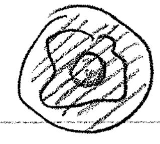
Thm (Laurent's Thm) $\Rightarrow$ Let f be analytic on a region D defined by $R_1 < |z-z_0| < R_2$. Then, for each $z \in D$

$$f(z) = \sum_{n \in \mathbb{Z}} c_n (z-z_0)^n$$

where, given a simple closed contour C in their annulus whose interior contains C_{R_1}

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \quad , n \in \mathbb{Z}$$

Usually, the series are produced by manipulating known series to determine c_n instead of calculating integrals ad infinitum.



Reality check: (recapturing Taylor ...)

Suppose that f is analytic on $|z-z_0| < R_2$. Then the Laurent series theorem still applies...

$$f(z) = \sum_{n\in\mathbb{Z}} c_n (z-z_0)^n, \quad c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

If $n = -1, -2, \ldots$, i.e. $n<0$, $n\in\mathbb{Z}$, then writing $-m = n+1$ has $m \geq 0$, $m \in \mathbb{Z}$.

$$\text{So, } c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-m}}\,dz = \frac{1}{2\pi i}\int_C f(z)(z-z_0)^m\,dz.$$

By properties of analytic functions, for $n<0$, $n\in\mathbb{Z}$

$$\frac{f(z)}{(z-z_0)^{n+1}} = f(z)(z-z_0)^m \text{ is analytic on } B_{R_1}(z_0)$$

and so $\boxed{c_n = 0}$, by Cauchy-Goursat.

In this case,
$$f(z) = \sum_{n\in\mathbb{Z}} c_n (z-z_0)^n = \sum_{n\in\mathbb{N}} c_n (z-z_0)^n$$
$\nearrow$ f is analytic

$$\text{where } c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{f^{(n)}(z_0)}{n!}$$

$\rightarrow$ we have recaptured Taylor series...

Ex Compute the Laurent series of $f(z) = e^{+1/z}$ about $z_0 = 0$...
For $w \in \mathbb{C}$, we have that

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}, \qquad \sum_{n=-\infty}^{0} \frac{1}{(-n)!}\frac{1}{z^n}$$

For $z \neq 0$, $0<|z|<\infty$, then we have

$$e^{1/z} = \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{1}{z^n}\frac{1}{n!}$$ By uniqueness of Laurent series,

this is the Laurent series for $e^{1/z}$ at $z_0 = 0$ with

$$c_n = \begin{cases} 0 & n \in \mathbb{N} \\ \dfrac{1}{(-n)!} & n = -0, -1, -2, \ldots \end{cases}$$

Application Consider a simple closed contour, which is (+) oriented whose interior contains 0. Compute

$$I = \int_C e^{1/z} z^{20}\, dz = ?$$

$$= \int_C \frac{e^{1/z}}{z^{-20}}\, dz = \int_C \frac{e^{1/z}}{z^{-21+1}}\, dz = 2\pi i\, C_{-21} \text{ of } e^{1/z} \text{ around } z_0 = 1$$

So $\boxed{I = (2\pi i)\dfrac{1}{21!}}$

what about ... $\displaystyle\int_C \frac{e^{1/z}}{z^3}\, dz = \int_C \frac{e^{1/z}}{z^{2+1}}\, dz = C_2 = \boxed{0}$

$\boxed{Ex}$ $\displaystyle f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$

by inspection, we see that f is analytic on the region $|z| < 1$, $1 < |z| < 2$, and $2 < |z| < \infty$

Laurent series expansion @ $|z| < 1$?

$\Rightarrow$ we focus inside the unit disk. We seek Taylor series...

$$f(z) = \frac{-1}{1-z} + \frac{1}{2-z} = \frac{-1}{1-z} + \frac{1}{1-z/2}\cdot\frac{1}{2}$$

For $|z| < 1$, $|z/2| < 1$ so $\displaystyle\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $\displaystyle\frac{1}{1-z/2} = \sum_{n=0}^{\infty} (z/2)^n$.

So, $\displaystyle f(z) = -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty} \frac{z^n}{2^n}$ $(|z| < 1)$

$$\boxed{f(z) = \sum_{n=0}^{\infty} z^n \left\{ \frac{1}{2^{n+1}} - 1 \right\}} \quad (|z| < 1)$$

What about $1 < |z| < 2$?? Get Laurent series...

