PH 241

Sept 6 2017

Modern Phy → Special Relativity
→ Quantum Mech.

Applications → Particle Creation (1)
→ Atomic Clocks (2)
→ GPS (atomic clocks & Relativity) (3)
→ Atomic Clock (as Detector) (4)
→ Quantum Information (5)
→ Quantum Biology (6)
→ Understanding the Universe (LIGO) (7)

I. RELATIVITY

{ Cosmological inquiry into space & time }

(A) Galilean & Newtonian Relativity

★ 1. Two central problems of relativity

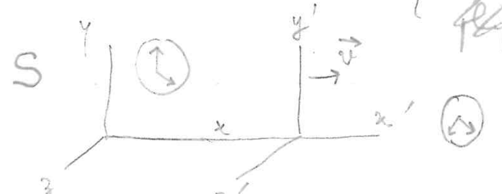
★ 2. Terms defined

{ Two observers in different inertial reference frames S & S' make detailed observations as physical pheno. occur → Each obs records the TIME & LOCATION of a set of events. How do their records compare? }

a. Events : { Sth that occurred independent of our description of it (time & location) }

not when & where the OBS "sees" the event → b. Observer { Someone who compiles measurements of where & when Events occur }

c. Inertial reference frame { Coordinate system in which the law of ~~physics~~ inertia is true (no $\vec{F}$, no $\vec{a}$ }



★ 3. Principle of Relativity

{ The laws of physics are the same in all inertial reference frames } ← mathematical form

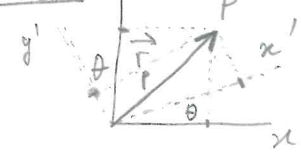
★ 4. Coordinate Transformation

↳ If "event" is observed in space-time coordinate (t, x, y, z) in S, how do we determine the space-time coordinate (t', x', y', z') in S'?

(First lab = build laser)

Buy Notebook!

a. Spatial Rotation



$$\begin{cases} S: \vec{r}_P = \langle x_p, y_p \rangle = x_p \hat{i} + y_p \hat{j} \\ S': \vec{r}_p = \langle x_p', y_p' \rangle = x_p' \hat{i}' + y_p' \hat{j}' \end{cases}$$

$$\vec{r}_p = x_p\hat{i} + y_p\hat{j} = x_p'\hat{i}' + y_p'\hat{j}'$$

★ Any vector $\vec{a} = a_x\hat{i} + a_y\hat{j} = a_x'\hat{i}' + a_y'\hat{j}'$

Unit Vector $\begin{cases} \hat{i}' = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \\ \hat{j}' = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \end{cases}$ $\xrightarrow{\text{inverse}}$ $\begin{cases} \hat{i} = \cos(-\theta)\hat{i}' + \sin(-\theta)\hat{j}' \\ \hat{j} = -\sin(-\theta)\hat{i}' + \cos(-\theta)\hat{j}' \end{cases}$ $\begin{cases} -\sin(-\theta) = \sin \\ \cos(-\theta) = \cos\theta \end{cases}$

Coefficient?
$$\vec{a} = a_x\hat{i} + a_y\hat{j} = a_x(\cos\theta\,\hat{i}' - \sin\theta\,\hat{j}') + a_y(\sin\theta\,\hat{i}' + \cos\theta\,\hat{j}')$$
$$= (a_x\cos\theta + a_y\sin\theta)\hat{i}' + (-a_x\sin\theta + a_y\cos\theta)\hat{j}'$$

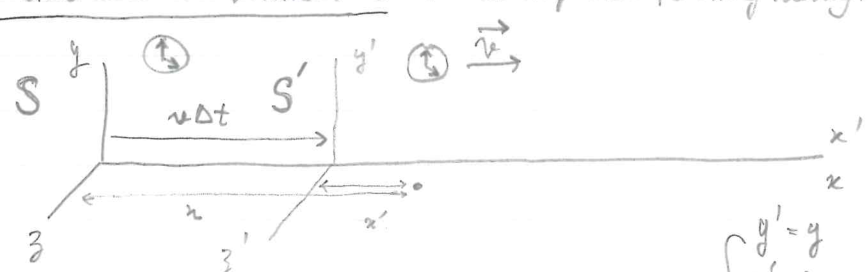
$$\boxed{a_x\hat{i} + a_y\hat{j} = (a_x\cos\theta + a_y\sin\theta)\hat{i}' + (-a_x\sin\theta + a_y\cos\theta)\hat{j}'}$$

$$\begin{cases} a_x' = a_x\cos\theta + a_y\sin\theta \\ a_y' = -a_x\sin\theta + a_y\cos\theta \end{cases}$$

feature of an "orthagonal" transformation

→ What do observers agree on? ⇒ $\boxed{\text{Magnitude}}$ → an "invariant" is $\boxed{a_x^2 + a_y^2 = a_x'^2 + a_y'^2}$ (length)

b. Galilean Transformations (relationship btwn relatively moving frames)



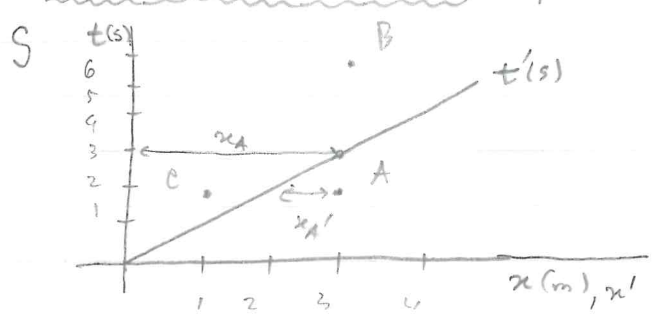
At $t = t' = 0 \rightarrow x = x' = 0$

(i.) Galilean transformation equation $\begin{cases} y' = y \\ z' = z \\ x' = x - vt \\ t' = t \end{cases}$ (clock run at the same rate ...)

$t' = t$ → hypothesis of universal time

$\xrightarrow{\text{inverse}}$ $\begin{cases} t = t' \\ y = y' \\ z = z' \\ x = vt + x' \end{cases}$

Visualization of Gal. trans.: (space-time diagrams)



Events A, B, C

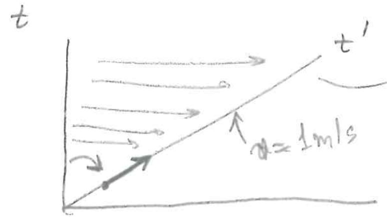
$t_A, x_A = (2s, 3m)$

$t_B, x_B = (6s, 3m)$

$t_C, x_C = (2s, 1m)$

$\boxed{\text{What about } S' ?}$ $\quad t' = t$

$x' = x - vt$, if $x' = 0 \rightarrow \boxed{x = vt}$.



$\boxed{\text{path of } x'}$ → because S' is moving in S → draw a path

For S', $x' = 0$

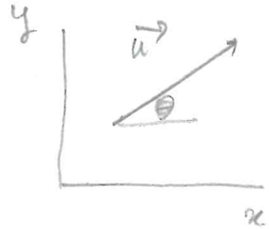
For S, $x = vt$

(ii) Galilean velocity transformation $\quad \vec{u} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$, $\vec{u}' = \left\langle \frac{dx'}{dt'}, \frac{dy'}{dt'}, \frac{dz'}{dt'} \right\rangle$

$$\frac{dx'}{dt'} = \frac{dx'}{dt} \cdot \overbrace{\frac{dt}{dt'}}^{1} = 1 \cdot \frac{d}{dt}(x - vt) = \left(\frac{dx}{dt}\right) - \frac{dt}{dt} \cdot v = \boxed{u_x - v} = u_x'$$

$$\boxed{\begin{aligned} u_x' &= u_x - v \quad ; \quad u_x = u_x' + v \\ u_y' &= u_y \\ u_z' &= u_z \end{aligned}}$$

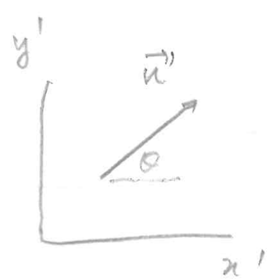
(iii) Direction



$$\begin{cases} u_x = u\cos\theta \\ u_y = u\sin\theta \end{cases}$$

$u = \sqrt{u_x^2 + u_y^2}$

$\tan\theta = \frac{u_y}{u_x}$



$$\begin{cases} u_x' = u'\cos\theta = u_x - v \\ u_y' = u'\sin\theta = u_y \\ u' = \sqrt{u_x'^2 + u_y'^2} \end{cases}$$

$\tan\theta' = u_x'/u_y'$

Speed in S' → $\quad u'^2 = u^2 + v^2 - 2uv\cos\theta$

$$\tan\theta' = \frac{u_y'}{u_x'} = \frac{u_y}{u_x - v} = \frac{u\sin\theta}{u\cos\theta - v}$$

//

$\boxed{\text{Sept 11, 2017}}$ Recap : Galilean transformation

$$\begin{cases} \Delta t' = \Delta t \\ \Delta x' = \Delta x - v\Delta t \\ \Delta y' = \Delta y \\ \Delta z' = \Delta z \end{cases}$$

(iii) Acceleration Transformation

$$a_x = \frac{du_x}{dt} \quad , \quad a_y = \frac{du_x'}{dt'} = \frac{du_x'}{dt} \cdot \overbrace{\frac{dt}{dt'}}^{1} = \frac{d}{dt}(u_x - vt) = \boxed{a_x = a_x'}$$

$(a_y = a_y' \ , \ a_z = a_z')$ $\qquad$ (true only if $v = \text{const}$)

★ 5. Newton's law and Galilean relativity

$\boxed{\vec{F} = m\vec{a}}$ Newton's laws are "invariant" under a Galilean transformation

(B) Electromagnetism & Galilean Relativity

★ 1. Maxwell's equations and light

summary of rules for $\vec{E}$ and $\vec{B}$

$\Rightarrow$ predicts waves travel w/ speed $c = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8$ m/s

★ 2. The "Luminiferous ether"

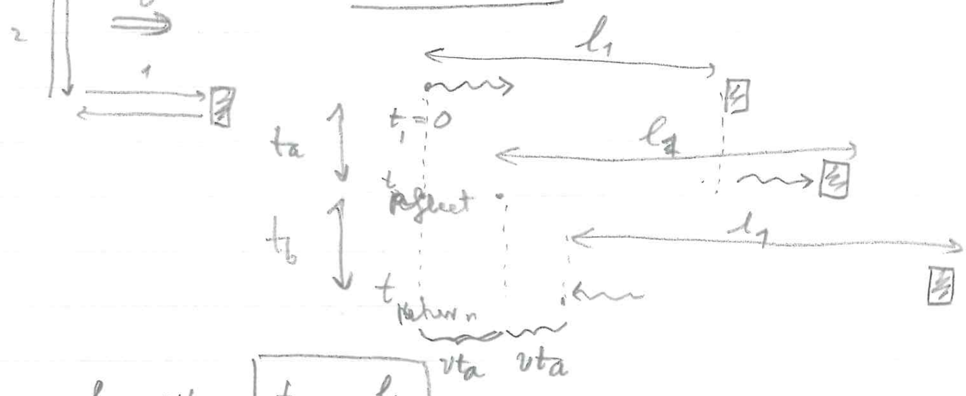
Why?

- → Maxwell's equations are wrong?
- → Galilean's transformations are wrong?
- → There is a medium in which $c = 3 \times 10^8$ m/s (luminiferous ether...)

★ 3. The Michelson-Morley experiment (a test of the ether hypothesis)

Experiment for measuring c

$\boxed{\text{<<Horizontal>>}}$ (along/against v)

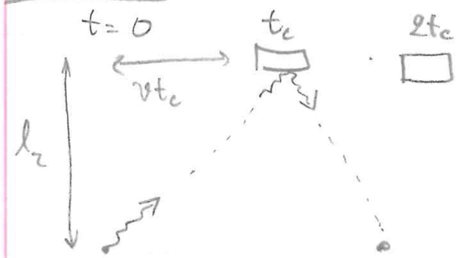


$ct_a = l_1 + vt_a \therefore \boxed{t_a = \dfrac{l_1}{c-v}}$

$ct_b = l_1 - vt_b \quad \boxed{t_b = \dfrac{l_1}{c+v}}$

$$\Delta t_1 = t_a + t_b = \frac{l_1}{c-v} + \frac{l_1}{c+v} = \frac{2l_1 c}{c^2 - v^2} = \left(\frac{2l_1}{c}\right)\frac{1}{\left(1 - v^2/c^2\right)}$$

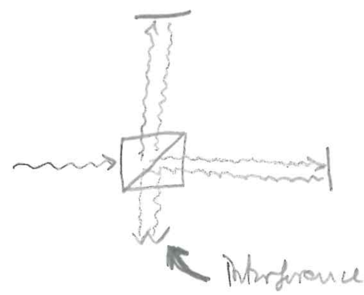
"Vertical" ($\perp$ to v)



$$ct_c = \sqrt{l_2^2 + (vt_c)^2} \therefore t_2 = 2t_c = \boxed{\frac{2l_2}{c} \cdot \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}}$$

a. Michelson's 1st brilliant observation

→ Interferometry



$$T_{light} = \frac{1}{f} = \frac{\lambda}{c} \simeq \boxed{2 \times 10^{-15} s}$$

λ: $600 \times 10^{-9} m$; c: $3 \times 10^{8} m$

b. Michelson's second brilliant observation

Rotate the apparatus (switch roles of mirrors)

(1) $\Delta t\,(\theta = 0) = t_2(\theta = 0) - t_1(\theta = 0) = \frac{2l_2}{c} \cdot \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{2l_1}{c} \cdot \frac{1}{1-\frac{v^2}{c^2}}$

(2) $\Delta t\,(\theta = 90°) = \frac{2l_2}{c} \frac{1}{1-\frac{v^2}{c^2}} - \frac{2l_1}{c} \cdot \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

$$\Delta T = \Delta t\,(\theta = 90°) - \Delta t\,(\theta = 0°) = \frac{2}{c}(l_1 + l_2)\left[\frac{1}{1-\frac{v^2}{c^2}} - \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right] \simeq \boxed{\frac{\cancel{2}(l_1+l_2)}{c} \cdot \left(\frac{v^2}{c^2}\right)}$$

↓

turns out $\Delta T = 0$

Approximate answer $(1+x)^p \simeq 1 + px$

$$\left(\frac{1}{1-\frac{v^2}{c^2}} \simeq 1 + \frac{v^2}{c^2} \qquad \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \simeq 1 + \frac{1}{2}\frac{v^2}{c^2} \right)$$

Sept 12, 2017

$$\boxed{\Delta t = \left(\frac{l_1 + l_2}{c}\right)\beta^2} \qquad \frac{\Delta T}{T} = \left(\frac{l_1+l_2}{c}\right)\beta^2 \cdot \left(\frac{c}{\lambda}\right) = \boxed{\left(\frac{l_1+l_2}{\lambda}\right)\beta^2}$$

if half-period → light → dark

want: $\frac{\Delta T}{T} = 0.5$ → light/dark

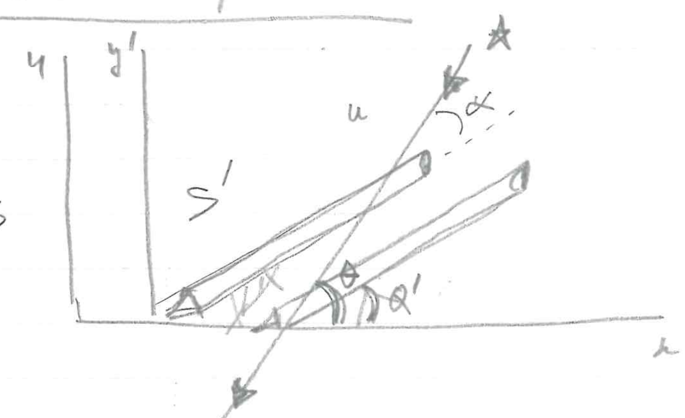
with $l_1 = l_2 = 11\,m$
$\lambda = 0.590\ \mu m$ (sodium lamp)

$\Bigg\} \rightarrow \boxed{\frac{\Delta T}{T} \simeq 0.4}$ (predict) if ether exists

$\boxed{\text{Result}}$: No change in interference pattern

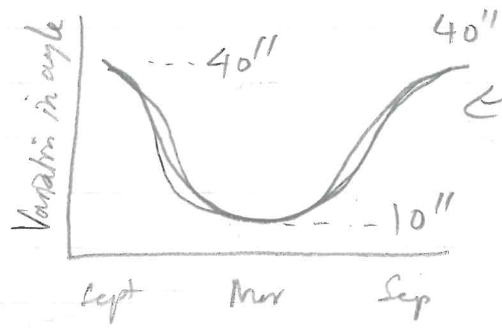
$\hookrightarrow \frac{\Delta T}{T} < 0.005 \rightarrow$ we can't detect Earth's motion thru ether

A different experiment (1725)



$\theta - \theta' = \alpha$

(Bradley's Stellar Abberation)



(stellar abberation)

matches "ether" theory → calculate speed of light

↓

Contradicts Michelson-Morley

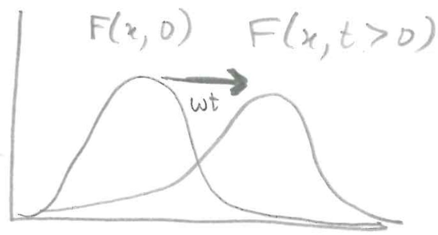
$$\frac{u_x'}{u} = \cos\theta' = \frac{u\cancel{\cos\theta} - v}{\sqrt{u^2+v^2-2uv\cos\theta}} \qquad (u=c)$$

And if $\cos\theta = 0 \Rightarrow \cos\theta' = \left(\frac{-v}{c}\right)\left(\frac{1}{\sqrt{1+v^2/c^2}}\right) \approx \frac{-v}{c}$
(star shining straight down)

and if $\theta \neq 90^\circ$ but $\frac{v}{c} << 1 \rightarrow \alpha = \frac{v}{c}\sin\theta$ (?)

Wave eq. $\boxed{\dfrac{\partial^2 E}{\partial x^2} = \mu_0 \epsilon_0 \dfrac{\partial^2 E}{\partial t^2}}$

$E(x,t)$ must have form $F(x - wt)$



$$\begin{cases} x' = x - vt \\ t' = t \end{cases}$$

Transform $F(x-wt)$ in S

$$\boxed{\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}} \longrightarrow E(x,t) = E\big(x(x',t'),\ t(x',t')\big)$$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x'} \cdot \underbrace{\frac{\partial x'}{\partial x}}_{1} + \frac{\partial F}{\partial t'} \cdot \underbrace{\frac{\partial t'}{\partial x}}_{0}$$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x'} \cdot \underbrace{\frac{\partial x'}{\partial t}}_{(-v)} + \frac{\partial F}{\partial t'} \cdot \underbrace{\frac{\partial t'}{\partial t}}_{1}$$

$\rightarrow$ 2nd derivative ...

$$\boxed{\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2 E}{\partial x'^2} + \frac{2v}{c^2}\frac{\partial^2 E}{\partial x' \partial t'} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t'^2}}$$

$\rightarrow$ solution $F(x - \underbrace{ct + vt}_{\text{different wave speed!}})$

$\rightarrow$ NO LONGER THE SAME FORM!!!

Sept 13, 2017

C Einstein's Postulates of Relativity

Can we reconcile the MM exp & stellar aberration?
$\hookrightarrow$ are laws of EM independent of the inertial frame?

1. Einstein's realization about ~~electrodyn~~ EM

Two exp.