But note that $\left|\frac{z}{2}\right| < 1$ but $1 < |z|$.

$\Rightarrow$ "$z/2$" expansion is Taylor, but "z" expansion needs modification

Here, $-\frac{+1}{z-2} = \frac{1}{2}\frac{1}{1-z/2} = \frac{1}{2}\sum_{n=0}^{\infty} z^n/2^n$ valid for $|z| < 2$ no problem.

For,

$$\frac{1}{z-1} = \frac{1}{z}\frac{1}{1-1/z} \quad \text{Here } 1/|z| < 1 \text{ since } |z| > 1$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}(1/z)^n = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} = \sum_{n=-\infty}^{n=-1} z^n \quad \text{valid for } |z| > 1$$

And so... for $1 < |z| < 2$ ...

$$f(z) = \sum_{n=-\infty}^{n=-1} z^n + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

$$= \sum_{-\infty}^{\infty} c_n z^n \quad \text{where } c_n = \begin{cases} 1 & n \le -1 \\ \frac{1}{2^{n+1}} & n \ge 0 \end{cases}$$

Nov 25, 2019

Reading topics...

Consider a power series $S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

Results:

+ If $S(z)$ converges @ some $z_1 \neq z_0$ then $S(z)$ converges on $B_R(z_0)$ where $|z_0 - z_1| \le R$ (converge on ball...)

+ The series converges uniformly & absolutely on every ball B properly contained in $B_R(z_0)$.

+ On $B_R(z_0)$, $S(z)$ is analytic, $S'(z) = \sum_{n=1}^{\infty} a_n n (z-z_0)^{n-1}$

+ If C is a s.c.c, g is cont on C & $C \subseteq B_R(z_0)$ then

$$\boxed{\int_C f \cdot g \, dz = \sum_n \int_C a_n g(z)(z-z_0)^n dz}$$

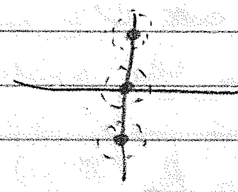
<u>Uniqueness of Laurent series</u>... If $S(z) = \sum_{n \in \mathbb{Z}} c_n (z-z_0)^n$ converges on an annulus on $R_1 \leq |z-z_0| \leq R_2$, then this is <u>precisely</u> the Laurent series of S at z_0

RESIDUES & POLES

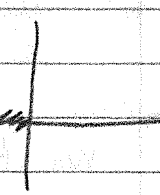
<u>Recall:</u> a point z_0 is called a singularity point for a function f if f fails to be analytic at z_0.

If z_0 is a singularity for f and further, $\exists \varepsilon > 0$ s.t. f is analytic on the punctured disk $B_\varepsilon(z_0) \setminus \{z_0\}$ we say that z_0 is an <u>isolated singularity</u> for f.

$\boxed{Ex}$ $f(z) = \dfrac{1}{z^2(z^2+1)}$ has isolated singularities at $z = 0, \pm i$



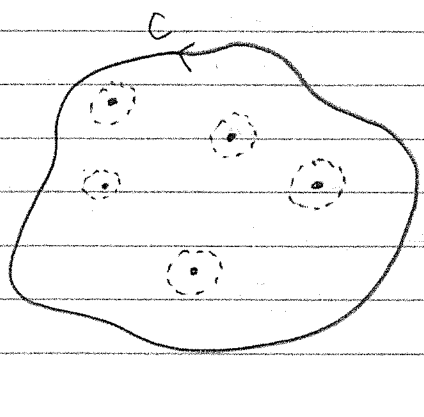
$Log(z)$ has a singularity @ $z_0 = 0$. ~~xxxxxx~~
this is <u>not</u> an isolated singularity.



<u>Remark</u>

Let C be a s.c.c (+) and let f have singularities at $z_1, z_2, \ldots, z_n \in \mathring{C}$ and no where else ...
int(C)

then $z_1, \ldots, z_n$ are isolated singularities, and $\exists$ punctured disks $B_1, B_2, \ldots, B_n$ inside C which are non-overlapping whose centers contain z_k respectively...



Residues Suppose that f has an isolated singularity @ z_0. Then f has a Laurent series expansion on an annulus $0 < |z - z_0| < R_2$ with

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

Further, for any s.c.c (+) C,

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-n+1}}\, dz \quad \forall\, n = 1, 2, 3, \ldots$$

In particular,

$$b_1 = \frac{1}{2\pi i}\int_C f(z)\, dz.$$

We shall call this coef of $\frac{1}{z-z_0}$ in the Laurent series expansion the residue of f @ z_0.

$$\boxed{b_1 = \operatorname*{Res}_{z=z_0}(f(z))}$$

We then have $$\int_C f(z)\, dz = 2\pi i \operatorname*{Res}_{z=z_0} (f(z)) = 2\pi i\, b_1$$

This gives us a way to compute integrals by finding Laurent series expansion...

Ex Let C be a simple c.c (+) containing $z_0 = 0$ in its interior.

$\int z^2 \sin\left(\frac{1}{z}\right) dz = ?$ <u>Note</u> $z^2 \sin\left(\frac{1}{z}\right)$ has an isolated singularity @ $z_0 = 0$.

$$z^2 \sin\left(\frac{1}{z}\right) = z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z}\right)^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z}\right)^{2n-1}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{1-2n}} = z - \frac{1}{6z} + \ldots$$

$\Rightarrow b_1 = \frac{-1}{3!} = -\frac{1}{6} \Rightarrow \operatorname*{Res}_{z=0}(f(z)) = b_1 = \frac{-1}{6}$.

Thus,

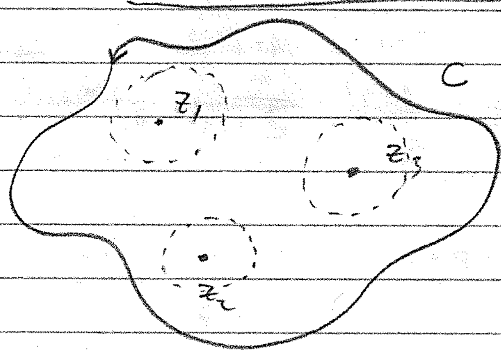
$$\boxed{\int_C z^2 \sin(1/z)\,dz = 2\pi i\left(\frac{-1}{6}\right) = -\frac{\pi i}{3}}$$

Thm (The Residue Thm)

Let C be a s.c.c (+) and suppose that f is analytic on C and interior to C except @ a finite number of points $z_1, z_2, \ldots, z_n$ all lying interior to C.

Then, $\displaystyle\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k}(f(z))$.

Dec 2, 2019



pf Take $C_1, C_2, \ldots, C_n$ to be non intersecting s.c.c (+) inside C where each contains only the singular point z_k (resp.)

Then f analytic on $\operatorname{Int}(C) \setminus \bigcup^n \operatorname{Int} C_k$

By Cauchy-Goursat, (multiply-connected ~~region~~)

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz$$

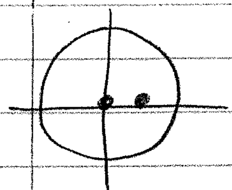
But for each k, $\displaystyle\oint_{C_k} f(z)dz = 2\pi i \operatorname*{Res}_{z=z_k} f(z)$.

So,

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

$\square$

Ex <u>Consider</u> $f(z) = \dfrac{z^2+1}{z(z-1)}$, $C: = |z| = 2$, $(+)$



$$\oint_C f(z)\,dz = 2\pi i \left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z) \right)$$

$$\frac{1+z^2}{z(z-1)} = \frac{1+z^2}{z} \cdot \frac{-1}{1-z} = \frac{1+z^2}{z}(-1)\left\{ 1 + z + \cdots \right\}$$

$$= -\left(z + \frac{1}{z} \right)\left(1 + z + \cdots \right)$$

$(|z|<1)$ $\longrightarrow$ $\boxed{\operatorname*{Res}_{z=0} f(z) = -1}$

$$\frac{1+z^2}{z(z-1)} = \frac{1+z^2}{z-1} \cdot \frac{1}{z} = \frac{1+z^2}{z-1} \cdot \frac{1}{1+(z-1)}$$