↳ Pass magnet over loop: change $\Phi_B \longrightarrow \mathcal{E} = \frac{-d\Phi}{dt}$

↳ Pass loop thru magnet: $\vec{F}_c = q\vec{v} \times \vec{B}$

↳ $\Delta V \Rightarrow$ same as calculation by 1

2. Postulates of relativity (Special) (inertial frames)

a. Einstein's postulates

(1) All laws of physics are the same (same mathematical form) in all inertial reference frames

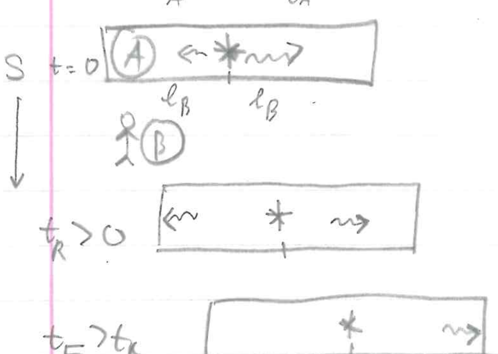
(2) Principle of constancy of the speed of light. (the s.o.l is the same in all inertial reference frames)

D. The fundamental consequences of the postulates

↳ prelude to the "Lorentz - Einstein Transformation"

1. Relativity of Simultaneity

Alice in a train moving at v relative to ground
Bob on ground

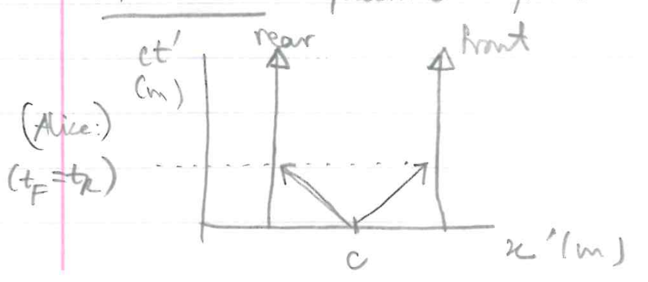


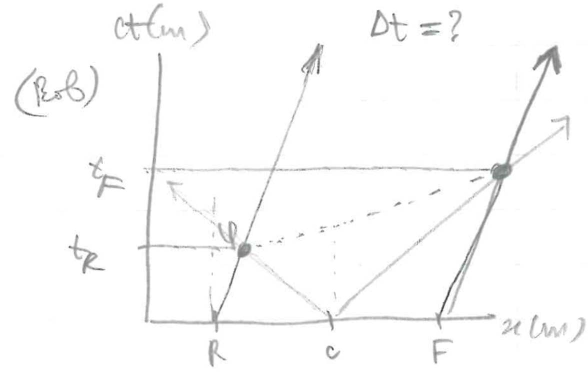
→ Things that are simultaneous in Alice's frame are NOT simultaneous in Bob's frame

$$\frac{\Delta x}{c\Delta t} = \frac{v}{c} = \beta$$

$\Delta t = ?$

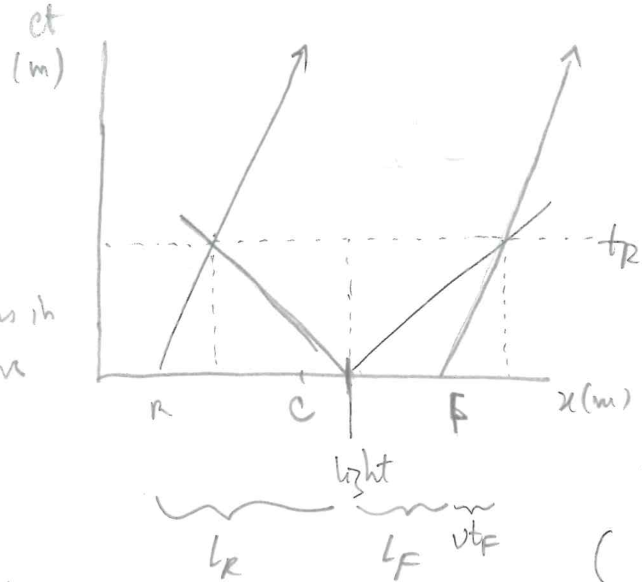
Alternate view Spacetime diagrams





$\boxed{\Delta t = ?}$

Set up so events are simultaneous in Bob's frame

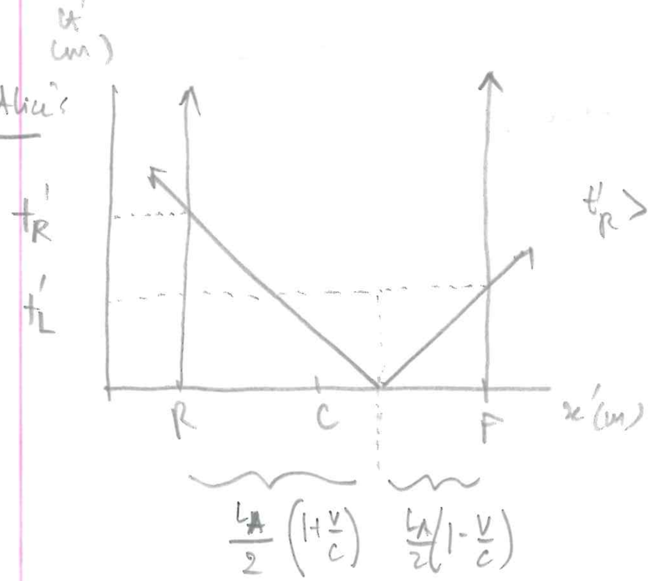


$$\begin{cases} l_R + l_F = L_{Bob} \\ t_R = t_F \\ c t_R = l_R - v t_R \\ c t_F = l_F + v t_F \end{cases}$$

$$t_R = \frac{l_R}{c+v} = t_F = \frac{l_F}{c-v}$$

$$\begin{cases} l_F = \frac{L_B}{2}\left(1 - \frac{v}{c}\right) \\ l_R = \frac{L_B}{2}\left(1 + \frac{v}{c}\right) \end{cases}$$

In Alice's



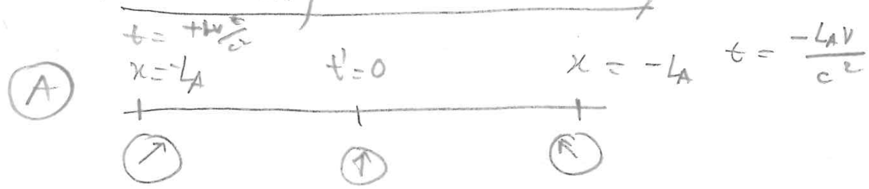
$t'_R > t'_L$

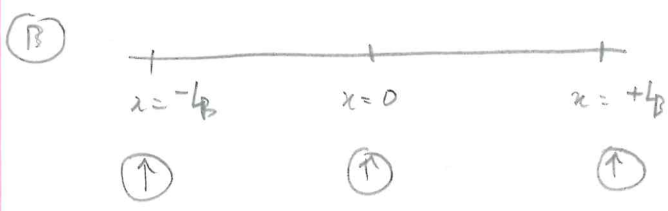
$t'_R - t'_F = ?$

$$\begin{cases} c t'_R = \frac{L_A}{2}\left(1 + \frac{v}{c}\right) \\ c t = \frac{L_A}{2}\left(1 - \frac{v}{c}\right) \end{cases}$$

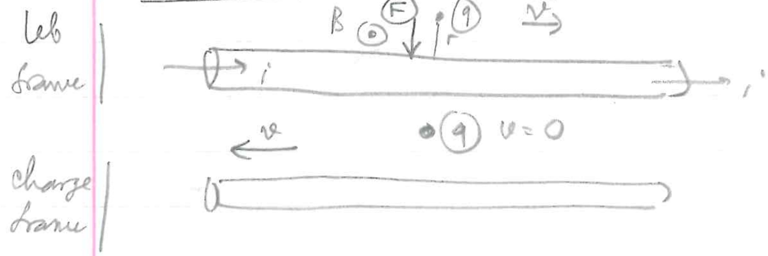
$$\Rightarrow \boxed{\Delta t' = \frac{L_A v}{c^2}}$$ (rear clock ahead)

a. Simultaneity and clock readings





b. Simultaneity and magnetism: an example



$$F_{Lorentz} = q\vec{v} \times \vec{B} = \frac{qv\mu_0 i}{2\pi r}$$

F

Experiment setup

In lab: Bob turns on the source = sink simultaneously

$\Rightarrow$ no net charge

But In charge the sink turns on before source $\Delta t' = \frac{L'v}{c^2}$

q' on wire in charge frame: $-i\Delta t' = \frac{-iL'v}{c^2}$

$$\lambda = \frac{q'}{L'} = \frac{-iv}{c^2}$$

$$E_{charged\ wire} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r} = \qquad F_q = qE$$

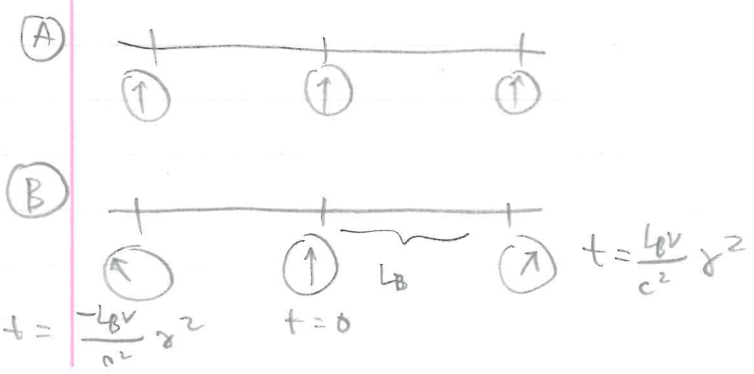
$$F_{Alice} = q\frac{1}{2\pi\varepsilon_0}\cdot\left(\frac{-iv}{c^2}\right)\cdot\frac{1}{r} \overset{?}{\neq} \boxed{qvi\frac{\mu_0}{2\pi r}} \quad \text{while } F = \frac{qiv}{2\pi r}\cdot\frac{1}{\varepsilon_0 c^2} = \boxed{\frac{qvi\mu_0}{2\pi r}}$$

c. Simultaneity and relativity: a cautionary tale

$$\left(\frac{1}{\sqrt{1-v^2/c^2}} = \gamma\right)$$

In Alice's frame

$$\Delta t_B = \frac{L_B v}{c^2}\left[\frac{1}{(1-v^2/c^2)}\right] = \frac{L_B v}{c^2}\gamma^2$$



2. Relativity and How to understand space & time

Rules: { All observers agree on events
The principle of relativity & constancy of speed of light
All situations need to be measured w/ real tools – sticks – clocks

Sept 18, 2017

Recap ① Simultaneity

$$\Delta t_A = \frac{L_A v}{c^2} \quad \longleftarrow \text{ simultaneous for Bob} \quad (\longrightarrow\!+\!\!-)$$

$$\Delta t_B = \frac{\gamma^2 L_B v}{c^2} \quad \longleftarrow \text{ simultaneous for Alice} \quad (\longrightarrow\!+\!\!-)$$

② Determining distances and time interval
a) Invariance of Events
b) Real measurements on the clocks and meterstick

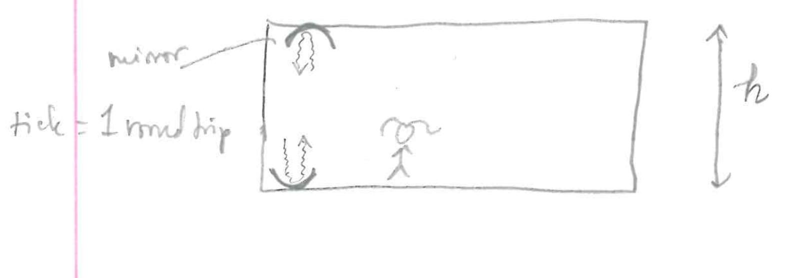
③ Transverse length measurements

Thought exp

Hyp 1 ↕ contracts → contradiction btwn 2 frames
Hyp 2 ↕ expand → "
Hyp 3 ↕ neither contracts / expand...

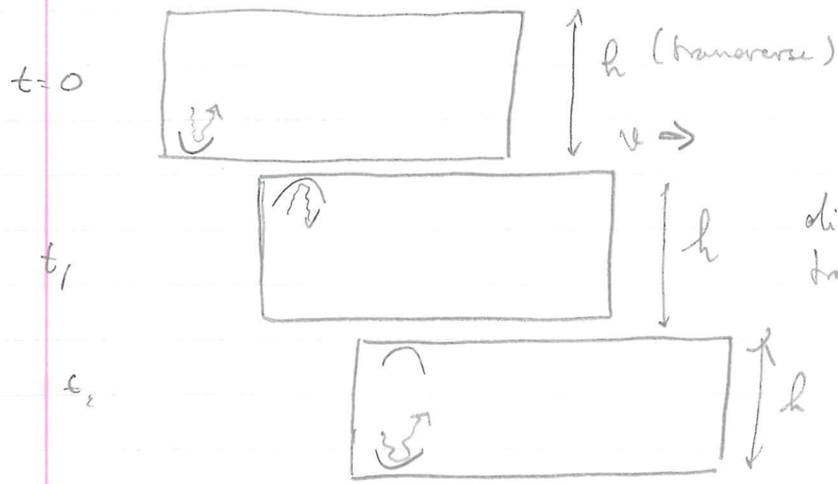
Conclusion – based on invariance of events → moving obj «retain» their stationary transverse dimensions

④ Time dilation
→ moving clocks run slow.
a) A classic proof: a light pulse clock



In Alice's frame $\boxed{\tau_A = \frac{2h}{c}}$

In Bob's frame



distance light travelled $= \sqrt{h^2 + v^2 t^2}$

distance light travelled $= \sqrt{h^2 + v^2 t^2}$

$$\Rightarrow \quad \tau_B = \frac{2\sqrt{h^2 + v^2 \tau_B^2}}{c}$$

$$(c\tau_B)^2 = 4h^2 + v^2\tau_B^2 \quad \therefore \quad \tau_B = \sqrt{\frac{4h^2}{c^2 - v^2}} = \frac{2h}{c} \cdot \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \boxed{\frac{2h}{c}\gamma} = \boxed{\gamma T_A}$$

$\Rightarrow$ $\boxed{\tau_B = \gamma T_A}$ Bob's tick time is <u>larger</u> than A's.

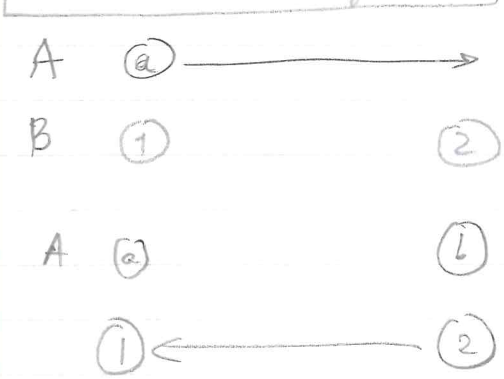
↳ From Bob's perspective → A's clock runs slow.

For Alice: $\boxed{\tau_A = \gamma \tau_B}$ → How?

↳ Because Bob uses 2 clocks to measure τ_A

↳ Comparison is asymmetric

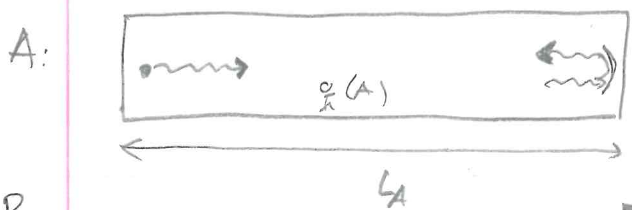
$\boxed{\text{How can this be symmetric?}}$



5 length contraction

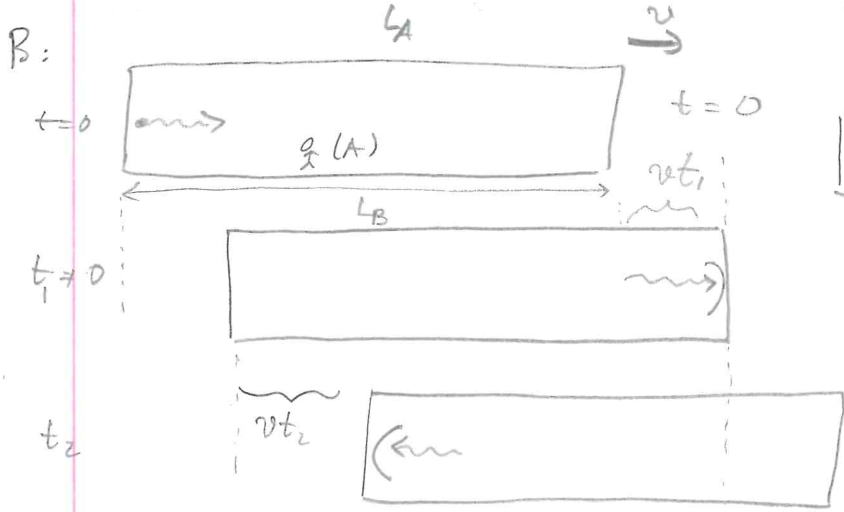
"moving objects are measured to be longitudinally shorter"

A:



$$\boxed{\tau_A' = \frac{2L_A}{c}}$$

B:



$$\boxed{ct_1 = L_B + vt_1} \qquad t_1 = \frac{L_B}{c-v}$$

$$\boxed{ct_2 = L_B - vt_2} \qquad t_2 = \frac{L_B}{c+v}$$

$$\tau_B' = \frac{2L_B c}{c^2 - v^2} = \frac{2L_B}{c}\left(\frac{1}{1 - v^2/c^2}\right) = \boxed{\left(\frac{2L_B}{c}\right)\gamma^2}$$

Since $\tau_B' = \gamma \tau_A' \implies \boxed{L_B = \frac{L_A}{\gamma}}$

6 Reflection on fundamental ~~effects~~ moving

What causes:
1. Clocks to be unsynchronized?
2. Moving clocks run slow?
3. Moving objects to get short?

PERSPECTIVE

7 Time dilation w/ regular clock

Ticks w/ period τ (sends out radio pulse...)

Assume that moving clock ticks with period $g_v \tau$

If clock moves away from me at speed v, I see ticks w/ interval $f_v\tau \neq g_v\tau$

$$f_v\tau = g_v\tau + \frac{v}{c}g_v\tau$$

(every tick adds a distance $vg_v\tau = \Delta d$
extra distance $= \frac{\Delta d}{c} = \frac{v}{c}g_v\tau$)

$$\boxed{f_v = g_v\left(1+\frac{v}{c}\right)}$$

If clock moves toward me

$$f_v\tau = g_v\tau - \frac{v}{c}g_v\tau$$

$$\boxed{f_{-v} = g_v\left(1-\frac{v}{c}\right)}$$

I will also hear light ticks with period τ

but since the ship is towards me $\rightarrow$ $\boxed{\tau'' = f_{-v}\tau'}$

For stationary clock: $\longrightarrow$ I will hear ticks @ period τ

ship hears $f_v\tau$ (moving away from Saturn)
$= \tau'$

$$\boxed{\tau'' = f_{-v}f_v\tau = \tau} \longrightarrow f_{-v}f_v = \left(1-\frac{v}{c}\right)\left(1+\frac{v}{c}\right)g_v^2 = 1$$

$$\boxed{g_v = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}}$$

$$\boxed{\tau_{moving} = \gamma\,\tau_{stat}}$$

Is this real?

half life of μ^-: 150 μs

2000m $\square \longrightarrow$ 570 μ/hr

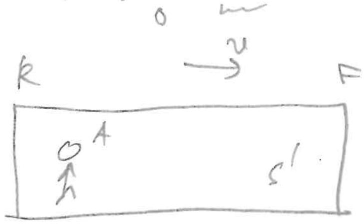
at 3×10^8 m/s
($\hookrightarrow$ 300m/μsec) $\longrightarrow 450 \simeq 500\text{m}/t_{1/2}$

$\square \longrightarrow$ 400 μ/hr

w/o rela			
	2000 m	570 μ/sec	(without time dilation)
	1500 m	285	
	1000 m	140	
	500 m	70	
	0 m	35	

because muon clocks run slow

35 << 4000 ft/hr



If 2 events are simultaneous for Rob → not Alice

$$\Delta t' = \boxed{t_R' - t_F' = \frac{L_A v}{c^2}} \quad (\text{RCA})$$

If 2 events are simultaneous for Alice → not Rob

$$\Delta t^* = \boxed{t_F - t_R = \gamma^2 \frac{L_A v}{c^2}}$$

From Rob

at $t=0$: $t'=0$ $\quad t' = \frac{L_A v}{c^2}$ (RCA)

L_B

F event won't happen until $t=0$

clock advanced by $\Delta t' = \frac{L_A v}{c^2}$

↳ time elapsed for Rob

$$= \Delta t = \gamma \Delta t' = \boxed{\gamma \frac{L_A v}{c^2}}$$

at $t = \frac{\gamma L_A v}{c^2}$: $\underbrace{\quad}_{vt}$ $\quad t' = \left(\frac{L_A v}{c^2} + ?\right)$ $\quad t' = 0$

Δx

$$\boxed{t_F - t_R = \gamma^2 \frac{L_B v}{c^2} = \frac{\gamma L_A v}{c^2}}$$

$$\leftrightarrow \boxed{L_B = \frac{1}{\gamma} L_A}$$

$\Delta x = ?$

$$\frac{\Delta x}{\gamma} = L_A \rightarrow \boxed{\Delta x = \gamma L_A}$$

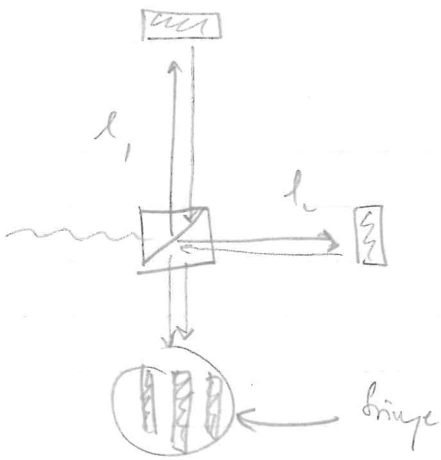
In Bob's frame: $\Delta x = v\Delta t + L_B$

$$= v\cdot\frac{\gamma^2 L_B v}{c^2} + L_B$$

$$\Delta x = L_B\left(\frac{v^2}{c^2}\gamma^2 + 1\right) = L_B\left(\frac{\frac{v^2}{c^2}+1-\frac{v^2}{c^2}}{1-v^2/c^2}\right) = L_B\cdot\gamma^2 = \boxed{\gamma L}$$

The Michelson interferometer and precision measurement

Beam splitter



$$\frac{\Delta T}{T} = \Delta N =$$

$l_1, l_2 \to$ switch roles

↳ fringe-shift

What can you do w/ MI?

1. Look for motion through ether $\frac{\Delta T}{T} = \left(\frac{l_1+l_2}{\lambda}\right)\cdot\frac{v^2}{c^2}$

2. Measure the wavelength of light

Expl

a) Move m1 by distance d $\to \Delta T = \frac{2d}{c} = \frac{2d}{c/n}$ → index of refraction of air

↳ $\frac{\Delta T}{T} = (\Delta N) = \frac{2dn}{c}\cdot\frac{c}{\lambda}$

$$\boxed{\Delta N = \frac{2dn}{\lambda}}$$ (derive this)

fringe shift

$$\lambda_{air}\cdot f_{air} = v_{\text{light in air}} = \frac{c}{n}$$

$$\boxed{\lambda_{air} = \frac{c}{n f_{air}} = \frac{1}{n}\cdot\lambda_{vacuum}}$$

1892 → using the standard meter (Paris) → $\boxed{\lambda_{air}\cdot[1,553,164.13(1)] = 1\ m}$

$\boxed{\lambda_{air} \simeq 643,846958(1)\ nm}$ → can be used to measure distances somewhere else...