$(|z-1|<1)$

$$= \frac{1+z^2}{z-1} \cdot \frac{1}{1-(1-z)}$$

$$= \frac{1+z^2}{z-1} \sum_{n=0}^{\infty} (1-z)^n$$

Now, $\dfrac{1+z^2}{z-1} = \dfrac{z^2 - z + z + 1}{z-1} = \dfrac{z^2 - z + z - 1 + 2}{z-1}$

$$= z + 1 + \frac{2}{z-1} = (z-1) + 2 + \frac{2}{z-1}$$

So $\dfrac{1+z^2}{z(z-1)} = \left\{ z - 1 + 2 + \dfrac{2}{z-1} \right\} \displaystyle\sum_{n=0}^{\infty} (1-z)^n$

So $\boxed{\operatorname*{Res}_{z=1} f(z) = 2}$

So $\displaystyle\oint_C f(z)\,dz = 2\pi i\,(2-1) = \boxed{2\pi i}$

CLASSIFICATION OF SINGULARITIES

Suppose that $f(z)$ has an isolated singularity at z_0.
By Laurent series thm, $\exists R>0$ s.t.

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k + \underbrace{\frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots}_{\text{the principal part of } f(z)\text{'s Laurent series.}}$$

valid for
$0 < |z-z_0| < R$

If the principal part of f's Laurent series contains a finite number of nonzero terms (≥ 1), then let

$$m = \max \{ k = 1, 2, \ldots : b_k \neq 0 \} \text{ exists}, \geq 1.$$

In this case, z_0 is said to be a pole of order m.

If $m = 1$, it is called a "simple pole".

Ex $\quad g(z) = \dfrac{\sin z}{z^3} = \dfrac{1}{z^3} \left\{ z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots \right\}$

So $\quad z_0 = 0$ is pole of order 2.

What about $f(z) = \dfrac{\sin z}{z}$? $\to$ no principal part.

Defn $\boxed{\text{If the principal part of the Laurent series expansion is identically zero, then } z_0 \text{ is said to be a removable singularity.}}$

Observe : If z_0 is an isolated removable singularity for f for $z \neq z_0$ but $0 < |z-z_0| < R$,

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k + 0.$$

At $z = z_0$, the LHS $= a_0$.

So, define
$$f_{ext}(z) = \begin{cases} f(z) & 0 < |z - z_0| < R \\ a_0 & z = z_0. \end{cases}$$

Then,

$$\boxed{f_{ext}(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \forall\, z \text{ s.t. } |z - z_0| < R}$$

extension of f. Note $f_{ext}(z)$ is analytic on $B_R(z)$.

We have removed the removable singularity.

Dec 4, 2014

Recall ... when z_0 is an isolated singularity ... then

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z - z_0)^n}_{} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}}_{\text{principal part of } f(z).}$$ Laurent series

Cases:

① When Principal part is nonzero & contains a finite # of summands
$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} = \frac{b_1}{(z - z_0)} + \dots + \frac{b_m}{(z - z_0)^m}$$
and $b_k = 0 \;\; \forall\, k \geq m+1$

then z_0 is a pole of order m for f.
when $m = 1$, z_0 is a simple pole.

$\boxed{Ex}$ $f(z) = \dfrac{z^2+1}{z(z-1)} \rightsquigarrow$ simple poles $z_0 = 1$, $z_0 = 0$.

$\boxed{Ex}$ $f(z) = \dfrac{\sin z}{z^3} \longrightarrow$ pole $z_0 = 0$ of order 2.

② If principal part is identically zero, then z_0 is a removable singularity thus, f can be extended via its valid Laurent-Taylor series expansion to an analytic function on $B_R(z_0)$

$\boxed{Ex}$ $$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

Here $z_0 = 0$ is a removable singularity.

(3) z_0 is said to be an <u>essential singularity</u> of f if it is not removable or a pole...

i.e. the principal part contains an infinite number of non-zero terms.

$\boxed{Ex}$ $$e^{1/z} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n \cdot \frac{1}{n!} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

$$= 1 + \underbrace{\sum_{n=1}^{\infty} \frac{(1/z)^n}{n!}}_{\text{principal part}}$$

So $z_0 = 0$ is an essential singularity.