(3) Reverse the process

$$\boxed{d = \frac{N \cdot \lambda_{air}}{2}}$$

(4) Mix #2 & #3   k = known, u = unknown

red

$$d = \frac{N_k \lambda_k}{2} \longrightarrow \lambda_u = \frac{2d}{N_u} = \frac{2N_k\lambda_k}{\cancel{N_k}\, 2N_u} = \boxed{\frac{N_k}{N_u}\cdot\lambda_k = \lambda_u}$$

green

UNCERTAINTY ANALYSIS → to determine the "best" value & an estimate of the variation from try to try.

$$\lambda_{best} = \bar{\lambda} = \lambda_{avg}$$

The scatter : $\propto_{\text{mean deviation}} \dfrac{\sum(|\lambda_i - \bar{\lambda}|)^2}{N}$

$\sigma \Longrightarrow$ (stder deviation)

$$\boxed{\text{Variance} := \sigma^2 = \frac{1}{N_{meas}} \sum_{i=1}^{n} (\lambda_i - \bar{\lambda})^2}$$

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(\lambda_i - \bar{\lambda})^2}{N}}$$

Data (d = 2.0 cm)

N	$\lambda = 2d/N$ (m)	$\lambda = 2d/N$ (nm)	$(\lambda - \lambda_{avg})^2$

6. Velocity "addition"

↳ Alice & Bob disagree on time intervals and distances, so we need to be careful about what they would measure as velocity of a particle, too!

Goal: Creat a "real" experiment and determine measurements that est. u' & u → find the relationship.

Two events
- Ball leaves rear of train
- Ball arrives @ front of train

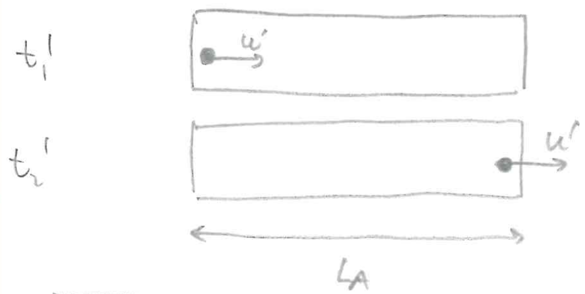
Alice measures $\Delta x'$, $\Delta t'$

$$u' = \frac{\Delta x'}{\Delta t'}$$

Bob measures Δx, Δt

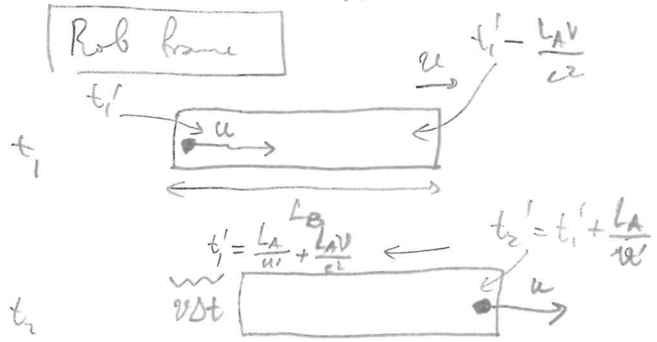
$$u = \frac{\Delta x}{\Delta t}$$

Alice's frame



$$\boxed{\Delta t' = \frac{L_A}{u'}}$$

Bob frame



$$v\Delta t + L_B = u\Delta t$$

$$\boxed{\Delta t = \frac{L_B}{u - v}}$$

Two relationships ① Length contraction: $L_B = \frac{L_A}{\gamma}$ $\quad \frac{L_B}{u-v}$

② Time dilation: → Bob's $(t_2 - t_1) = \gamma\left(t_1' - \left[t_1' + \frac{L_A}{u'} + \frac{L_A v}{c^2}\right]\right)$

relationship between time elapsed on one moving clock vs. difference btwn times readings on stationary clocks...

→ You have to compare 1 clock interval vs. 2 clocks sync

(loss of simultaneity)

$\downarrow$

have to adjust

Bob: $\Delta t = \gamma \Big[\underbrace{\frac{L_A}{u'} + \frac{L_A v}{c^2}}_{\text{time difference of Alice's front clock as corresponding to } \Delta t} \Big] = \frac{L_B}{u-v} = \frac{L_A}{\gamma(u-v)}$

$\therefore \; \gamma\left(\frac{1}{u'} + \frac{v}{c^2}\right) = \frac{1}{\gamma(u-v)}$

$\therefore \; \frac{c^2 - u'v}{u'c^2} = \frac{1}{\gamma^2} \cdot \frac{1}{u-v} = \frac{1}{u-v} \cdot \left(1 - \frac{v^2}{c^2}\right)$

$\therefore \; (c^2 - u'v)(u-v) = u'c^2\left(1 - \frac{v^2}{c^2}\right)$

$\therefore \; c^2u - c^2v - u'uv + u'v^2 = u'c^2 - u'v^2$

$\therefore \; \ldots$

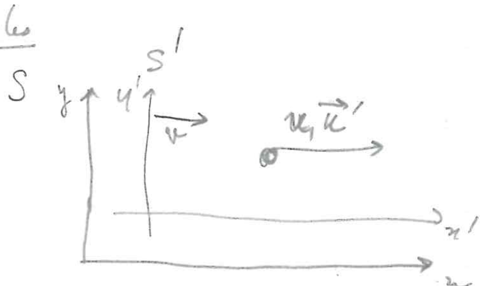
$\therefore \; \frac{1}{u'} = \left(1 - \frac{uv}{c^2}\right) \Big/ u-v \;\; \longrightarrow$

relativistic velocity addition formula

$$\boxed{u' = \frac{u-v}{1-\frac{uv}{c^2}}}$$

$$\boxed{u = \frac{u'+v}{1+\frac{u'v}{c^2}}}$$

Examples



1) $u' = 0.9c$
$v = 0.9c$ $\qquad u = \frac{0.9c + 0.9c}{1 + \frac{0.81c^2}{c^2}} = \frac{1.8c}{1.81} = 0.9945c$

What if $u' = c$? $\qquad u = \frac{c+v}{1+\frac{cv}{c^2}} = \frac{c+v}{1+\frac{v}{c}} = \frac{c(1+v/c)}{1+v/c} = \boxed{c}$ $\leftarrow$ true $\rightarrow$ this works!

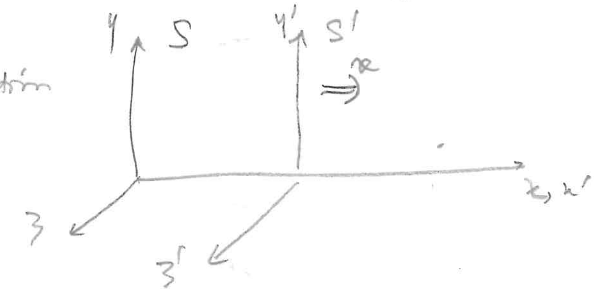
if $u' = 0$? $\qquad u = \frac{v}{1+0} = \boxed{v} \leftarrow$ makes sense.

Sept 22, 2017

E. Lorentz - Einstein Transformation Equations

Goal: find $(t, x, y, z)_S \longrightarrow (t', x', y', z')_{S'}$

Standard Configuration



① $x = x' = 0$ - origins occurs at $t = t' = 0$

$\boxed{\text{Linear transformation of the coordinates}}$ → Why linear?

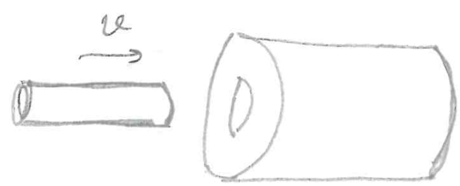
Bcz if not, then constant v in S does not imply constant u' in S'

$$\begin{cases} x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t \\ y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t \\ z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t \\ t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t \end{cases}$$

② Derivation

a) Transverse dimension

→ Thought expt.



$$\rightarrow \begin{cases} x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t \\ y' = y \\ z' = z \\ t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t \end{cases}$$

a) x-t transformation

i) Dependence of x' & t' on y & z

Theorem x', t' can't depend on y & z

Proof Since y & z appear linearly & they must behave the same then

$$a_{12} = a_{13} = a_{42} = a_{43} = 0$$

$$\begin{cases} x' = a_{11}x + a_{14}t \\ y' = y \\ z' = z \\ t' = a_{41}x + a_{44}t \end{cases}$$

ii) Dependence of x' & t' on x, t

$\boxed{\text{Step 1}}$ Consider a light pulse $\boxed{\text{along } +x}$ $\rightarrow \begin{cases} x = ct \\ x' = ct' \end{cases} \rightarrow \begin{cases} x - ct = 0 \\ x' - ct' = 0 \end{cases}$

(1) $\rightarrow \lambda(x - ct) = (x' - ct')$ — arbitrary #

$\boxed{\text{along } -x}$

$\begin{cases} x = -ct \\ x' = -ct' \end{cases} \Rightarrow \begin{cases} x + ct = 0 \\ x' + ct' = 0 \end{cases}$

(2) $\rightarrow \mu(x + ct) = (x' + ct')$

$$\begin{cases} (2) - (1) & 2ct' = (\mu - \lambda)x + c(\mu + \lambda)t \\ (1) + (2) & 2x' = (\mu + \lambda)x + c(\mu - \lambda)t \end{cases}$$

$$\begin{matrix} (3) \\ (4) \end{matrix} \begin{cases} ct' = -bx + act \\ x' = ax - bct \end{cases} \quad \text{where } \begin{cases} a = \left(\dfrac{\mu + \lambda}{2}\right) \\ b = -\left(\dfrac{\mu - \lambda}{2}\right) = \left(\dfrac{\lambda - \mu}{2}\right) \end{cases}$$

(2) Consider the motion of origin of S' in S

$x' = 0 \iff x = vt$

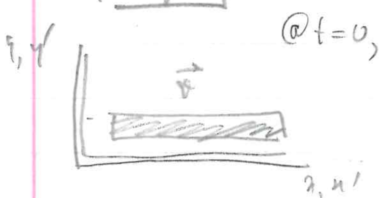
$x' = 0 = a(vt) - b(ct) \rightarrow av = bc \Rightarrow \boxed{b = \frac{v}{c} a}$

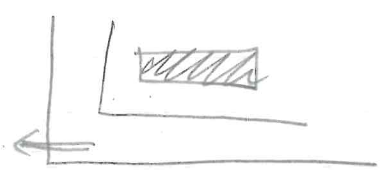
$$(5) \begin{cases} x' = a(x - v/c \cdot ct) = a(x - \beta ct) & (5) \\ ct' = a(ct - \frac{v}{c} x) = a(ct - \beta x) & (6) \end{cases}$$

(3) Apply the principle of relativity

$\boxed{\text{Exp 1}}$

@ $t = 0$, $x_L' = 0$, $x_R' = l_0$





@ $t = 0$ in S $\begin{cases} x_L' = 0 \\ x_R' = l_0 \end{cases} \Rightarrow x_R' = a x_R = \boxed{a l_0 = \frac{1}{a} l'}$ $(\star)$

$\gamma\left(L - v\frac{L}{c}\right) = \gamma L\left(1 - \frac{v}{c}\right) = \gamma L\,\frac{c-v}{c} =$

$= \gamma(L - \beta L) = \frac{1}{\sqrt{1+\beta}\sqrt{1-\beta}}(1-\beta)L = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}L$

at $t'=0$, in s'

$x_L = 0$, $x_R = l_0$

$x_L' = 0$ what is x_R' at $t_1' = 0$?

$S \to S'$ $\begin{cases} ct' = \gamma(ct - \beta x) \\ x' = \gamma(x - \beta ct) \end{cases}$

$S' \to S$ $\begin{cases} ct = \gamma(ct' + \beta x') \\ x = \gamma(x' + \beta ct') \end{cases}$

Goal Solve for t' in terms of t

$(6) \to ct = \frac{ct'}{a} + \frac{v}{c}x$

$\Rightarrow x_R' = a\left(x_R - \frac{v^2}{c^2}x_R - vt'\right)$

at $t'=0$, with $x_R' = l'$, and $x_R = 0$

$$\boxed{l' = a(l_0)\left(1 - \frac{v^2}{c^2}\right)} \quad (\star)$$

by isotropy of space $\to$ $(\star) = (\star)$ i.e. $\frac{l_0}{a} = a l_0\left(1 - \frac{v^2}{c^2}\right)$

$\therefore \quad a^2 = \frac{1}{1-\frac{v^2}{c^2}} \to \boxed{a = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}} = \gamma$

3 The Lorentz-Einstein equations:

$$\begin{cases} x' = \gamma(x - \beta ct) \\ y' = y,\ z' = z \\ ct' = \gamma(ct - \beta x) \end{cases} \quad \text{OR} \quad \begin{cases} x' = \gamma(x - vt) \\ y' = y,\ z' = z \\ t' = \gamma\left(t - \frac{v}{c^2}x\right) \end{cases}$$

4 Some properties of the Lorentz-Einstein Transformations

a) spatial separation, time interval between events.

$\begin{cases} \text{Event 1: } (ct_1, x_1, y_1, z_1)_S \\ \text{Event 2: } (ct_2, x_2, y_2, z_2)_S \end{cases}$

$\Delta x = x_2 - x_1$

$\Delta x' = x_2' - x_1'$

$= \gamma(x_2 - \beta ct_2) - \gamma(x_1 - \beta ct_1)$

$= \gamma(x_2 - x_1) - \gamma\beta c(t_2 - t_1)$

$$\boxed{\Delta x' = \gamma(\Delta x - \beta c\Delta t)}$$

b. Non-relativistic limit

↪ as $\frac{v}{c} \to 0$; as $c \to \infty$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \boxed{\lim_{c\to\infty} \gamma = 1} \to \boxed{x' = x - vt}, \ \boxed{t' = t}$$

≡ Galilean Transform

c. The limiting speed

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \geqslant 1$$

| for $0 \leq v < c$
| for $v = c \to \gamma \to \infty$
| for $v > c \to \gamma \to$ imaginary

There's a speed limit. $\to \boxed{c}$

(No 2 frames can travel at c relative to each other...)

↪ No physical object (with mass) can travel at c/beyond.

d. The limiting speed - really!

↪ c is the limiting speed of any signal of any type (known/unknown)

Underlying Assumptions:

↪ Physics describes sequence of events that are CAUSALLY related.

$\boxed{\text{Experiment}}$

#1 causes #2 by sending a signal with speed u_{signal}

#1 x_1, t_1 #2 x_2, t_2

$$\begin{cases} \Delta x = x_2 - x_1 = \Delta t \cdot u_{signal} \\ \Delta t = \dfrac{\Delta x}{u_{signal}} \end{cases}$$

In S' $\to$ $c\Delta t' = \gamma(c\Delta t - \beta \Delta x)$

$$= \underbrace{\gamma}_{\geq 1}\left(c\Delta t - \frac{v}{c}\cdot \Delta t \cdot u_{signal}\right)$$

$$\boxed{c\Delta t' = \gamma c \Delta t \left(1 - \frac{v}{c^2} u_{signal}\right)}$$

>0

≤ 0 if $u_{sig} > c$ for some $v < c$

If $u_{sig} > c$

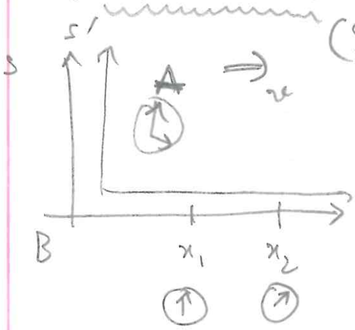
$$\hookrightarrow \frac{v}{c} \cdot \frac{u_{signal}}{c} > 1 \to \frac{v}{c}\left(\frac{u_{sig}}{c}\right) > 1$$

There are some frame S' with $\frac{v}{c} < 1$ in which the order of events are reversed...

↪ violate causality

5 The fundamental effects: consequence of the L-E transformation

a) Time dilation



(S) $\Delta x = x_2 - x_1 = v\Delta t = v(t_2 - t_1)$

(S') $\Delta x' = \gamma\,(\Delta x - \beta c\,\Delta t)$

$= \gamma\left(\Delta x - \frac{v}{c}\cdot c\cdot\frac{\Delta x}{v}\right) = 0$ (in Alice's the clock is at rest)

(S') $c\Delta t' = \gamma(c\Delta t - \beta\Delta x)$

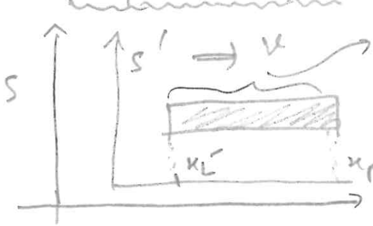
$= \gamma\left(c\Delta t - \frac{v}{c}\cdot v\Delta t\right)$.

$\rightarrow \Delta t' = \gamma\left(\Delta t - \frac{v^2}{c^2}\Delta t\right) = \gamma\Delta t\left(1 - \frac{v^2}{c^2}\right) = \gamma\cdot\Delta t\cdot\left(\frac{1}{\gamma^2}\right) = \boxed{\frac{\Delta t}{\gamma} = \Delta t'}$

time dilation

b). Length contraction

$\boxed{l' = x_R' - x_L'}$, $l' = l_0$ (length in rest frame)

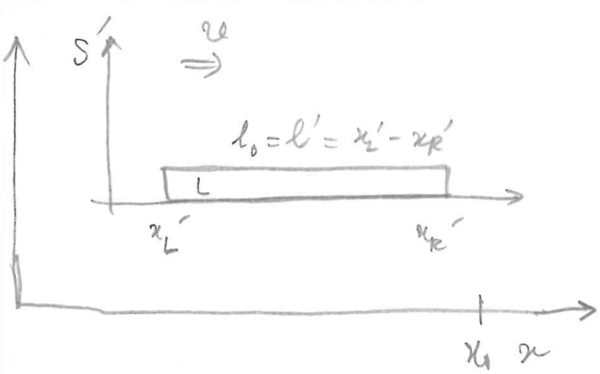


length in (S) is $\boxed{x_R - x_L}$ but when the measurements are made SIMULTANEOUSLY in S $\rightarrow \Delta t = 0$

$\Delta x' = \gamma(\Delta x - \beta c\cancel{\Delta t}^{\,0}) = \gamma\Delta x \rightarrow \boxed{\Delta x' = \gamma\Delta x} \rightarrow \boxed{l = \frac{l_0}{\gamma}}$ (length contraction)

Sept 26

length contraction from another frame



Bob measures 2 events:

\# R @ x_0 } happen at $x = x_0$ but
\# L @ x_0 } @ different times t_1, t_2

But we know:

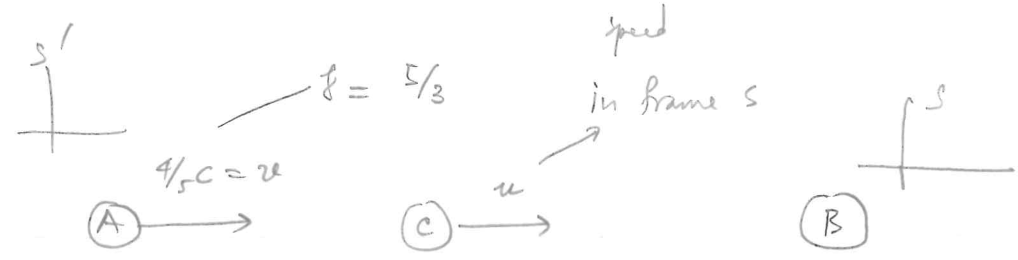
$\frac{l}{v} = \Delta t \Rightarrow l = v\Delta t = v(t_2 - t_1)$

length in S

$\Delta x' = \gamma(\cancel{\Delta x}^{\,0} - \beta c\Delta t) \Rightarrow \Delta x' = \boxed{-l_0} = +\gamma\beta c\Delta t = +\gamma L \longrightarrow \boxed{l = l_0/\gamma}$

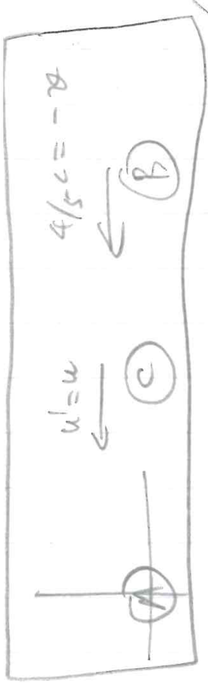
$\hookrightarrow x_L' - x_R' = x_L' - x_R' < 0 = -l_0$

$\boxed{\text{Morin } 1.42}$



What C wants is also have u' in S' $= -u$

For Alice's frames



Velocity Transform $u_x' = \dfrac{u_x - v}{1-\frac{u_x v}{c^2}}$, $u_x = \dfrac{u_x' + v}{1+\frac{u_x v}{c^2}}$

Want : $u_x' = -u_x$

$$u' = \frac{u-v}{1-\frac{uv}{c^2}} = -u \rightarrow u - v = -u + \frac{u^2 v}{c^2}$$

$$\hookrightarrow \frac{v}{c^2}u^2 - 2u + v = 0$$

$$\rightarrow u = \frac{+2 \pm \sqrt{4 - 4^{v^2}/_{c^2}}}{2v/c^2} = \boxed{\frac{c^2}{v}\left(1 \pm \sqrt{1-v^2/c^2}\right) = u}$$

have to pick $(-) \rightarrow u < c$

$$\Rightarrow \boxed{u = \frac{c^2}{v}\left(1 - \sqrt{1-\frac{v^2}{c^2}}\right)} \quad \text{or} \quad \boxed{u = \frac{c^2}{v}\left(1 - \frac{1}{\gamma}\right)}$$

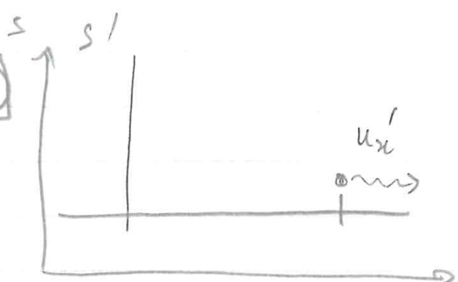
Consider $v \ll c$

$$\hookrightarrow u = \frac{c^2}{v}\left(1 \pm \left(1-\frac{v^2}{c^2}\right)^{1/2}\right) \cong \frac{c^2}{v}\left(1 \pm \cancel{\tfrac{1}{2}}\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)\right) \quad \text{(choose } (-))$$

$$\cong \frac{c^2}{v}\left(1 - 1 + \frac{1}{2}\frac{v^2}{c^2}\right) = \boxed{\frac{1}{2}v}$$

Makes sense $\boxed{u \cong \frac{1}{2}v}$

$\boxed{\text{Bob lemma 8}}$



What is $x(t)$?

$\hookrightarrow x(t) = u_x t + x_0$

In S'

$\boxed{x' = x_0' + u_x' t'}$

$$\underbrace{x' = \gamma\left(x - \beta c t\right)}_{x'} = \cancel{\gamma x}$$

$$= \frac{u_x'}{c}\underbrace{\gamma\left(ct - \beta x\right)}_{ct'} + x_0'$$

$$\gamma(x-\beta ct) = \frac{u_x'}{c}\gamma(ct-\beta x) + \cancel{\gamma}\, x_0'$$

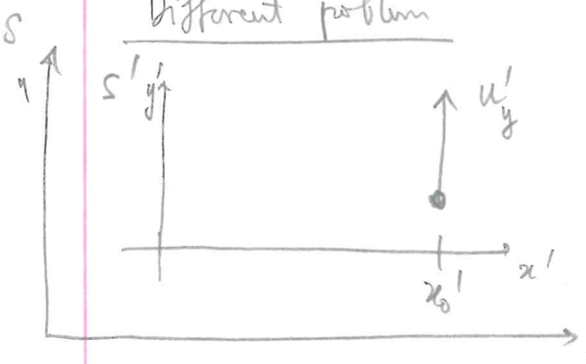
$$\to\ x - vt = \frac{u_x'}{c}\left(ct - \frac{v}{c}x\right) + \frac{x_0'}{\gamma}$$

$$\to\ x - vt = u_x' t - \frac{u_x' v}{c^2}\cdot x + \frac{x_0'}{\gamma}$$

$$\to\ x\left(1+\frac{u_x' v}{c^2}\right) = (u_x'+v)\,t + \frac{x_0'}{\gamma}$$

$$\to\ \boxed{x = \frac{(u_x'+v)}{\left(1+\frac{u_x' v}{c^2}\right)}\,t + \frac{x_0'}{\gamma\left(1+\frac{u_x' v}{c^2}\right)}} \ \to\ @\,t=0 \to x = \frac{x_0'}{\gamma\left(1+\frac{u_x' v}{c^2}\right)}$$

★ Different problem



$$\begin{cases} x'(t') = x_0' \\ y'(t') = y_0' + u_y' t' \end{cases}$$

$$\begin{cases} y' = y \\ \frac{ct'}{c} = \frac{\gamma(ct-\beta x)}{c} \\ \to t' = \gamma\left(t - \frac{v}{c^2}x\right) \end{cases}$$

$$x' = x_0' = \gamma(x - vt) \Rightarrow \boxed{x = \frac{x_0'}{\gamma} + vt}$$

$$y' = y_0' + u_y' t' \Rightarrow y = y' = y_0' + u_y' \cdot \frac{\gamma}{c}(ct - \beta x)$$

$$y = u_y \cdot t + y_0$$

? ?

$$\Rightarrow y = y_0' + u_y'\gamma\left(t - \frac{v}{c^2}x\right)$$

$$\Rightarrow y = y_0' + u_y'\gamma\left(t - \frac{v}{c^2}\left[\frac{x_0'}{\gamma} + vt\right]\right)$$

$$\to y = u_y'\gamma\underbrace{\left(1-\frac{v^2}{c^2}\right)}_{1/\gamma^2}t + u_y'\cancel{\gamma}\left(\frac{-v}{c^2\cancel{\gamma}}\right)x_0' + y_0'$$

$$\boxed{y = \left(\frac{u_y'}{\gamma}\right)t + \left[y_0' - \frac{u_y' v}{c^2}x_0'\right]}$$

Form: $y = u_y t + y_0$

rear clock ahead term (due to loss of simultaneity)

PROPAGATION OF ~~ERROR~~ UNCERTAINTY

mean + uncertainty

Lab $\bar{x} \pm \sigma_x$

$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$ $\qquad \sigma_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2}$

What if we want a number that depends on ≠2 measurements?, each with uncertainty

Examples $A = l \cdot w$ (table) $\longrightarrow$ $\boxed{P = iV}$ $\longrightarrow$ what is σ_P?

$\boxed{\sigma_A \neq \sigma_l \cdot \sigma_w}$ $\qquad \sigma_i \quad \sigma_V$

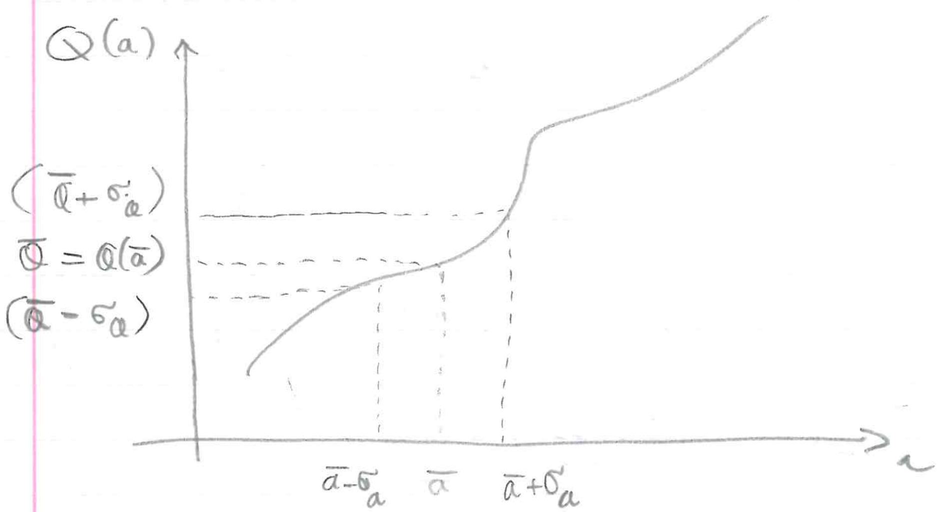
What is σ_A? $\qquad E = \frac{V}{d}$

Assumptions: Uncertainties are uncorrelated (p. 96–101 ; Stat. Data ...)

In general: $Q = Q(a, b, c, \dots)$

$\bar{Q} \pm \sigma_Q = ?$

Example $A = \frac{\pi D^2}{4}$, $Q = Q(a)$



Taylor linear approximation

If $\sigma_a \ll a \rightarrow$ $\boxed{\bar{Q}(a) \cong Q(\bar{a}) + \left.\frac{dQ}{da}\right|_{\bar{a}} (a - \bar{a})}$ $\simeq Q(\bar{a} + \sigma_a)$

$Q(a)$ σ_Q

$$\boxed{Q(\bar{a}+\sigma_a) - Q(\bar{a}) = \left.\frac{dQ}{da}\right|_{\bar{a}} \cdot \sigma_a}$$

$a - \bar{a} = \sigma_a$

<u>Special case</u> $Q = a^n$

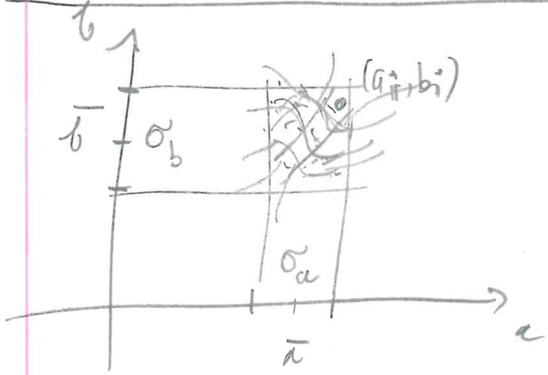
$a \to \bar{a} \pm \sigma_a$

$\bar{Q} \to \bar{a}^n$

$$\sigma_Q = \left(\left.\frac{dQ}{da}\right|_{\bar{a}} \cdot \sigma_a\right) = \left. n a^{n-1}\right|_{\bar{a}} \cdot \sigma_a = (n\bar{a}^{n-1})\,\sigma_a$$

$$\boxed{\sigma_Q = \underbrace{(n\bar{a}^{n})}_{n\bar{Q}} \cdot \underbrace{\left(\frac{\sigma_a}{\bar{a}}\right)}_{\langle \text{fractional uncertainty} \rangle}}$$

What about functions with 2 variables?



$Q = Q(a,b)$

$a = \bar{a} \pm \sigma_a \leftarrow \{a_i\}$

$b = \bar{b} \pm \sigma_b \leftarrow \{b_i\}$

$$\boxed{dQ_i = \left.\frac{\partial Q}{\partial a}\right|_{\bar{a},\bar{b}} \cdot da_i + \left.\frac{\partial Q}{\partial b}\right|_{\bar{a},\bar{b}} \cdot db_i}$$

assumption
⟨Q ≃ linear over range of data⟩

against, linear approximation

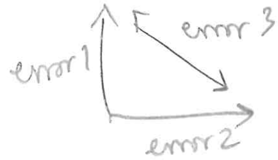
$$\boxed{\Delta Q_i = \left.\frac{\partial Q}{\partial a}\right|_{\bar{a},\bar{b}} \cdot \Delta a_i + \left.\frac{\partial Q}{\partial b}\right|_{\bar{a},\bar{b}} \cdot \Delta b_i}$$

$$(\sigma_Q)^2 = \left[\frac{1}{N}\sum_{i=1}^{N} \Delta Q_i^2\right]$$

$$\boxed{\sigma_a^2 = \frac{1}{N}\sum_{i=1}^{N}\left[\left(\left.\frac{\partial Q}{\partial a}\right|_{\bar{a},\bar{b}} \Delta a_i\right)^2 + 2\left(\left.\frac{\partial Q}{\partial a}\right|_{\bar{a},\bar{b}} \cdot \Delta a_i\right)\left(\left.\frac{\partial Q}{\partial b}\right|_{\bar{a},\bar{b}} \cdot \Delta b_i\right) + \left(\left.\frac{\partial Q}{\partial b}\right|_{\bar{a},\bar{b}} \cdot \Delta b_i\right)^2\right]}$$

$$= \underbrace{\left(\frac{1}{N}\sum_{i=1}^{N}\Delta a_i^2\right)}_{(\sigma_a^2)}\underbrace{\left(\left.\frac{\partial Q}{\partial a}\right|_{\bar{a},\bar{b}}\right)^2} + \underbrace{\left(\frac{1}{N}\sum_{i=1}^{N}\Delta b_i\right)^2}_{(\sigma_b)^2}\underbrace{\left(\left.\frac{\partial Q}{\partial b}\right|_{\bar{a},\bar{b}}\right)^2}_{(\sigma_b)^2} + \underbrace{2\frac{1}{N}\sum_{i=1}\left.\left(\frac{\partial Q}{\partial a}\right)\left(\frac{\partial Q}{\partial b}\right)\right|_{\bar{a},\bar{b}} \cdot \Delta a_i \Delta b_i}_{\text{Co-variance}}$$

(assume $= 0$)



<u>Example</u> ~~If~~ sometimes, covariance $\neq 0$

$$\Rightarrow \boxed{(\sigma_Q)^2 = (\sigma_a)^2 \cdot \left(\frac{\partial Q}{\partial a}\Big|_{\bar{a},\bar{b}}\right)^2 + (\sigma_b)^2 \left(\frac{\partial Q}{\partial b}\Big|_{\bar{a},\bar{b}}\right)^2} \quad (2\text{-var})$$

like Pythagorean

$$\boxed{(\sigma_Q^2) = (\sigma_a)^2 \left(\frac{dQ}{da}\Big|_{\bar{a}}\right)^2} \quad (1\text{-var})$$

↳ Similarly, for n-var ...

<u>Example</u> $Q = a^n b^m$

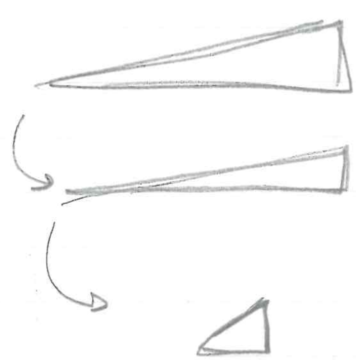
$$\hookrightarrow (\sigma_Q)^2 = (\sigma_a)^2 \left(\bar{a}^{n-1} \bar{b}^m\right)^2 \cdot n^2 + (\sigma_b)^2 \left(\bar{a}^n \bar{b}^{m-1}\right)^2 \cdot m^2$$

$$\hookrightarrow \boxed{\sigma_Q^2 = \left[\frac{(\sigma_a)^2}{\bar{a}^2} \cdot n^2 + \left(\frac{\sigma_b}{\bar{b}}\right)^2 \cdot m^2\right] \cdot \bar{Q}^2}$$

<u>Example</u>

$E = \frac{V}{d}$

$$\sigma_E = \bar{E} \sqrt{\left(\frac{\sigma_V}{V}\right)^2 + \left(\frac{\sigma_d}{d}\right)^2}$$



↳ choose where to improve measurement...

Scratch

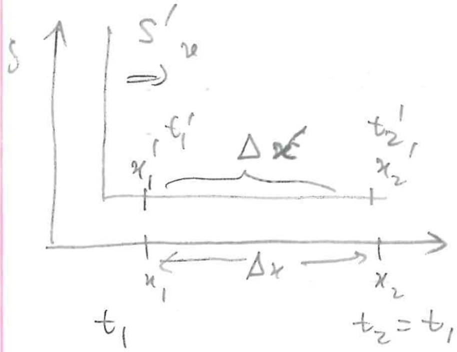
$$\Delta L' = \gamma(L - \beta L)$$

$$= \gamma L (1-\beta)$$

$$= \frac{L\sqrt{1-\beta}}{\sqrt{1+\beta}}$$

Sept 27, 17

c. Relativity of simultaneity



if 2 events simultaneous in S, what's the time interval between S'?

$$\Delta x' = \gamma(\Delta x) = \gamma(\Delta x - \beta c \cancel{\Delta t}^{\,\to 0})$$

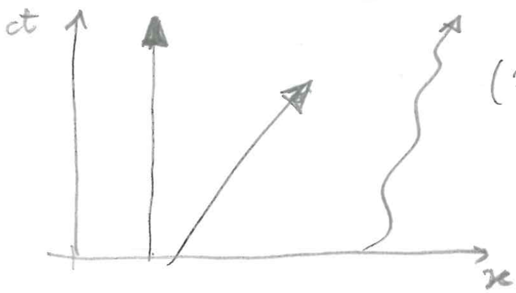
$$c\Delta t' = \gamma(c\cancel{\Delta t}^{\,\to 0} - \beta \Delta x)$$

$$\Delta t' = \frac{-\gamma\beta\Delta x}{c} = -\frac{v}{c^2}\Delta x' = \Delta t'$$

$$\boxed{\Delta t' = \frac{-v\Delta x'}{c^2}}$$

Rear clock ahead.

F. Relativistic Kinematics (description of motion of particles)



$(x(t), y(t), z(t))_S$

$\hookrightarrow (x'(t'), y'(t'), z'(t'))_{S'}$

Given $(u_x(t), u_y(t), u_z(t)) \longleftrightarrow (u_x'(t'), u_y'(t'), u_z'(t'))_{S'}$?

Approach 1

Convert finite intervals to differentials

$$\left\{\begin{array}{l} c\,dt' = \gamma(c\,dt - \beta\,dx) \\ \boxed{dx = u_x\,dt} \\ dx' = \gamma(dx - \beta c\,dt) \\ dy' = dy \\ dz' = dz \end{array}\right. \longrightarrow \left\{\begin{array}{l} c\,dt' = \gamma\left(c\,dt - \frac{v}{c}u_x\,dt\right) = \gamma\left(c - \frac{v}{c}u_x\right)dt \\ dx' = \gamma(u_x - v)\,dt \\ dy' = dy \\ dz' = dz \end{array}\right.$$

a. Longitudinal velocity transform

$$u_x' = \frac{dx'}{dt'} = \frac{(u_x - v)}{1 - \frac{v u_x}{c^2}} \longrightarrow u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}}$$

b. Transverse velocity transform

$$u_y' = \frac{dy'}{dt'} = \frac{dy}{\gamma\left(1-\frac{vu_x}{c^2}\right)dt} = u_y\cdot\frac{1}{\gamma\left(1-\frac{vu_x}{c^2}\right)} = \frac{u_y/\gamma}{\left(1-\frac{vu_x}{c^2}\right)}$$

depends on longitudinal velocity

$$\boxed{u_y = (u_y')\cdot\gamma\left(1-\frac{vu_x}{c^2}\right)}$$

same for dz'

USE THIS $\Longrightarrow$

$$\boxed{u_y = \frac{u_y'}{\gamma\left(1+\frac{u_x' v}{c^2}\right)}}$$

Approach 2 $\quad \frac{dx}{dt} = \frac{dx}{dt'}\cdot\frac{dt'}{dt} = \ldots$

c. Relativistic speed transform

$$\underbrace{\sqrt{u_x^2+u_y^2+u_z^2}}_{u} \xrightarrow{?} \sqrt{u_x'^2+u_y'^2+u_z'^2}\ ?$$

$$u \longrightarrow u'$$

(Solution) $$\underbrace{\left(1-\frac{v^2}{c^2}\right)}_{1/\gamma^2(v)}\underbrace{\left(1-\frac{u^2}{c^2}\right)}_{1/\gamma^2(u)} = \underbrace{\left(1-\frac{u'^2}{c^2}\right)}_{1/\gamma^2(u')}\left(1-\frac{u_x v}{c^2}\right)^2$$

(solution) $$\boxed{\gamma(u') = \gamma(u)\gamma(v)\left(1-\frac{u_x v}{c^2}\right)^2}$$

d. Relativistic acceleration transform

$$a_x = \frac{du_x}{dt}, \quad a_x' = \frac{du_x'}{dt'}$$

$$a_y = \frac{du_y}{dt}, \quad a_y' = \frac{du_y'}{dt'}$$

$$\boxed{(a_x) = (a_x')\frac{1}{\gamma^3\left(1+\frac{u_x' v}{c^2}\right)^3}}$$

$$\boxed{(a_y) = \frac{1}{\gamma^2\left(1+\frac{u_x' v}{c^2}\right)}\left\{a_y' - a_x'\frac{u_y'(v/c^2)}{\left(1+\frac{u_x' v}{c^2}\right)}\right\}}$$

Sept 29, 2017

2) The Relativistic Doppler Effect

a) The Doppler effect for sound

From PH141/142 $\quad \nu = \nu_0 \left[\dfrac{v \mp v_D}{v \pm v_S} \right]$ $\quad \nu$ = frequency

ν: freq. hear $\quad \nu_0$: freq. emitted

$v = v_{sound}$

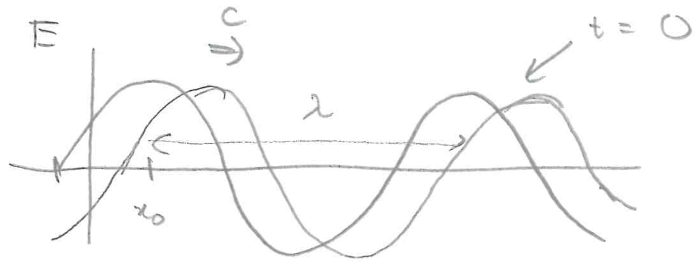
$v_D = v_{Detector}$

$v_S = v_{source}$

frame dependent $\rightarrow$ NOT same as light

b) Light propagation

(i) 1-D



$$E(x, t=0) = E_0 \cos\left(\frac{2\pi}{\lambda}(x - x_0)\right)$$

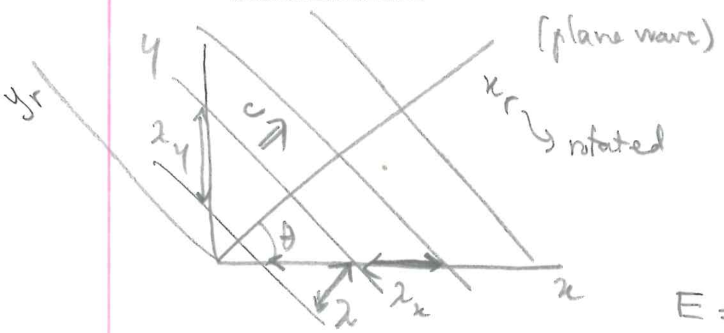
$$E(x,t) = E_0 \cos\left(\frac{2\pi}{\lambda}(x - ct - x_0)\right)$$

$$= E_0 \cos\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{\lambda}ct - \frac{2\pi x_0}{\lambda}\right)$$

- let $k = \dfrac{2\pi}{\lambda}$ $\quad \omega = \dfrac{2\pi c}{\lambda} = \dfrac{2\pi}{T} = 2\pi\nu$
- $-\dfrac{2\pi x_0}{\lambda} = +\varphi_0$

$$\Rightarrow E(x,t) = E_0 \cos\left(kx - \omega t + \varphi_0\right)$$

(ii) 2-D $\quad$ light waves in 2-D



$$E(x_r, t) = E_0 \cos\left(kx_r - \omega t + \varphi_0\right)$$

$x_r = x\cos\theta + y\sin\theta$ $\}$ rotation transform

$$E = E_0 \cos\left[\frac{2\pi}{\lambda}\left[(x\cos\theta + y\sin\theta) - ct\right] + \varphi_0\right]$$

$\lambda_x = \dfrac{\lambda}{\cos\theta}$, $\lambda_y = \dfrac{\lambda}{\sin\theta}$ $\qquad$ Goal: $E(x,y,t) \rightarrow E'(x',y',t')$

c. Doppler Effect for light $\quad$ Given plane wave in S, how to express in S'?