<u>Thm</u>: Let z_0 be an isolated singularity of f. Then z_0 is a pole of order m <u>iff</u> $\exists$ a function $\phi(z)$ which is nonzero at z_0, analytic at z_0 and for which

$$f(z) = \frac{\phi(z)}{(z-z_0)^m} \quad \text{for } z \text{ in a nbh of } z_0.$$

In this case,

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

$\boxed{\text{Ex}}$ Compute the residue $\operatorname*{Res}_{z_0=1} \dfrac{z^2+1}{z(z-1)}$.

Well... $f(z) = \dfrac{z^2+1}{z(z-1)} = \dfrac{1}{(z-1)}\left\{\dfrac{z^2+1}{z}\right\} = \dfrac{\phi(z)}{(z-1)^1}$

Then
$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(0)}(z_0)}{(1-1)!} = \phi^{(0)}(z_0) = \phi(1) = \frac{2}{1} = \boxed{2}.$$

because $\phi(z) = \dfrac{z^2+1}{z}$ is analytic & nonzero at $z=1$.

$\boxed{\text{Pf of Thm}}$ Suppose that $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic

$(\Rightarrow)$

at z_0 and $\phi(z_0) \neq 0$. We know that $\phi(z)$ has a valid Taylor series in some $B_R(z_0)$
$$\phi(z) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n$$

Then
$$f(z) = \frac{1}{(z-z_0)^m}\sum_{n=0}^{\infty}\frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n$$
$$= \sum_{n=0}^{\infty}\frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m}$$
$$= \underbrace{\sum_{n=0}^{m-1}\frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m}}_{\text{the principal part.}} + \underbrace{\sum_{n=m}^{\infty}\frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m}}_{\text{Taylor}}$$

$(k = m-n)$
$$= \sum_{k=1}^{m}\frac{\phi^{(m-k)}(z_0)}{(m-k)!}\frac{1}{(z-z_0)^k} + (\text{Taylor})$$

So, z_0 is a pole of order m, since $\phi^{(0)}(z_0) \neq 0$.

And also, $\operatorname*{Res}_{z=z_0} f(z) = \dfrac{\phi^{(m-1)}(z_0)}{(m-1)!}$.

$\square$

($\Leftarrow$) Conversely, assume that f has a pole at z_0 of order m.
<u>Then</u>

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{m} \frac{b_n}{(z-z_0)^n} + 0 \dots$$

So, $$f(z) = \frac{1}{(z-z_0)^m} \left\{ \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + \sum_{n=1}^{m} \frac{b_n}{(z-z_0)^{n-m}} \right\}$$ ($b_n \neq 0$ by hyp)

So,
$$f(z) = \frac{1}{(z-z_0)^m} \underbrace{\left\{ \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + \sum_{n=1}^{m} b_n (z-z_0)^{m-n} \right\}}_{=: \phi(z)}$$

With this def^n of $\phi(z)$, we see that it is analytic at z_0 and

$$\phi(z_0) = 0 + b_m \neq 0 \text{ by hyp.}$$

$\square$

ec 5, 2014

$\boxed{Ex}$

$$f(z) = \frac{1}{\sin z} \quad \text{at } z_0 = 0$$

$$= \frac{1}{z \cdot \frac{\sin z}{z}} = \frac{1}{z\left(1 - \frac{z^2}{3!} + \dots\right)} = \frac{\phi(z)}{z}$$

<u>Check</u> $\phi(z)$ is analytic & $\phi(0) = 1 \neq 0$.

By our thm, $z_0 = 0$ is a simple pole and

$$\operatorname*{Res}_{z=0} f(z) = \frac{\phi^{(1-1)}(0)}{(1-1)!} = \phi(0) = 1$$

$\cancel{\#}$

Thm (Hint: Final)

Let p, q be analytic at z_0. If $p(z_0) \neq 0$, $q(z_0) \neq 0$ and $p'(z_0) = 0$, then

$$f(z) = \frac{p(z)}{q(z)}$$

has a simple pole at z_0 and

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)} \qquad (\text{pf: use } \phi)$$

What happens near singularities?