$$E(x,y,t) \rightarrow E\left(x(x',y',t'),\, y(x',y',t'),\, t(x',y',t')\right)$$

$y'=y$

$$E = E_0 \cos\left(\frac{2\pi\gamma}{\lambda}(x'+\beta c t')\cos\theta + \frac{2\pi}{\lambda} y'\sin\theta - \gamma(ct'+\beta x') + \varphi_0\right)$$

$$\boxed{E(x',y',t') = E_0 \cos\left(\frac{2\pi}{\lambda}\gamma(\cos\theta-\beta)x' + \frac{2\pi}{\lambda}\sin\theta\, y' - \frac{2\pi}{\lambda}\gamma(1-\beta\cos\theta)\,ct' + \varphi_0\right)}$$

plane wave traveling thru space in S' too

$$\boxed{E'(x',y',t') = E_0 \cos\left(\frac{2\pi}{\lambda'} x'\cos\theta' + \frac{2\pi}{\lambda'} y'\sin\theta' - \frac{2\pi}{\lambda'} ct' + \varphi_0'\right)}$$

Equate term by term ct' term $\quad -\frac{2\pi}{\lambda}\gamma\cdot(1-\cos\theta\cdot\beta) = \frac{-2\pi}{\lambda'} \Rightarrow \boxed{\lambda' = \frac{\lambda}{\gamma(1-\beta\cos\theta)}} = \boxed{\frac{\lambda\sqrt{1-\beta^2}}{(1-\beta\cos\theta)}}$

and $\boxed{\lambda = \frac{\lambda'\sqrt{1-\beta^2}}{1+\beta\cos\theta'}}$

$$\Rightarrow \boxed{\nu' = \frac{\nu(1-\beta\cos\theta)}{\sqrt{1-\beta^2}}} \longleftrightarrow \boxed{\nu = \frac{\nu'(1+\beta\cos\theta)}{\sqrt{1-\beta^2}}} \quad \text{(Doppler Equations)}$$

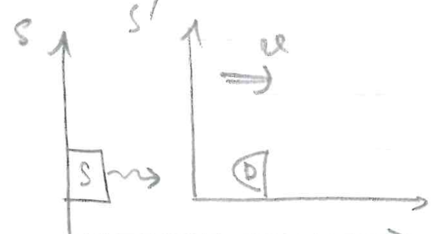
y-term

$$\frac{2\pi}{\lambda'} y'\sin\theta' = \frac{2\pi}{\lambda} y'\sin\theta^* \rightarrow \boxed{\frac{\sin\theta'}{\sin\theta} = \frac{\lambda'}{\lambda} = \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta}}$$

$$\boxed{\cos\theta' = \frac{\cos\theta-\beta}{1-\beta\cos\theta}} \Rightarrow \text{In } S' \text{ to } S \rightarrow \left\{\begin{matrix}\text{angle} \\ \lambda,\nu\end{matrix}\right. \text{ change!}$$

Special cases

Longitudinal



$\theta = 0 \rightarrow \cos\theta' = \frac{1-\beta}{1-\beta} = 1 \rightarrow \boxed{\theta'=0}$

"Red Shift"
{ longer wavelength
shorter freq.

$$\lambda' = \frac{\lambda\sqrt{1-\beta^2}}{1-\beta} \Rightarrow \boxed{\lambda' = \lambda\sqrt{\frac{1+\beta}{1-\beta}}} \rightarrow \boxed{\nu' = \nu\sqrt{\frac{1-\beta}{1+\beta}}}$$

$(\lambda' > \lambda) \qquad (\nu' < \nu) \qquad$ (Receding)

<u>Approaching</u> $\cos\theta = -1 \rightarrow \begin{cases} \lambda' = \lambda\sqrt{\frac{1-\beta}{1+\beta}} & (\lambda' < \lambda) \\ \nu' = \nu\sqrt{\frac{1+\beta}{1-\beta}} & (\nu' > \nu) \end{cases} \rightarrow$ (BLUE SHIFT)

<u>If transverse:</u> $\begin{cases} \nu' = \nu \cdot \gamma \\ \lambda' = \frac{\lambda}{\gamma} \end{cases} \rightarrow$ light traveling y-direction in S

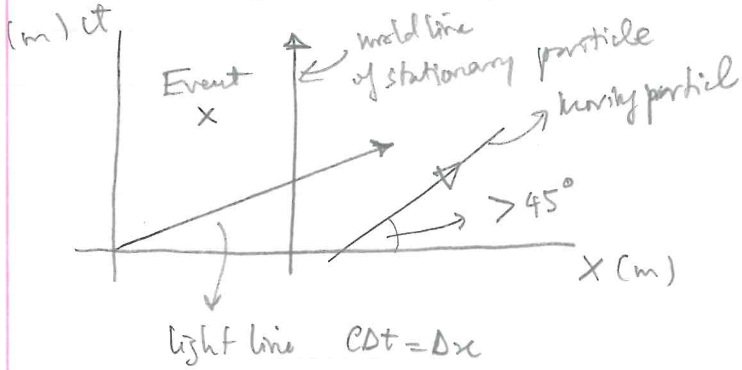
What if light travel in y'-axis in S'? $(\cos\theta' = 0)$

~~Sept~~ $\boxed{\text{Oct 2}}$ ~~Sept~~ $\boxed{\text{G. Spacetime}}$

1.) <u>Minkowski spacetime diagrams & the Lorentz-Einstein transformations</u>

↳ graphical representation of relationship between measurements in dif. frames.

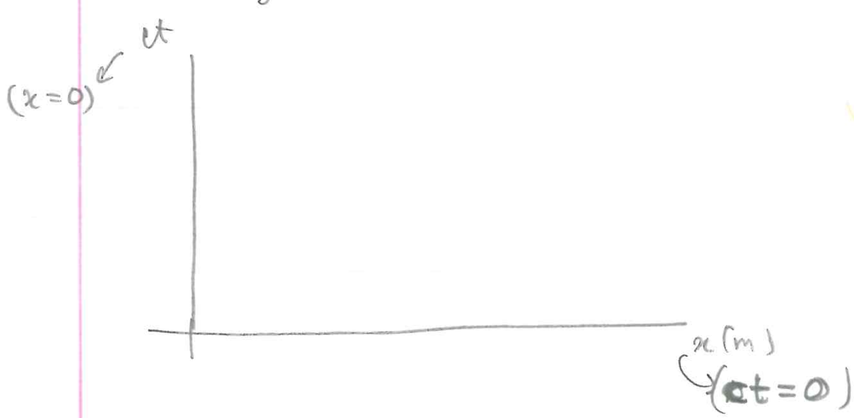
a. spacetime axes, events, and world lines

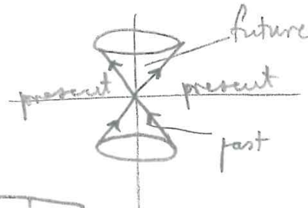


$u = \frac{\Delta x}{\Delta t} \rightarrow \frac{u}{c} = \frac{\Delta x}{c\Delta t} = \frac{1}{\text{slope}}$

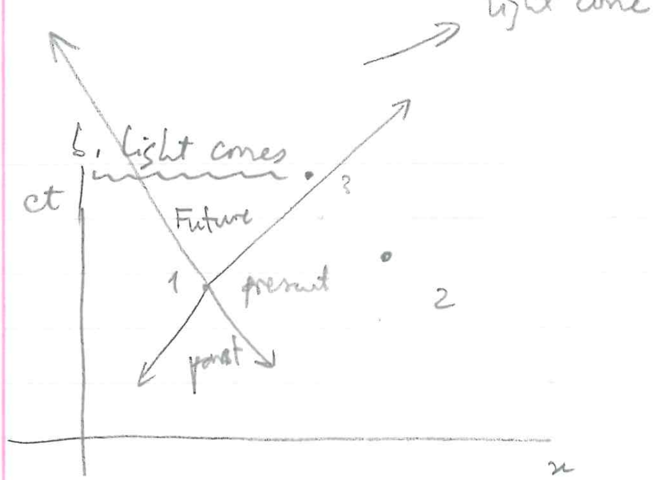
→ line of const x → <u>parallel</u> w (ct)

↳ line of const ct → <u>parallel</u> w (X)





b. light cones



Events 1 & 2

- $\Delta x_{12} = x_2 - x_1 > c\Delta t_{12} = c(t_2 - t_1)$
  ↳ 1 & 2 too far even for light to reach
  ↳ 1 & 2
  ↳ NOT causally related

$\boxed{\text{«spacelike» separated events}}$

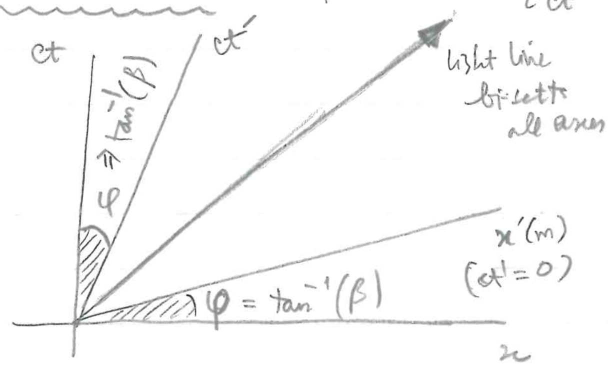
Events 1 & 3

$\Delta x_{13} < c\Delta t_{13} \rightarrow$ light could go from $1 \rightarrow 3$
↳ a sub-luminal signal can be at both events
↳ 1 & 3 can be causally related
$\Rightarrow \boxed{\text{«timelike» separated events}}$

c. The S' axes

→ represent S' $\begin{cases} x' \\ ct' \end{cases}$ on ct, x axes

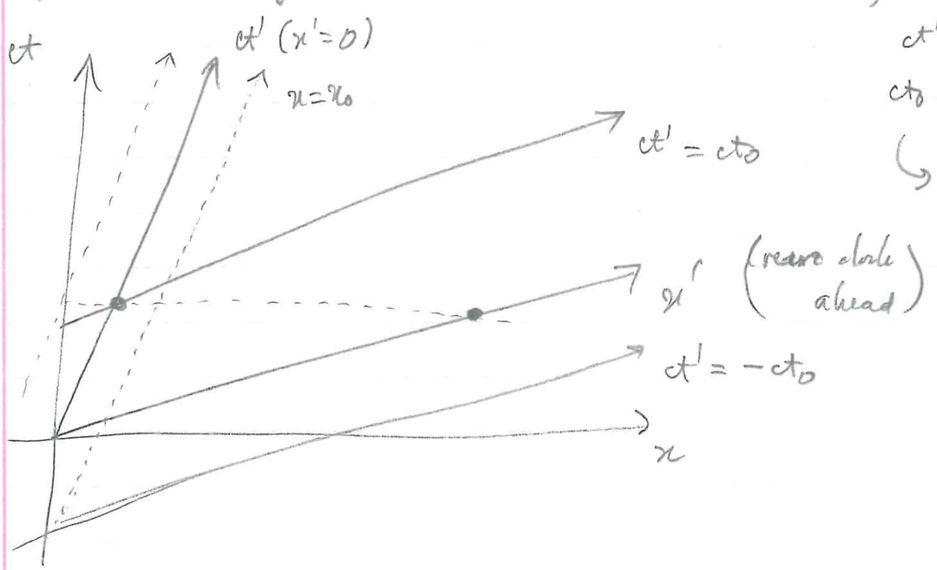
$$\begin{cases} x' = \gamma(x - \beta ct) \\ c\Delta t' = \gamma(c\Delta t - \beta x) \end{cases}$$



ct':
@ $x' = 0 \rightarrow x = \beta ct$
↳ $ct = \frac{1}{\beta}x \rightarrow \left(\text{slope} = \frac{1}{\beta}\right)$

x':
@ $ct' = 0 \rightarrow ct = \beta x$
$\frac{1}{\text{slope}} = \frac{x}{ct} = \frac{1}{\beta} \rightarrow \left(\text{slope} = \beta\right)$

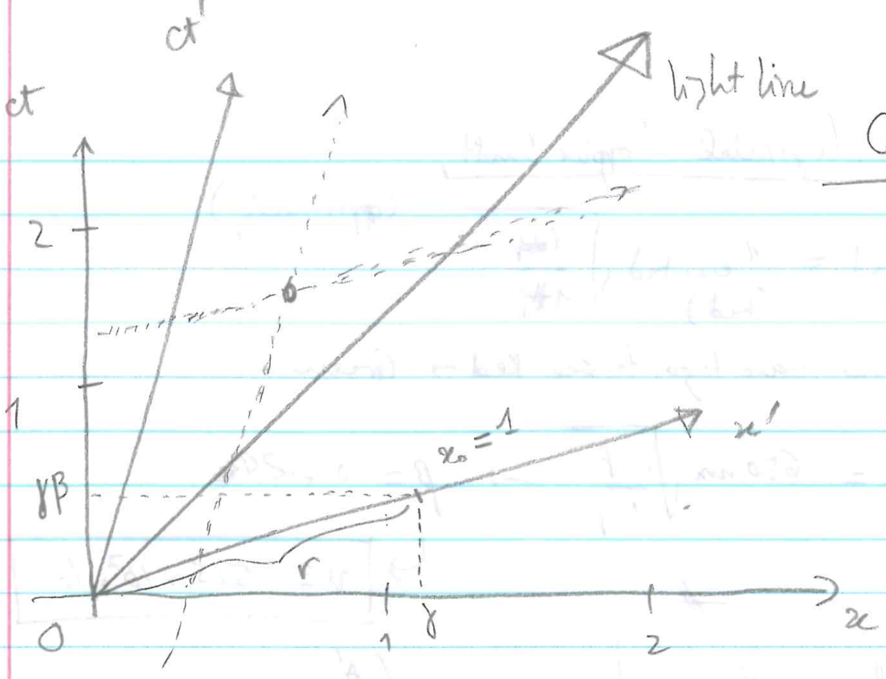
→ NOT an orthogonal transformation (non-Euclidean)



$ct' = ct_0$
$ct_0 = \gamma(ct - \beta x)$
↳ $ct = \frac{ct_0}{\gamma} + \gamma\beta x$

$x' = x_0 = \gamma(x - \beta ct)$
$x = \frac{x_0}{\gamma} + x_0\beta ct$

Calibrating scales



Where is $x' = x_0$ along the x-axis? $\longrightarrow$ $x = \gamma(x' + \beta \cancel{ct'})$

~~$x' = x_0 = \gamma(x - \beta ct)$~~

~~Along x-axis → ct = 0 → $x_0 = \gamma x$~~

$\hookrightarrow \boxed{x = \gamma x_0}$

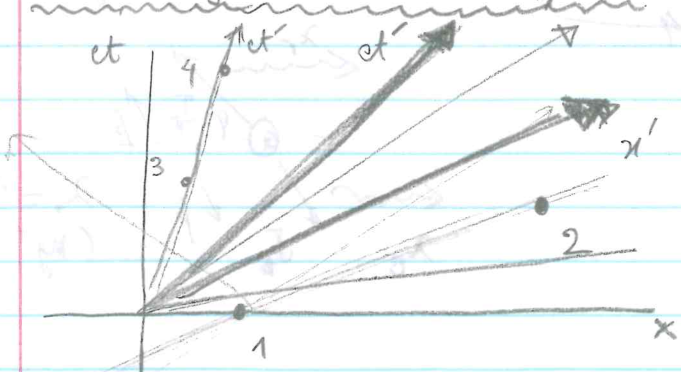
Where $x = \gamma x_0$

$ct = \gamma(ct' + \beta x') = \boxed{\gamma\beta x_0 = ct}$

$r = x_0\sqrt{\gamma^2 + (\gamma\beta)^2} = \cancel{\sqrt{\ldots}} = x_0\sqrt{\gamma^2(1+\beta^2)} = \boxed{x_0\sqrt{\frac{1+\beta^2}{1-\beta^2}} = r}$

x_0 in x'

d. Causality and Minkowski diagrams



$\boxed{1,2}$

These two events are "spacelike" separated (NOT causally related)

↳ in some frame they can be simultaneous

and in other frames the time order can be reversed

$\boxed{3,4}$ → "time like" separated (can be causally related)

↳ can be @ same place in some frame

↳ but time order cannot be reversed

Oct 3

8 Sept

Example of Longitudinal Doppler Shift

$$\lambda_{\text{observed}} = \lambda_{\text{emitted}} \sqrt{\frac{1-\beta}{1+\beta}} \quad \text{(approaching)}$$

(Green) (Red)

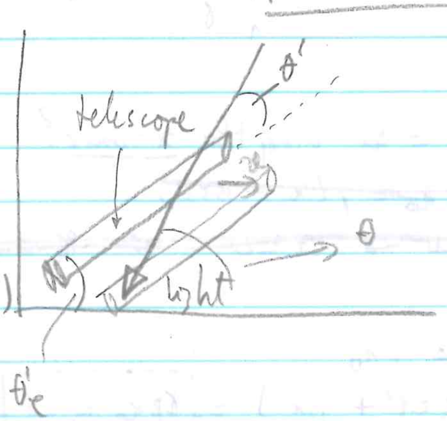
How fast do you have to go to see Red → Green

$$540\,\text{nm} = 650\,\text{nm}\sqrt{\frac{1-\beta}{1+\beta}} \rightarrow \beta = 0.184$$

$$\boxed{v \cong 5.5\times10^{7}\,\text{m/s}}$$

How does Relativity explain Stellar Abberation?

With an ether

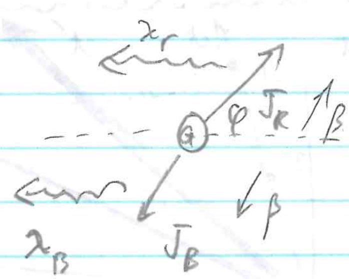


$$\cos\theta'_e = \frac{\cos\theta + \beta}{(1+2\beta\cos\theta+\beta^2)^{1/2}} = \cos(\theta - \theta'_e)$$

With Einsteinean postulates

$$\cos\theta' = \frac{\cos\theta + \beta}{(1-\beta\cos\theta)}$$

→ 2 theories has different prediction → but indistinguishable



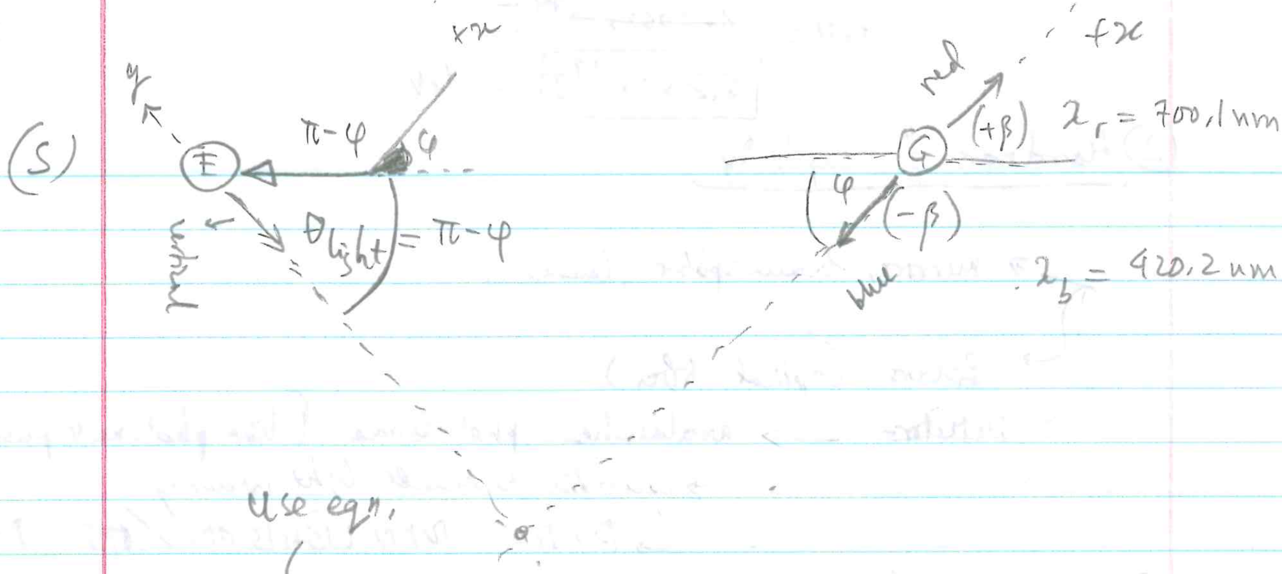
Ⓔ $\lambda_b = 420.2\,\text{nm}$

$\lambda_r = 700.1\,\text{nm}$

$\lambda_0 = 448.1\,\text{nm}$ (Mg^+ line)

φ can be found

What is φ, $v_{jet}(\beta)$? → SOLUTION below



Use eqn.

In S: $\boxed{\lambda' = \frac{\lambda}{\gamma} \cdot \frac{1}{1-\beta\cos\theta}} \Rightarrow \gamma\lambda' = \lambda \cdot \frac{1}{1+\beta\cos\varphi}$

$\Rightarrow \lambda = \gamma\lambda'(1+\beta\cos\varphi)$

$\Rightarrow \begin{cases} \lambda_b = \lambda_0\gamma(1-\beta\cos\varphi) & \text{(approach)} \\ \lambda_r = \lambda_0\gamma(1+\beta\cos\varphi) & \text{(recede)} \end{cases}$

$\Rightarrow \boxed{\lambda_b + \lambda_r = \gamma 2\lambda_0} \Rightarrow \gamma = \frac{\lambda_b+\lambda_r}{2\lambda_0} = \frac{5}{4} \Rightarrow \beta = \frac{3}{5}$

Find φ: $\lambda_r = \lambda_0\gamma(1+\beta\cos\varphi)$

$\Rightarrow \boxed{\varphi = 65^\circ}$

//

Lab Oct 3 — **Clicks, Coincidences, Photons**

① Goal of next 3 experiments

Explore the "fundamental mystery of quantum mechanics"

WAVE – PARTICLE DUALITY

$E(x,t) = E_0\cos(kx-\omega t)$

light $\Rightarrow$ { light is a stream of particles

~~KEeV~~ $E = qV$

$1eV = \cancel{1.9062\times10^{-19}}\,J$

$\boxed{6.2\times10^{-19}J} = 1eV$

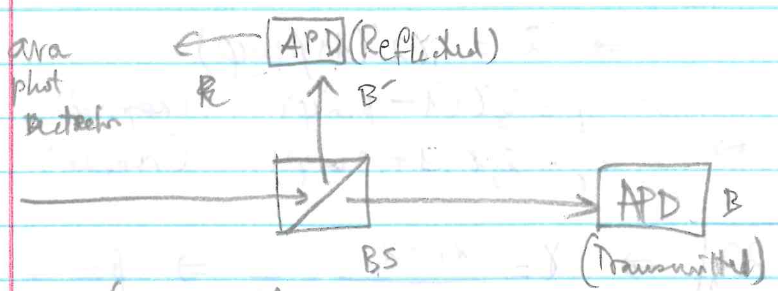
② Hardware for exploring

- → mirrors, beam splitter, lenses
- → fibers (optical fiber)
- → Detectors → avalanche photodiode (like photomultiplier)
  - ↳ sensitive to small light intensity
    - ↳ DO NOT TURN LIGHTS ON / OPEN DOOR

1 count ≢ 1 photon

↳ All we know is that there's enough energy to excite electrons!

③ Goal of #1 Experiment



@ B': Rate of clicks = $R_{B'}$

@ B: Rate of clicks = R_B

$\tau_c << \frac{1}{R_B}$ → ~~time between~~ time sensitivity limit CCC time between clicks

τ_c ($\sim 10^{-8}$ s)

R_B: i can run at high rates... ⇒ I can tell ~~if~~ that events B', B do not happen @ the same time?

- ↳ if wave → same
- ↳ if particle → NOT same

2 photons same place → (••)

But I can accidentally get simultaneous clicks? → YES

ACCIDENTAL COINCIDENCES

What is the probability that (2) is high? → $\boxed{P_2 = \tau_c \cdot R_2}$

$\boxed{\text{What is the rate of accidental coincidences?}}$

~~→ What is the probability that both is high~~?

$P_{12} = \underbrace{P_2}_{\text{prob of } R_2 \text{ high}} \cdot R_1 \rightarrow \boxed{R_{12} = R_1 R_2 \tau_c}$

$R_{accidental}$, in this case

$\boxed{\text{The anticorrelation parameter?}}$

coincidence rate

| measured value on electronics |

$$\alpha_{2D} = \frac{R_{12}}{R_{acc}} = \frac{R_{12}}{R_1 R_2 \tau_c}$$

2-detector

| measured single rates |

expected accidental coincidences.

If $\alpha_{2D} > 1 \rightarrow$ correlated more than by random chance

If $\alpha_{2D} = 1 \rightarrow$ coincidences explainable by random

If $\alpha_{2D} < 1 \rightarrow$ coincidence < random

Expect $\boxed{\alpha_{2D} < 1}$ (if light = wave)

$\boxed{\text{Another view of } \alpha_{2D}}$ Define $\left.\begin{array}{l} P_1 = R_1 \tau_c \\ P_2 = R_2 \tau_c \end{array}\right\} \rightarrow P_{12} = R_{12}\tau_c$

$$\hookrightarrow \alpha_{2D} = \frac{R_{12}}{R_1 R_2 \tau_c} = \frac{P_{12}/\tau_c}{\frac{P_1}{\tau_c} \cdot \frac{P_2}{\tau_c} \cdot \tau_c} = \boxed{\frac{P_{12}}{P_1 P_2}}$$

$\boxed{\text{probability of 2 clicks within } \tau_c}$

Semiclassical Theory of "clicks"

$P_1 \propto$ energy deposited by the light

$$\boxed{P_1 = \underbrace{\eta_1}_{\text{detector efficiency}} \cdot \underbrace{\tau_c}_{\text{how long I wait}} \cdot \langle i_n^{(1)} \rangle}$$

$$\underbrace{i_n^{(1)}}_{\text{average intensity over } \tau_c} = \frac{1}{\tau_c} \int_{t_n}^{t_n + \tau_c} I_1(t)\, dt$$

$$\underbrace{\langle i_n^{(1)} \rangle}_{\text{average over many } \tau_c} = \frac{1}{N} \sum_{n=1}^{N} i_N^{(1)}$$

$$\boxed{P_2 = \underbrace{\eta_2}_{\text{detector efficiency}} \cdot \underbrace{\tau_c}_{\text{how long I wait}} \langle i_N^{(2)} \rangle}$$

detector indices

$$\Rightarrow \boxed{P_{12} = \underbrace{\eta_1 \eta_2}_{\text{total eff.}} \cdot \tau_c^2 \cdot \langle \underbrace{i_N^{(1)} i_N^{(2)}}_{\text{average of product of average intensities}} \rangle} \neq P_1 \cdot P_2$$

$$\Rightarrow \frac{P_{12}}{P_1 P_2} = \frac{\eta_1 \eta_2 \tau_c^2 \langle i_N^{(1)} i_N^{(2)} \rangle}{\eta_1 \tau_c \langle i_N^1 \rangle \cdot \eta_2 \tau_c \langle i_N^2 \rangle} =$$

anti-correlation parameter

$$\boxed{\frac{P_{12}}{P_1 P_2} = \frac{\langle i_N^{(1)} \cdot i_N^{(2)} \rangle}{\langle i_N^1 \rangle \langle i_N^2 \rangle} = \alpha_{2D}}$$

DOES NOT depend on η_1, η_2, and τ_c

$\alpha_{2D} = 1 \Leftrightarrow i_N^1, i_N^2 = \text{constant} \rightarrow \alpha_{2D} \neq 1$ if i fluctuates!

<u>Proof</u> $\langle i_N^{(1)} \rangle = \frac{1}{N} \sum_{n=1}^{N} i_n^{(1)}$ $\langle i_N^{(2)} \rangle = \frac{1}{N} \sum_{n=2}^{N} i_n^{(2)}$

$$\langle i_N^{(1)} i_N^{(2)} \rangle = \frac{1}{N} \sum_{n=1}^{N} i_n^{(1)} \cdot i_n^{(2)}$$

$\boxed{\text{let}}$ $i_n^{(1)} = R \cdot i_n$, $i_n^{(2)} = T \cdot i_n$, $\boxed{R + T = 1}$

constants

$$\boxed{\alpha_{2D} = \frac{\langle RT i_n^2 \rangle}{\langle R i_n \rangle \langle T i_n \rangle} = \frac{\langle i_n^2 \rangle}{\langle i_n \rangle^2}}$$

average of squares of averages

squared of averges

$$\langle i_n^2 \rangle = \frac{1}{N} \sum_{n=1}^{N} i_N^2$$

average of i_n

~~$\langle i_n \rangle^2$~~ $\boxed{\sigma^2 = \frac{1}{N} \sum_{n=1}^{n} (i_N - \langle i_n \rangle)^2}$

1

$$= \underbrace{\frac{1}{N} \sum_{n=1}^{N} i_N^2} - 2 \langle i_n \rangle \underbrace{\frac{1}{N} \sum_{n=1}^{n} i_n}_{\langle i_n \rangle} + \langle i_n \rangle^2 \frac{1}{N} \sum_{i=1}^{N} 1$$

$$= \langle i_N^2 \rangle - 2 \langle i_n \rangle^2 + \langle i_n \rangle^2 \geq 0$$

$$\boxed{\sigma^2 = \langle i_n^2 \rangle - \langle i_n \rangle^2} \geq 0 \rightarrow \boxed{\langle i_N^2 \rangle \geq \langle i_N \rangle^2}$$

<u>Classical wave</u> $\Rightarrow \alpha_{2D} \geq 1$ ~~Since α_{2D}~~

<u>Particle</u> $\Rightarrow \alpha_{2D} < 1 \rightarrow$ <u>signature of photon</u>

Oct 4, 2017

③. $\boxed{\text{The Space-time interval}}$ An Invariant Is $c\Delta t >< = \Delta x$?

→ Is there a number that uniquely and in a frame independent way identifies the "kind" of separation between events?

$\boxed{\text{SPACE-TIME INTERVAL}}$

not really

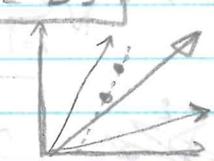
$$\Delta S^2 = c\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

frame Independent (scalar) can take +, -

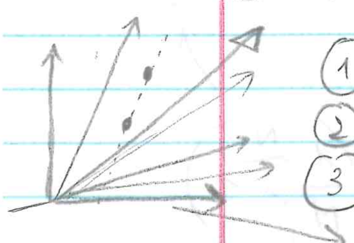
Frame independent → $\Delta S^2 = \Delta s'^2$

$$\begin{aligned}
\Delta s'^2 &= (c\Delta t')^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2 \\
&= \gamma^2(c\Delta t - \beta\Delta x)^2 - \gamma^2(\Delta x - \beta c\Delta t)^2 - \Delta y^2 - \Delta z^2 \\
&= \gamma^2\left((c\Delta t)^2 - \cancel{2c\Delta t\beta\Delta x} + (\beta\Delta x)^2\right) - \gamma^2\left(\Delta x^2 - \cancel{2\beta c\Delta t\Delta x} + \beta^2(c\Delta t)^2\right) - \Delta y^2 - \Delta z^2 \\
&= (\Delta t)^2\cdot(\gamma^2c^2 - \gamma^2v^2) + (\Delta x)^2(\gamma^2\beta^2 - \gamma^2) - \Delta y^2 - \Delta z^2 \\
&= (c\Delta t)^2\cdot(1-\beta^2) - (\Delta x)^2(1-\beta^2)\gamma^2 - \Delta y^2 - \Delta z^2
\end{aligned}$$

$$\boxed{\Delta s' = (c\Delta t)^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = \Delta s}$$



Time-like separated events : $\Delta s^2 > 0$



① $\Delta s^2 > 0$ in all frames → $|c\Delta t| > |\Delta x| \quad \forall s$

② Events can be causally related.

③ a) $\exists$ a frame where the events are colocated

b) Spatial arrangements can be reversed

④ a) No frame in which they are simultaneous (time-like separated)

b) They always have the same time order.

$\boxed{\text{Definitions}}$ The time interval in the frame where the events happen at the same place

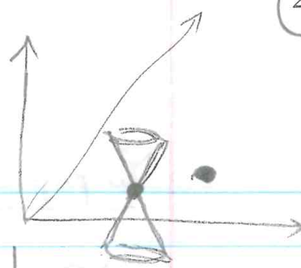
Proper time

$$\boxed{(c\Delta\tau)^2 = \Delta s^2}$$

$$\rightarrow \Delta\tau = \frac{\sqrt{+\Delta s^2}}{c} = \frac{\Delta s}{c}$$

"Proper distance" → Distance between events in the frame where they happen simultaneously

$$\Delta\tau = \sqrt{-\Delta s^2}$$



Spacelike separations $\Delta s^2 < 0$

1. $\Delta s^2 < 0$ in all frames $\rightarrow |c\Delta t| < |\Delta x|$
2. Events can NOT be causally related
3. a) There is NO frame where they are colocated
   b) Spatial arrangements can NOT be reversed
4. a) $\exists$ a frame in which they are simultaneous
   b) Time order can be reversed in some frames

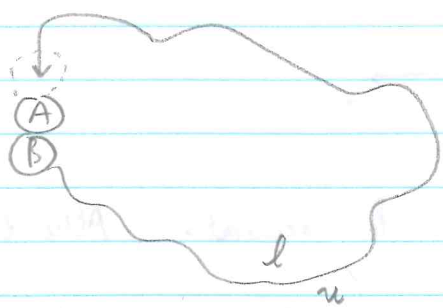
Light-like separation $\Delta s^2 = 0$

1. $\Delta s^2 = 0$ in all frames $\rightarrow |c\Delta t| = |\Delta x|$
2. Can be causally connected only by a signal w/ $v = c$
3. Can neither be SIMULTANEOUS nor COLOCATED in any frame

4 The twin paradox

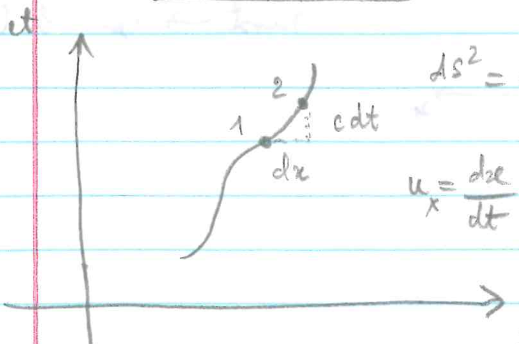
a) Einstein's clock paradox $\rightarrow$ 1905 paper

B younger!



Paradox $\Delta t_{lag} = \dfrac{tu^2}{2c^2} = \dfrac{ul}{2c^2}$ (for $u \ll c$)

b) Elapsed proper time



$$ds^2 = (cdt)^2 - (dx)^2 \rightarrow ds^2 = (cdt)^2 - u_x^2 (dt)^2$$

$$u_x = \frac{dx}{dt} \rightarrow (dx)^2 = u_x^2 dt^2 \rightarrow \boxed{ds^2 = c^2 (dt)^2 \left(1 - \frac{u_x^2}{c^2}\right)}$$

$\rightarrow$ In 3-D

$$\rightarrow \boxed{ds^2 = c^2 (dt)^2 \left(1 - \frac{u^2}{c^2}\right)}$$

<u>The proper time interval between any 2 events</u>

$$\Delta\tau = \int_{t_1}^{t_2} d\tau = \int_{t_1}^{t_2} \sqrt{\frac{ds^2}{c^2}} = \int_{t_1}^{t_2} (dt)\sqrt{1-\frac{u^2}{c^2}} = \int_{t_1}^{t_2} \frac{1}{\gamma(u)}\, dt$$

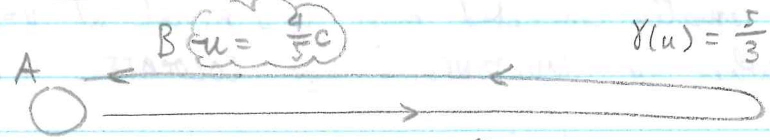
$u = u(t)$

→ Einstein result

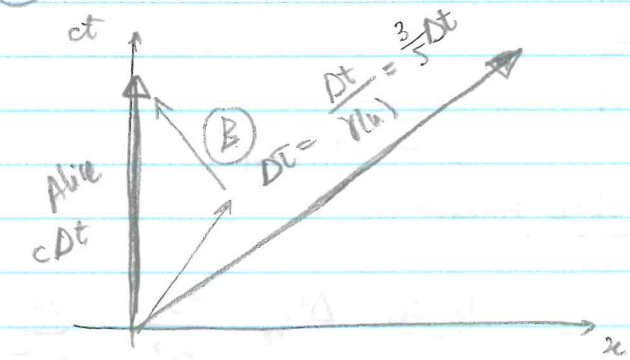
$$\Delta t_{lag} = \frac{l}{u} - \sqrt{1-\frac{u^2}{c^2}} \underbrace{\int_{t_1}^{t_2} dt}_{l/u} = \frac{l}{u}\underbrace{\left(1-\sqrt{1-\frac{u^2}{c^2}}\right)}_{\text{Taylor expand}} \simeq \frac{l}{u}\left(\cancel{1}-\left(\cancel{1}-\frac{1}{2}\frac{u^2}{c^2}\right)\right)$$

$$= \left(\frac{l}{u}\cdot\frac{1}{2}\frac{u^2}{c^2}\right) = \boxed{\frac{lu}{2c^2}}$$

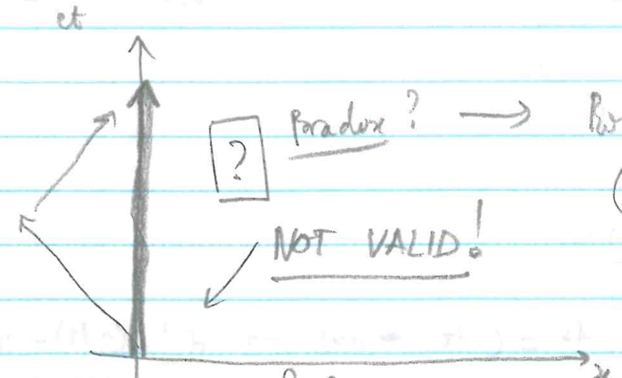
The twin paradox



Alice's frame



Bob's frame



Paradox? → Bob accelerates, Alice doesn't

→ The person who accelerates the most → ages the least

NOT VALID!

since Bob accelerates!

Oct 6, 2017

New goal: Relativistic Dynamics

Galilean dynamics: $\vec{F} = m\vec{a}$, $\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$

Relativistic Dynamics? $\vec{p} = m\vec{u}$ NOT Lorentz invariant

H. Four-vectors

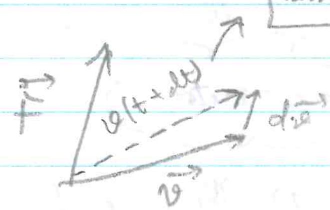
1. Three-vectors

a. The power of vector notation

$\vec{F} = m\dfrac{d\vec{u}}{dt}$ → abbreviation

$F_x = m\,du_x/dt$
$F_y = m\,du_y/dt$
$F_z = m\,du_z/dt$

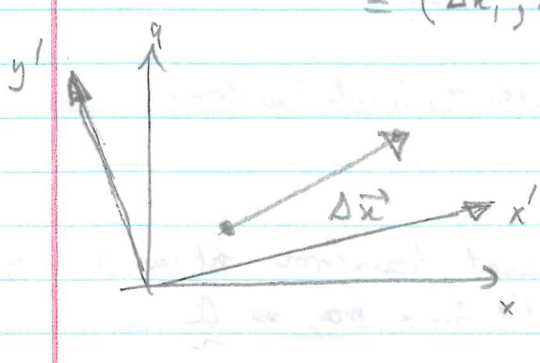
$d\vec{u} = \dfrac{\vec{F}}{m}\,dt$ → frame independent



b. The prototype vector

$\Delta\vec{X} = (\Delta x, \Delta y, \Delta z) = \Delta\vec{r}$
$= (\Delta x_1, \Delta x_2, \Delta x_3)_S = (\Delta x_1', \Delta x_2', \Delta x_3')_{S'}$

(displacement vector)



$\|\Delta\vec{x}\|$ const

$\Delta x_1 \leftrightarrow \Delta x_1'$

Δx invariant under translation/rotation of frames

c. What is a 3-vector?

↳ Is an "object" that transforms the same way as displacements under transformations of (1) rotation of axes
(2) displacement of the origin.

d. What is a scalar?

↳ A number that doesn't change when you change coordinate system

Ex $\|\vec{r}\|^2 = (\vec{r}\cdot\vec{r})$

time

mass

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cdot\cos\varphi$

e) Combination of vectors and scalars

$\vec{F} = m\vec{a}$

$\vec{u} = \frac{d\vec{x}}{dt}$

2. Four-vectors

a) The power of 4-vector notation → simplification $\underset{\sim}{P}_{1i} + \underset{\sim}{P}_{2i} = \underset{\sim}{P}_{1f} + \underset{\sim}{P}_{2f}$
→ frame independent (Lorentz transform independent)

b) The prototype 4-vector

$$\Delta \underset{\sim}{S} = (c\Delta t, \Delta x, \Delta y, \Delta z)_S$$

these transform with the Lorentz transform

c) What is a 4-vector

↳ A set of 4 numbers that transform between relatively moving frames in the same way as $\Delta\underset{\sim}{S}$

$$\boxed{\underset{\sim}{A} = (A_0, A_1, A_2, A_3)_S = (A_0', A_1', A_2', A_3')_{S'}}$$

transform rules

$$\begin{cases} A_0' = \gamma(u)\left(A_0 - \beta A_1\right), & A_2' = A_2 \\ A_1' = \gamma(u)\left(A_1 - \beta A_0\right), & A_3' = A_3 \end{cases}$$

d. What is a four-scalar?

→ A number that doesn't change between frames.