Thm If z_0 is a pole of order m for f, then the Laurent series of f

$$\lim_{z \to z_0} f(z) = \infty.$$

Thm If z_0 is a removable singularity for f then f is bounded and analytic on a punctured nbhd of z_0

~~Lemma~~ (Converse of ↑)

Let f be analytic on $0<|z-z_0|<\delta$ for some δ, ~~then~~ and if f is also bounded on $0<|z-z_0|<\delta$, then if z_0 is a singularity for f, it is a removable one.

Pf By assumption, f has a Laurent series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

where b_n in particular is $(2\pi i)^{-1} \oint_C \frac{f(z)}{(z-z_0)^{-n+1}} dz$

where C is a circle(+) in the annulus of analyticity.

In particular, if $0 < \rho < \delta$, and $C_\rho = \{z : |z - z_0| = \rho\}$, (+)
then

$$|b_n| = \left| \frac{1}{2\pi i} \oint_{C_\rho} \frac{f(z)}{(z-z_0)^{-n+1}} \, dz \right|$$

and if M is s.t. $|f(z)| \le M \;\forall\; 0 < |z - z_0| < \delta$ then
(f bounded)

$$|b_n| \le \frac{1}{2\pi} \oint_{C_\rho} \frac{M}{(z-z_0)^{-n+1}} \, dz = \frac{1}{2\pi} \frac{M}{\rho^{-n+1}} \cdot 2\pi\rho = M\rho^{+n}$$

Since this is valid $\forall \rho < \delta$, we must have $b_n = 0 \;\forall n$.

So done :) $\square$

Thm (Casorati-Weierstrass)

Let f have essential singularity at z_0. Then $\forall\, w_0 \in \mathbb{C}$ and $\forall\, \varepsilon > 0$,

$$|f(z) - w_0| < \varepsilon \text{ for some } z \in B_\delta(z_0) \;\forall\, \delta > 0$$

That is, f is arbitrarily close to every complex number on every nbh of z_0.

i.e. $\forall\, \delta > 0$, $f(B_\delta(z_0) \setminus \{z_0\})$ is dense in $\mathbb{C}$

i.e.

f gets close to every single point in ball $\forall$ ball.

i.e. (we're not going to prove this)
If z_0 is an essential singularity for f, then f attains, except for at most one value, every complex number an infinite number of times on every nbh of z_0

Pf Assume (to reach a contradiction) that ~~there~~ $\exists\, w_0 \in \mathbb{C}$, $\varepsilon, \delta > 0$ s.t.

$$|f(z) - w_0| \geq \varepsilon \quad \forall\ 0 < |z - z_0| < \delta$$

Consider $g(z) = \dfrac{1}{f(z) - w_0}$, which is bounded & analytic on a punctured disk $0 < |z - z_0| < \delta$. At worst, z_0 is a removable singularity for g. Also note $g(z) \not\equiv 0$ since f non constant (since f ~~has~~ singularity).

So, $g(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$, which allows us to extend g to ~~...~~ z_0. Let $m = \min\{k = 0, 1, 2, \dots\}$ s.t. $a_k \neq 0$, which exists because $g \not\equiv 0$.

Then $g(z) = (z - z_0)^m \sum_{k=0}^{\infty} a_k (z - z_0)^{k-m}$

$$= (z - z_0)^m \underbrace{\sum_{k=0}^{\infty} a_{k+m} (z - z_0)^k}_{h(z),\quad h(z_0) = a_m \neq 0}$$

So, on $0 < |z - z_0| < \delta$,

$$f(z) = w_0 + \frac{1}{g(z)}.$$

Done!

(i) If $g(z_0) \neq 0 \Leftrightarrow m = 0$. Then this formula allows us to extend f to z_0, which is then analytic there $\Rightarrow z_0$ is a removable singularity.

This is a contradiction.

(ii) If $g(z_0) = 0$, $m \geq 1$, and $f(z) = w_0 + 1/g(z) = \dfrac{w_0 g(z) + 1}{g(z)}$

We see that $\phi(z_0) \neq 0$, $\phi(z)$ analytic, $= \dfrac{w_0 g(z) + 1}{(z - z_0)^m h(z)} := \dfrac{\phi(z)}{(z - z_0)^m}$

$\Rightarrow z_0$ is a pole of order $m \Rightarrow \boxed{\text{CONTRADICTION}}$ $\square$