Example

→ c → NOT dt

→ $\Delta\tau$ (proper time)

→ $\boxed{\Delta s^2 = \underset{\sim}{S}\cdot\underset{\sim}{S}}$

$$\boxed{\underset{\sim}{S}\cdot\underset{\sim}{S} = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2}$$

$$\boxed{\underset{\sim}{A}\cdot\underset{\sim}{A} = A_0^2 - A_1^2 - A_2^2 - A_3^2} \rightarrow \text{TRUE}$$

$$\boxed{\underset{\sim}{A}\underset{\sim}{B} = A_0B_0 - A_1B_1 - A_2B_2 - A_3B_2} \longrightarrow \text{Same in all frames}$$

d) Combination of 4-vectors = 4-scalars

$\underset{\sim}{A} \rightarrow c\underset{\sim}{A} = \underset{\sim}{B} \leftarrow$ new four vector

4-velocity

$\left(\underset{\sim}{u} \neq \frac{d\underset{\sim}{S}}{dt}\right)$ ← not a scalar

$$\underset{\sim}{u} = \frac{d\underset{\sim}{S}}{d\tau}$$

3. 4-velocity

$(dt = \gamma(u)\, d\tau) \rightarrow \frac{dt}{d\tau} = \gamma(u)$

$$\underset{\sim}{u} = \frac{d\underset{\sim}{S}}{d\tau} = \frac{d\underset{\sim}{S}}{dt}\underbrace{\left(\frac{dt}{d\tau}\right)}_{\gamma(u)} = \gamma(u)\left(\frac{d}{dt}\underset{\sim}{S}\right) = \gamma(u)\left(c, u_x, u_y, u_z\right)$$

$$\underset{\sim}{u} = \Big(\overset{\text{time}}{\gamma(u)c},\ \overset{\text{space}}{\gamma(u)\vec{u}}\Big)_S \qquad \underset{\sim}{u}' = \Big(\gamma(u')c,\ \gamma(u')\vec{u}'\Big)_{S'}$$

Oct 9, 2017

a) **Four-velocity transform**

transform rule

$$(1)\quad \begin{cases} u_0' = \gamma(v)\left[u_0 - \beta u_1\right] = \gamma(v)\left[\gamma(u)c - \beta\gamma(u)u_x\right] = \gamma(u')c \\ u_1' = \gamma(u)\left[u_1 - \beta u_0\right] = \gamma(v)\left[\gamma(u)u_x - \beta\gamma(u)c\right] = \gamma(u')u_x' \\ u_2' = u_2 = \gamma(u)u_y = \gamma(u')u_y' \\ u_3' = u_3 = \gamma(u)u_z = \gamma(u')u_z' \end{cases}$$

$$(1) \rightarrow \gamma(u') = \gamma(v)\gamma(u)\left(1 - \frac{\beta u_x}{c}\right)$$

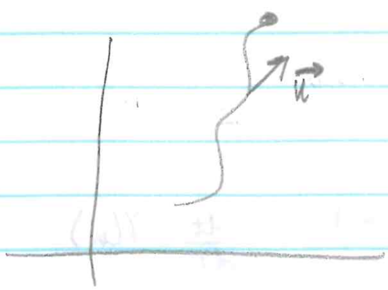
$$\boxed{\left(1 - \frac{u'^2}{c^2}\right)\left(1 - \frac{v u_x}{c}\right) = \left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u^2}{c^2}\right)}$$

b) **Meaning of $\underset{\sim}{u}$**

Magnitude: $(\underset{\sim}{u}\cdot\underset{\sim}{u})^{1/2} = \left[\gamma^2(u)c^2\left(1 - \frac{u^2}{c^2}\right)\right]^{1/2}$

$= c$

direction: along the world line (tangent to the world line)



$dt = \gamma(u)\, d\tau$ $\qquad \frac{d}{dt}\gamma(u) = \frac{d}{dt}\left[1 - \frac{\vec{u}\cdot\vec{u}}{c^2}\right]^{-1/2} = \gamma^3(u)\, \vec{u}\cdot\vec{a}$

3-vector

4. Four acceleration

$$\underset{\sim}{A} = \frac{d\underset{\sim}{u}}{d\tau} = \frac{d\underset{\sim}{u}}{dt} \cdot \frac{dt}{d\tau} = \gamma(u) \cdot \frac{d}{dt}\underset{\sim}{u} = \gamma(u) \cdot \left[\frac{d}{dt}\gamma(u)c,\ \frac{d}{dt}\gamma(u)\vec{u}\right]$$

$$= \gamma(u) \cdot \left[c \cdot \frac{d\gamma(u)}{dt},\ \vec{u}\frac{d\gamma(u)}{dt} + \gamma(u) \cdot \underbrace{\frac{d\vec{u}}{dt}}_{\vec{a}}\right]$$

$$= \gamma(u) \cdot \left[c \cdot \left(-\frac{1}{2}\right) \cdot \left(1 - \frac{\vec{u}\cdot\vec{u}}{c^2}\right)^{-3/2} \cdot (-2\vec{u}\cdot\vec{a}),\ \vec{u}\left(-\frac{1}{2}\right)\left(1 - \frac{\vec{u}^2}{c^2}\right)^{-3/2} \cdot (-2\vec{u}\cdot\vec{a}) + \gamma(u) \cdot \vec{a}\right]$$

$$\underset{\sim}{A}\ \boxed{= \gamma(u) \cdot \left[c\,\gamma^3(u)\vec{u}\cdot\vec{a},\ \gamma^3(u)\,\vec{u}\cdot\vec{a} + \gamma(u) \cdot \vec{a}\right]}$$

I. Relativistic Dynamics

What happens when obj interact?

1. Classical Mechanics

(1) $\vec{F} = m\vec{a}$, $\vec{F}_{12} = -\vec{F}_{21}$

(2)

$$\begin{cases} \sum_{i=1}^{n} m_i \vec{v}_i = \sum_{i=1}^{n} m_i \vec{v}_i{}' = \text{const} & \text{(cons. of momentum)} \\ \sum_{i=1}^{n} m_i = \text{const} & \text{(conservation of mass)} \end{cases}$$

2. Four-momentum

↪ We know that $\boxed{m_1 u_{1i} + m_2 v_{2i} + \dots = m_3 v_{3f} + m_4 v_{4f} + \dots}$

$\Downarrow$ Lorentz transform

$$\boxed{m_1 u_{1i}{}' + m_1 u_{1i}{}' \neq m_3 v_{3f} + m_4 v_{4f}}$$

a) Four-vector momentum

$$\underset{\sim}{P} = m\underset{\sim}{U} = (P_0, \vec{P}) = (\gamma(u)mc, \gamma(u)m\vec{u})_S$$

P_0 · $\vec{P}$ (relativistic 3-momentum)

$\underset{\sim}{P}$ is a four-vector

if $\underset{\sim}{P_1} + \underset{\sim}{P_2} = \underset{\sim}{P_3} + \underset{\sim}{P_4}$

$\Downarrow$ Lorentz-transform

$\underset{\sim}{P_1'} + \underset{\sim}{P_2'} = \underset{\sim}{P_3'} + \underset{\sim}{P_4'}$

$\boxed{\text{3. Interpretation of 4-momentum?}}$ → Meaning of $(P_0, \vec{P})$?

a. Non-relativistic reduction?

let $v \ll c$, $\gamma(u) \cong \left(1 + \frac{1}{2}\frac{u^2}{c^2}\right)$

Spacial $\vec{P} = \gamma(u)m\vec{u} \cong \left(1 + \frac{1}{2}\frac{u^2}{c^2}\right)m\vec{u} \cong \boxed{m\vec{u} = \vec{P}}$

Time $P_0 = \gamma(u)mc \cong \left(1 + \frac{1}{2}\frac{u^2}{c^2}\right)mc \cong mc + \frac{1}{2}mu^2/c$

$$\boxed{cP_0 = mc^2 + \frac{1}{2}mu^2}$$

energy for just having mass. · KE

b). Interpretation of P_0, $\vec{P}$:

$\boxed{\vec{P} = \gamma(u)m\vec{u}}$ ← relativistic momentum

$\boxed{cP_0 = \gamma(u)mc^2}$ ← relativistic total energy

$\hookrightarrow \simeq mc^2 + \frac{1}{2}mv^2 \quad (v \ll c)$

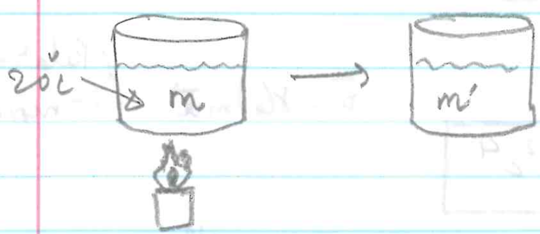
Oct 11, 2017

c. The equivalence of mass & energy

The interpretation of ~~mass~~ cP_0 as total energy implies

$\boxed{E(u=0/\gamma=1) = mc^2}$ ← mass-energy equivalence

Implications

(1) 

$$m' = m + \Delta E / c^2$$

(2) Binding reduces mass

$$p^+ \, e^- \;(m_H) \xrightarrow{13.6\,eV} p^+ + e^- \qquad m_p + m_e = m_H + \frac{13.6\,eV}{c^2}$$

(3)

$$\underbrace{p\,n}_{d} \xrightarrow{+2.23\,MeV} p + n$$

$$m_p + m_n = m_d + \frac{2.23\times10^6\,eV}{c^2}$$

$m_p c^2 \simeq 931\ MeV \qquad m_d c^2 \cong 2 m_p c^2$

$m_n c^2 \simeq 931\ MeV$

$\boxed{\text{The theory works!}}$

d) Relativistic kinetic Energy

$E = \gamma(u) mc^2 \leftarrow$ total energy

$E_0 = mc^2 \leftarrow$ rest energy

$$\boxed{k = E - E_0 = (\gamma(u) - 1) mc^2} \quad (\text{Relativistic } kE)$$

e) The energy-momentum invariant

$$\boxed{\underset{\sim}{P_1} \cdot \underset{\sim}{P_2} = \underset{\sim}{P_1'} \cdot \underset{\sim}{P_2'}} \rightarrow \text{invariant}$$

For a single particle $\boxed{\underset{\sim}{P} \cdot \underset{\sim}{P} = P_0^2 - \vec{P}^2} = \gamma^2(u) m^2 c^2 - \gamma^2(u) m^2 (\underbrace{\vec{u} \cdot \vec{u}}_{u^2})$

$= \gamma^2(u) m^2 c^2 \left(1 - \frac{u^2}{c^2}\right) \leftarrow 1/\gamma^2(u)$

$= \boxed{m^2 c^2}$

$\left(\frac{E}{c}, \vec{p}\right)$

$\parallel$

$$\boxed{p_0^2 - \vec{p}^{\,2} = m^2c^2}$$

$$p = \left(\gamma(u)mc, \gamma(u)m\vec{u}\right)$$

$$\frac{E^2}{c^2} - \vec{p}^{\,2} = m^2c^2$$

$$\vec{p} = \gamma(u)m\vec{u} \quad \begin{pmatrix}\text{Relativistic} \\ \text{3-momentum}\end{pmatrix}$$

$$\boxed{E^2 = c^2p^2 + m^2c^4}$$

Aside on units

$1.6 \times 10^{-19}\,\text{J}$

$\gamma(u)mc^2 \longrightarrow [E] \Rightarrow (\text{J}, \text{eV}, \text{keV}, \text{MeV}, \text{GeV}, \text{TeV})$

$m \rightarrow \left[\frac{E}{c^2}\right] \quad \left(\text{kg}, \frac{\text{eV}}{c^2}, \frac{\text{keV}}{c^2}, \frac{\text{MeV}}{c^2}, \ldots\right)$

People say : mass = 931 eV

means : ~~mass~~ $\boxed{\text{mass} = \frac{931\,\text{eV}}{c^2}}$

$cp \rightarrow (\text{J}, \text{eV}, \ldots)$

$p \rightarrow \left(\frac{\text{kg m}}{\text{s}}, \frac{\text{eV}}{c}, \ldots\right)$

People say : $p = ?\,\text{eV}$ $\longrightarrow$ means $\boxed{p = \frac{?\,\text{eV}}{c}}$

5. Energy-momentum transformation

$$\begin{cases} ct' = \gamma(ct - \beta x) \\ x' = \gamma(x - \beta ct) \end{cases}$$

$$cP_0' = \gamma(v)(cP_0 - c\beta P_1)$$

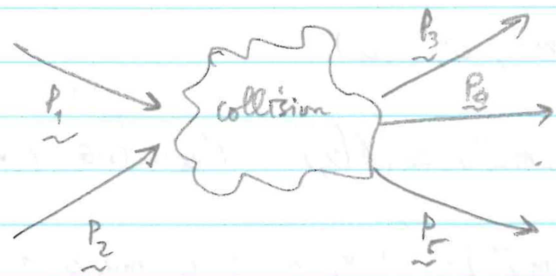
$$(ct, x, y, z)$$
$$(cP_0, P_1, P_2, P_3)$$

$$\boxed{E' = \gamma(v)(E - \beta c P_1)}$$

$$\boxed{P_1' = \gamma(v)\left(P_1 - \beta \cdot \frac{E}{c}\right)}$$

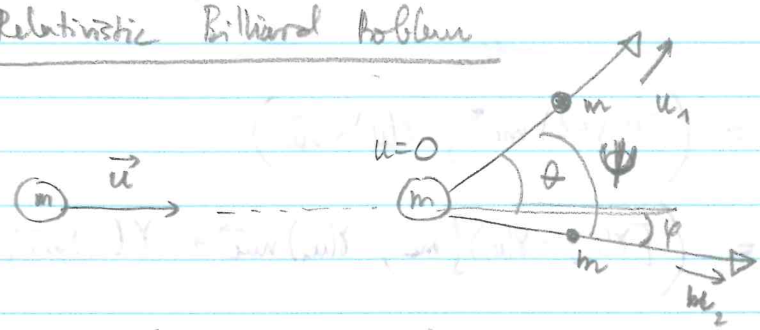
4. Testing four-momentum conservation

↳ Is conservation of 4-momentum a thing?



a) Relativistic Billiard Problem

(elastic collision)



kinematics (conservation laws)
don't alone determine $u_1, u_2, \theta, \varphi$

i) Newtonian mechanics results

$\vec{p}$: $m\vec{u} = m\vec{u}_1 + m\vec{u}_2$

k : $\frac{1}{2}mu^2 = \frac{1}{2}mu_1^2 + \frac{1}{2}mu_2^2 \longrightarrow \boxed{u_1^2 + u_2^2 = u^2}$

$(\vec{P})^2 \Rightarrow (\vec{u})^2 = (\vec{u}_1)^2 + (\vec{u}_2)^2 + 2\vec{u}_1\vec{u}_2$

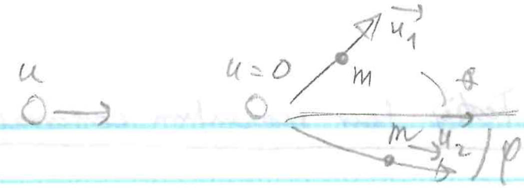
$u^2 = u_1^2 + u_2^2 + 2u_1u_2 \cdot \cos(\vec{u}_1, \vec{u}_2)$

$(\vec{u}_1, \vec{u}_2) = \frac{\pi}{2} \Rightarrow \boxed{\psi = 90^\circ}$

or ($\vec{u}_1$ or $\vec{u}_2 = \vec{0}$)

(Newtonian physics!)

ii.) <u>Relativistic Result</u>



$$\underset{\sim}{P_1} = (\gamma(u) mc, \gamma(u) m\vec{u}) = (\gamma(u) mc, \gamma(u) mu, 0, 0)$$

$$\underset{\sim}{P_2} = (mc, \vec{0}) = (mc, 0, 0, 0)$$

$$\underset{\sim}{P_3} = (\gamma(u_1) mc, \gamma(u_1) m\vec{u_1}) = (\gamma(u_1) mc, \gamma(u_1) m u_1 \cos\theta, \gamma(u_1) m u_1 \cdot \sin\theta, 0)$$

$$\underset{\sim}{P_4} = (\gamma(u_2) mc, \gamma(u_2) m\vec{u_2}) = (\gamma(u_2) mc, \gamma(u_2) m u_2 \cos\varphi, \gamma(u_2) m u_2 \cdot \sin\varphi, 0)$$

(turns out $\varphi + \theta < 90°$)

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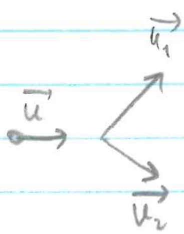
$$\underset{\sim}{P}_{\substack{total \\ initial}} = \left((\gamma(u) + 1) mc, \gamma(u) m\vec{u} \right)$$

$$\underset{\sim}{P}_{\substack{total \\ final}} = \left([\gamma(u_1) + \gamma(u_2)] mc, \gamma(u_1) m\vec{u_1} + \gamma(u_2) m\vec{u_2} \right)$$



$\boxed{\text{Vector (space-like) part}}$:

$$\gamma(u) m\vec{u} = \gamma(u_1) m\vec{u_1} + \gamma(u_2) m\vec{u_2}$$

$$\hookrightarrow \gamma^2(u) \frac{u^2}{c^2} = \gamma^2(u_1) \frac{u_1^2}{c^2} + \gamma^2(u_2) \cdot \frac{u_2^2}{c^2} + 2\gamma(u_1)\gamma(u_2) \cdot \frac{\vec{u_1} \cdot \vec{u_2}}{c^2}$$

$$\boxed{\gamma^2 \frac{u^2}{c^2} = \gamma^2 - 1}$$

$$\hookrightarrow (\gamma^2(u) - 1) = (\gamma^2(u_1) - 1) + (\gamma^2(u_2) - 1) + 2\gamma(u_1)\gamma(u_2) \cdot \frac{\vec{u_1} \cdot \vec{u_2}}{c^2}$$

$$\hookrightarrow \gamma^2(u) = \gamma^2(u_1) + \gamma^2(u_2) - 1 + 2\gamma(u_1)\gamma(u_2) \frac{\vec{u_1} \cdot \vec{u_2}}{c^2} \qquad (\star)$$

$$\frac{\gamma u}{c} = \sqrt{\gamma^2 - 1} =$$

$\boxed{\text{Scalar (time-like) part}}$ $\boxed{(\gamma(u) + 1) = \gamma(u_1) + \gamma(u_2)}$

$$(\star) \Rightarrow \boxed{\gamma(u_1)\gamma(u_2) \cdot \frac{\vec{u_1} \cdot \vec{u_2}}{c^2} = (\gamma(u_1) - 1)(\gamma(u_2) - 1)} \neq 0 \ldots$$

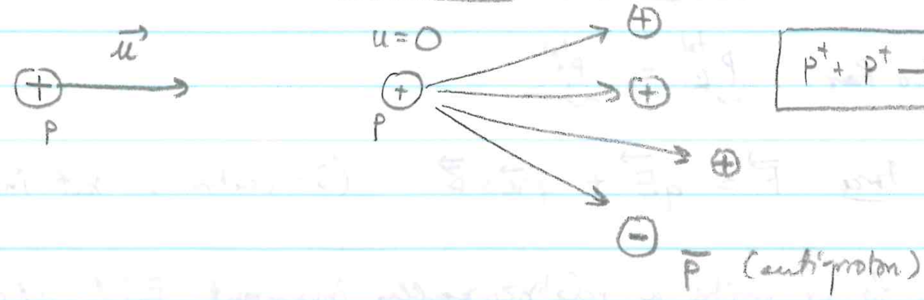
$$\rightarrow \left[\gamma(u_1) \frac{u_1}{c}\right]\left[\gamma(u_2) \cdot \frac{u_2}{c}\right] \cdot \cos\varphi = (\gamma(u_1) - 1)(\gamma(u_2) - 1)$$

$$\boxed{\varphi < 90°}$$

$$\hookrightarrow \boxed{\cos\varphi = \sqrt{\frac{(\gamma(u_1) - 1)(\gamma(u_2) - 1)}{(\gamma(u_1) + 1)(\gamma(u_2) + 1)}}} \Longleftarrow (\star)$$

6. Collisions & Particle Creation

$m_p c^2 = 931.5\ \text{MeV}$



$\boxed{p^+ + p^+ \to 3p^+ + \bar{p}^-}$

energy to mass

Segre & Chamberlain at Berkeley (1955)

↳ "BeVatron" → Target → $E_p = 6.0\ \text{GeV}$

$$\begin{cases} \underset{\sim}{P}_{\substack{initial\\ total}} = \left((\gamma(u_1)+1)mc,\ \gamma(u_1)m\vec{u} \right) \\ \underset{\sim}{P}_{\substack{final\\ total}} = \left([\gamma(u_1)+\gamma(u_2)+\gamma(u_3)+\gamma(u_4)]mc,\ m[\gamma(u_1)\vec{u}_1+\gamma(u_2)\vec{u}_2+\gamma(u_3)\vec{u}_3+\gamma(u_4)\vec{u}_4] \right) \end{cases}$$

What's the threshold $\gamma(\vec{u}^{min})$?

Insight from the "center of mass" frame

$\vec{P}_0 = \vec{0}$

At threshold

↳ $\underset{\sim}{P}_{final} = \left(4\gamma(u_f)mc,\ 4\gamma(u_f)m\vec{u}_f \right)$ along $\vec{u}$

$$\begin{cases} \vec{P}: & \gamma(u)m\vec{u} = 4\gamma(u_f)m\vec{u}_f \\ \frac{cP}{c}: & (\gamma(u)+1)mc^2 = 4\gamma(u_f)mc^2 \end{cases} \longrightarrow \ldots \longrightarrow \boxed{\gamma(u) = 7}$$

$\rightarrow E = \gamma(u)mc^2 = 7\ \text{GeV} \rightarrow$ kinetic energy $= 1\ \text{GeV}$

$$\boxed{\gamma(u)\frac{u}{c} = \sqrt{\gamma^2 - 1}}$$

Oct 18

5. Forces and relativistic Dynamics

So far: $\underset{\sim}{p}_f^{tot} = \underset{\sim}{p}_i^{tot}$

Force $\vec{F} = q\vec{E} + q\vec{u}\times\vec{B}$ (3-vector ... not invariant)

→ How do we write a relativistically invariant EM interaction?

a. Acceleration, velocity, mass

Newtonian: $\vec{F} = m\vec{a} = m\dfrac{d\vec{u}}{dt} = \dfrac{d}{dt}(m\vec{u}) = \dfrac{d\vec{p}}{dt}$

by def: $\vec{u} = \dfrac{d\vec{r}}{dt}$, $\vec{a} = \dfrac{d\vec{u}}{dt}$

Relativistic: 3-vector → 4-vector

$\boxed{\text{3-vector}}$ $\vec{r} = (x,y,z)_S$, $\vec{u}\left(\dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt}\right)_S$, $\vec{a} = \left(\dfrac{du_x}{dt}, \dfrac{du_y}{dt}, \dfrac{du_z}{dt}\right)_S$

$\boxed{\text{4vector}}$ $\underset{\sim}{S} = (ct, x, y, z)_S$

3-vector part ≠ 3-velocity (spatial)

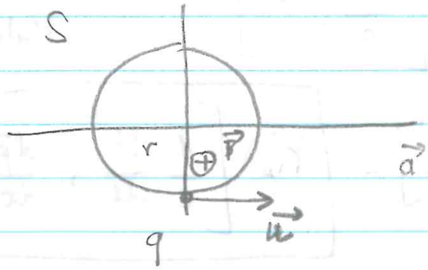
$\underset{\sim}{U} = \dfrac{d\underset{\sim}{S}}{d\tau} = \gamma(u)\dfrac{d\underset{\sim}{S}}{dt} = \left(\gamma(u)c, \gamma(u)\vec{u}\right)_S$

propertime

$$\underset{\sim}{A} = \gamma(u)\frac{d\underset{\sim}{U}}{dt} = \Bigg[\underbrace{\gamma^4(u)\frac{\vec{u}\cdot\vec{a}}{c}}_{\text{time}}, \underbrace{\gamma^4(u)\left(\frac{\vec{u}}{c}\cdot\vec{a}\right)\frac{\vec{u}}{c} + \gamma^2(u)\,\vec{a}}_{\text{space}}\Bigg]$$

$$\underbrace{\vec{a}\cdot\gamma^2(u)\cdot\underbrace{\left(\gamma^2\frac{u^2}{c^2}+1\right)}_{\gamma^2(u)}}_{\gamma^4\cdot\vec{a}}$$

Example Circular motion — 3-acceleration vs. 4-acceleration

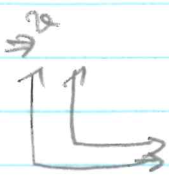


$\vec{a} = \uparrow$ $\qquad \vec{u} = (u, 0, 0)_S \qquad \vec{a} = \left(0, \frac{u^2}{r}, 0\right)_S \qquad \vec{u}\cdot\vec{a} = 0$

$$\left\{\begin{aligned} \underset{\sim}{U} &= \left(\gamma(u)c,\ \gamma(u)u,\ 0,\ 0\right)_S \\ \underset{\sim}{A} &= \left(0,\ 0,\ \gamma^2(u)\frac{u^2}{r},\ 0\right)_S \qquad (\vec{u}\cdot\vec{a} = 0) \end{aligned}\right.$$

(S) is lab frame

(S') frame follows particle!



transform to new frame with $u' = 0$
instantaneously co-moving (S') moving with speed v

$$v = u \Rightarrow \quad \left\{\begin{aligned} A_0' &= \gamma(v)\left(A_0 - \beta A_1\right) = \boxed{0} \\ A_1' &= \gamma(v)\left(A_1 - \beta A_0\right) = \boxed{0} \end{aligned}\right. \qquad \left\{\begin{aligned} A_2' &= A_2 = \boxed{\gamma^2(u)\cdot u^2/r} \\ A_3' &= A_3 = 0 \end{aligned}\right.$$

$$\underset{\sim}{A}' = \left(0,\ 0,\ \gamma^2(u)\frac{u^2}{r},\ 0\right)_{S'} \qquad \begin{matrix}(u' = 0) \\ (v = u)\end{matrix}$$

$$\underset{\sim}{A}_{rest} = \left(\gamma^4(u'=0)\cdot\frac{\vec{0}\cdot\vec{a}}{c},\ \gamma^4(0)\cdot\frac{\vec{0}\cdot\vec{a}}{c}\cdot\vec{0} + \gamma^2(u=0)\cdot\vec{a}'\right)_{S'}$$

$$= \left(0,\ \vec{a}\right)_S,$$

$$\rightarrow \boxed{a' = \gamma^2(u)\frac{u^2}{r}} \qquad \boxed{a'_y \neq a_y}$$

b) The four-~~vector~~ force (Minkowski force)

$$\underset{\sim}{F} = \frac{d\underset{\sim}{P}}{d\tau} = \gamma(u) \cdot \frac{d\underset{\sim}{P}}{dt} = \gamma(u)\frac{d}{dt}\left[\frac{E}{c}, \vec{p}\right]$$

$\gamma(u) m c^2$

$$\hookrightarrow \underset{\sim}{F} = \gamma(u) \frac{d}{dt}\left[\frac{1}{c}\frac{E}{1}, \vec{p}\right] = \boxed{\gamma(u)\left[\frac{1}{c}\frac{dE}{dt}, \frac{d\vec{p}}{dt}\right] = \underset{\sim}{F}}$$

For constant mass particle $\boxed{\underset{\sim}{F} = m\frac{d\underset{\sim}{U}}{dt} = m\underset{\sim}{A}}$

$$\underset{\sim}{F} = \gamma(u)\left[\frac{1}{c}\frac{dE}{dt}, \vec{F}\right]$$ (relativistic 3-force)

Example

$\vec{F} = q\vec{u} \times \vec{B}$ (magnetic analysis of particle)

$$q\vec{u}\times\vec{B} \overset{?}{=} \begin{cases} \dfrac{d\vec{p}}{dt} \leftarrow \boxed{\text{relativistic 3-force}} \\ \cancel{\gamma(u)\dfrac{d\vec{p}}{dt}} \leftarrow \text{✗ vector component of the four-force} \end{cases}$$

$$\underset{\sim}{F} = m\gamma(u)\left[\gamma^3(u)\frac{\vec{u}\cdot\vec{a}}{c}, \gamma^3(u)\frac{\vec{u}\cdot\vec{a}}{c}\frac{\vec{u}}{c} + \gamma(u)\vec{a}\right]$$

For relativistic particles in a $\vec{B}$ field

$q\vec{u}\times\vec{B}$ = "still" makes circular motion

$$\hookrightarrow \frac{d\vec{p}}{dt} = m\gamma(u)\vec{a}$$

$$\Rightarrow \|q\vec{u}\times\vec{B}\| = m\gamma(u)\frac{u^2}{r} \Rightarrow \boxed{r = \frac{m\gamma(u)u}{qB}}$$

only difference from $r = \dfrac{mu}{qB}$

→ can measure r, meaning p rela

J. The General theory of relativity – A brief introduction

Special relativity → consider observers in inertial frames

General relativity → observations in relatively accelerating frames.
↳ and it's a theory of gravity!

(1) Why Einstein included gravity in theory of general relativity?

(a) Newton's gravitation $\vec{F} = m_I \vec{a}$ — $\vec{F} = m_g \vec{g} = m \frac{GM}{r^2}\hat{r}$

gravitational "charge" w/ $\vec{F} = q\vec{E}$

In reality → $m_I = m_g$ → all object fall w/ the same acceleration

Experimentally $m_I = m_g$ to precision of 10^{-11}

(b) Problems with Newton's gravitation (1) $\vec{F} = m \frac{GM}{r^2}\hat{r}$

→ Is not Lorentz – invariant

(2) You can't eliminate gravitational forces → there's always gravitational frame

only attractive g-force →

↳ there's only one sign for m_g ← gravitational charge

(2) The Equivalence principle

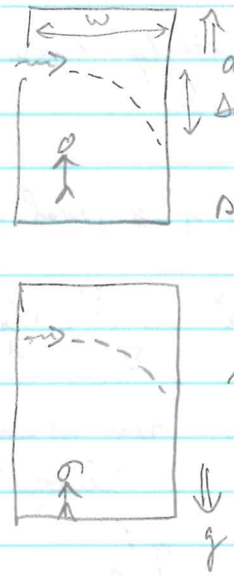
→ In a freely falling frame, you do eliminate gravity
↳ rules of special relativity hold

→ There's no observable difference between a real acceleration + gravity

The "strong" equivalence principle → In freely falling frames, all of the laws of physics obey the rules of special relativity

⇒ $\boxed{\text{Inferences from the equivalent principle}}$

a) freely falling light ← light's path in a gravitational field is bent / curved



$$\Delta y = \frac{1}{2} g t^2 = \boxed{\frac{1}{2} \cdot a \left(\frac{w}{c}\right)^2}$$

$$\Delta t = \frac{w}{c}$$

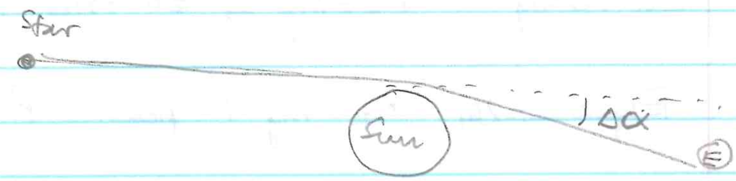
Equivalence principle

$$\Delta y = \frac{1}{2} g t^2 = \boxed{\frac{1}{2} g \left(\frac{w}{c}\right)^2}$$

→ this is wrong by a factor of 2

WRONG

Eddington's Eclipse

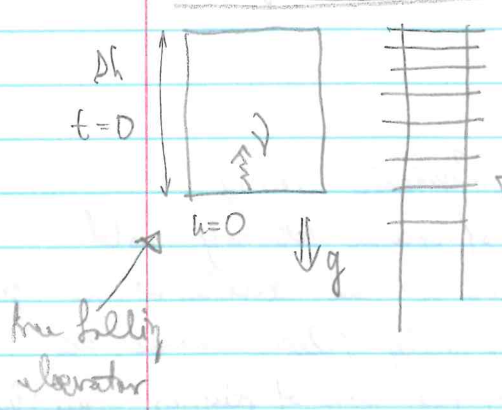


Prediction by GR → $\Delta\alpha = 1.75''$

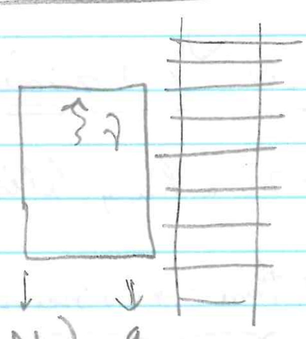
→ Eddington measured $\underline{1.98 \pm 0.2''}$ (1919)

→ Formalont & Sramek $\underline{1.775 \pm 0.02''}$ (1977)

(b) $\boxed{\text{Gravitational time dilation (gravitational red shift)}}$



$$t = \Delta t = \frac{\Delta h}{c}$$



$$\left(v = g \frac{\Delta h}{c}\right)$$

(Observation from elevator) $\rightarrow$ ν is constant

(Observation from earth) $\rightarrow$ $\nu_{bottom} = \nu$ $\quad$ $\nu_{top} = \nu \left[\dfrac{1-v/c}{1+v/c}\right]^{1/2} = \nu\left(1-\dfrac{v}{c}\right)^{1/2}\left(1+\dfrac{v}{c}\right)^{-1/2}$

not (r)

Taylor expand

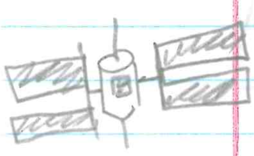
$$\Rightarrow \nu_{top} \simeq \boxed{\nu\cdot\left(1-\frac{1}{2}\frac{u}{c}\right)\left(1-\frac{1}{2}\frac{u}{c}\right)} = \boxed{\nu\left(1-\frac{u}{c}+\underbrace{\frac{u^2}{4c^2}}_{\approx 0}\right)}$$

$$\nu_{top} = \nu\left(1-\frac{u}{c}\right) \rightarrow \boxed{\nu_{top} = \nu_{bottom}\left(1-\frac{g\Delta h}{c^2}\right)}$$ → smaller freq @ top ↳ (red shifted)

$$\tau_{top} = \frac{1}{\nu_{top}}, \quad \tau_{bottom} = \frac{1}{\nu_{bottom}} \Rightarrow \boxed{\tau_{top}} = \frac{1}{\nu}\left(\frac{1+v/c}{1-v/c}\right)^{1/2} = \boxed{\tau_{bottom}\cdot\left(1+\frac{g\Delta h}{c^2}\right)}$$

★ (clocks go faster at higher h) $\rightarrow$ $\boxed{\tau_{top} = \tau_{bottom}\left(1+\dfrac{g\Delta h}{c^2}\right)}$

↳ $\tau_{top} - \tau_{bottom} = \boxed{\Delta\tau = \tau_{bottom}\left(\dfrac{g\Delta h}{c^2}\right)}$ $\rightarrow$ $\Delta\tau = \tau\left(\dfrac{g\Delta h}{c^2}\right)$



Consider ISS: $h = 400\,km$, $R_E = 6.37\times10^6\,m$ } $\rightarrow$ $r_{iss} = 6.77\times10^6\,m = 1.06 R_E$

$$v = \sqrt{\frac{GM}{r}}, \quad T = \frac{2\pi r}{v} = 2\pi\sqrt{\frac{r^3}{GM}} \rightarrow v = 7680\,m/s, \quad G = 6.67\times10^{-11}\,Nm^2/kg^2$$

$(m_E = 5.98\times10^{24}\,kg)$

$\Rightarrow$ { the ISS has $\vec{u}$ → clocks run slow!
the ISS has Δh → clocks run fast! }

$t_B = \dfrac{t_A}{\gamma}$

Special: $$\tau_{ISS} = \left(\tau_g/\gamma(u)\right)\cdot\left(1+\frac{g\Delta h}{c^2}\right) = \tau_g\sqrt{1-u^2/c^2}\cdot\left(1+\frac{g\Delta h}{c^2}\right)$$

$$\simeq \tau_g\left(1-\frac{1}{2}\frac{u^2}{c^2}\right)\left(1+\frac{g\Delta h}{c^2}\right) \simeq \tau_g\left(1-\underbrace{\frac{1}{2}\frac{u^2}{c^2}}_{10^{-10}}+\underbrace{\frac{g\Delta h}{c^2}}_{10^{-11}}-\underbrace{\frac{1}{2}\frac{u^2}{c^2}\frac{g\Delta h}{c^2}}_{0}\right)$$

⇒ For ISS → special relativity wins!

Find h... $\Delta\tau \simeq \tau \frac{g\Delta h}{c^2} \to d\tau = g\frac{\tau}{c^2}\cdot dh \Rightarrow \int_{\tau_{bottom}}^{\tau_{top}} \frac{d\tau}{\tau} = \int_{h=0}^{h} \frac{g}{c^2}\, dh$

$\boxed{e^x \simeq 1+x}$

$\to \ln\left(\frac{\tau_{top}}{\tau_{bottom}}\right) = \begin{cases} g = const \longrightarrow \boxed{\frac{g}{c^2}h} \to \tau_{top} = \tau_{bottom}\, e^{\frac{gh}{c^2}} \\ g = \frac{GM}{h^2} \longrightarrow \int_{h=0}^{h} \frac{1}{c^2}\cdot\frac{GM_e}{(r_e+h)^2}\, dh = \int_{r_e}^{r_e+h} \frac{1}{c^2}\frac{GM_e}{z^2}\, dz \end{cases}$

$= \frac{-GM_e}{c^2}\left(\frac{1}{r_e+h} - \frac{1}{r_e}\right) = \boxed{\frac{GM_e}{c^2}\left(\frac{1}{r_e}-\frac{1}{r_e+h}\right)} = \ln\left(\frac{\tau_{top}}{\tau_{bottom}}\right)$

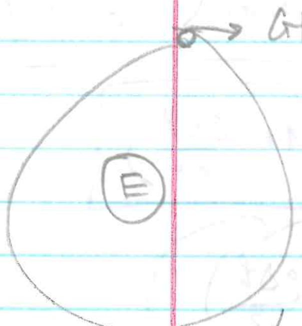
↳ for non-constant g

↳ $\tau_{high} = \tau_{low}\, e^{\frac{GM_e}{c^2}\left(\frac{1}{r_e}-\frac{1}{r_e+h}\right)}$

$\boxed{e^x \simeq 1+x}$

$\tau_{high} \simeq \tau_{low}\cdot\left[1+\frac{GM_e}{c^2}\left(\frac{1}{r_e}-\frac{1}{r_e+h}\right)\right]$

GPS orbit



$r_{GPS} = 4.2\, r_e$

$v = \frac{2\pi r}{12hr}$, $h = 20,200$ km $\to \begin{cases} r_{low} = ? \\ r_{high} = ? \end{cases}$

$\boxed{\tau_{GPS} \simeq \tau_{earth}\left(1+\frac{GM_e}{c^2}\left(\frac{1}{r_e}-\frac{1}{r_e+4.2r_e}\right) - \frac{1}{2}\frac{v^2}{c^2}\right)}$

general relativity (10^{-10})

special relativity (10^{-11})

General Relativity WINS here

⇒ Where is the GOLDEN spot where $\boxed{general = special}$????

Oct 23, 2017

II. QUANTUM PRELIMINARIES

→ new observations and ~~ex~~ old unanswered questions

→ Why do glowing objects have the colors that they do?

$\boxed{\text{Quantum theory}}$ is based on wave-particle duality → probability distribution

→ requires a probability interpretation.

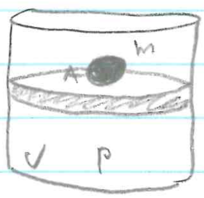
(A) Origins of the quantum theory — the physics of gases

→ from macro to micro

1) The Ideal Gas Law

MACRO RULES

P_0



$$P = P_0 + \frac{mg}{A}, \quad PV = nRT$$

n: no. of moles

R: ideal gas const $8.314472 \frac{J}{mol.K}$

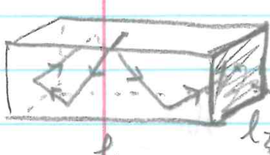
T: absolute temperature (K)

2) The Kinetic Molecular Theory of Gases

→ have a fundamental understanding of the gas law

a) Model of gas → contain a large number of widely separated atoms ~~that~~ exert forces through elastic collisions.

b) Derivation of pressure formula



$$\Delta t_{\text{between 2 collisions}} = \boxed{\frac{2l_x}{v_x}} \quad \rightarrow \quad \vec{F} = \frac{\Delta \vec{p}}{\Delta t}$$

$$J_{(\text{impulse on wall})} = \boxed{\Delta \vec{p} = 2mv_x} \text{ per collision}$$

$$\Rightarrow \langle F_{\text{on the wall, on average}} \rangle = \frac{\Delta \vec{p}}{\Delta t} = \frac{2mv_x}{2l_x/v_x} = \boxed{\frac{mv_x^2}{l_x} = \vec{F}}$$

$$\rightarrow \text{Pressure on right wall} = \langle \vec{P}_n \rangle = \frac{\langle \vec{F} \rangle}{A} \Rightarrow \langle P \rangle = \frac{mv_x^2}{l_x \cdot (l_z \cdot l_y)} = \boxed{\frac{mv_x^2}{V}}$$

$\langle \cdot \rangle$: time average

mv_x^2: 2KE

For many atoms

$$\boxed{\langle P\rangle_{\text{on right wall}}} = \sum_{n=1}^{N} \frac{m v_{xn}^2}{V} = \frac{m}{V}\sum_{n=1}^{N} v_{xn}^2 = \boxed{\frac{Nm}{V}\cdot\langle v_{xn}^2\rangle}$$ N average

Assume isotropy $\Rightarrow \langle v_{xn}^2\rangle = \langle v_{yn}^2\rangle = \langle v_{zn}^2\rangle$

Since $\langle v^2\rangle = \langle v_{xn}^2\rangle + \langle v_{yn}^2\rangle + \langle v_{zn}^2\rangle \Rightarrow \boxed{\langle v^2\rangle = 3\langle v_{xn}^2\rangle}$

$$\langle P\rangle_{\text{right wall}} = \frac{Nm}{V}\cdot\frac{1}{3}\langle v^2\rangle = \boxed{\frac{2}{3}\frac{N}{V}\left(\frac{1}{2}m\langle v^2\rangle\right)} \quad \langle \bar{k}\rangle$$

$$\boxed{\langle P\rangle = \frac{2}{3}\frac{N}{V}\langle \bar{k}\rangle}$$

c) Ideal gas law revisited

$$\underbrace{PV}_{\text{macro}} = \underbrace{\frac{2}{3}N\bar{k}}_{\text{micro}} = \underbrace{nRT}_{\text{macro}} \Rightarrow \boxed{\bar{k} = \frac{3}{2}\left(\frac{n}{N}\right)RT = \frac{3}{2}\left(\frac{R}{N_A}\right)T}$$

Boltzman const

$$\boxed{\langle\bar{k}\rangle = \frac{3}{2}k_B\cdot T} \quad 1{,}381\times10^{-23}\,J/$$

Total energy $\boxed{N\bar{k} = \frac{3}{2}nRT}$

d) Consequences of the kinetic theory

i) Gas diffusion $\boxed{\bar{k} = \frac{1}{2}m\langle v^2\rangle \rightarrow \langle v^2\rangle = \frac{3RT}{N_A m}}$

$\frac{3k_BT}{m}$

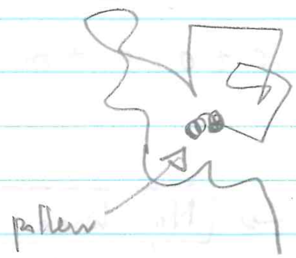
$\|$

$N_A m \downarrow M$

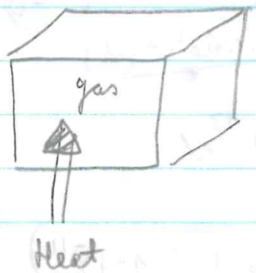
lighter → faster

$$rms = \sqrt{\langle v^2\rangle} = \sqrt{\frac{3RT}{M}}$$

ii) Brownian Motion (1827) — Einstein (1905)



iii) Heat capacity



Can we understand heat capacity with the kinetic theory

Heat capacity: $\begin{cases} C_V = \frac{1}{n} \cdot \frac{\Delta Q}{\Delta T} \Big|_{V=\text{const}} \\ C_p = \frac{1}{n} \cdot \frac{\Delta Q}{\Delta T} \Big|_{p=\text{const}} \end{cases}$

Model $\Delta K_{total} = \frac{3}{2} N k_B \Delta T$

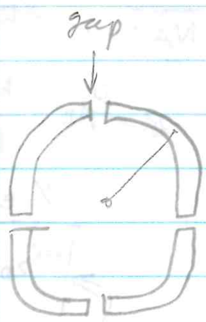
constant?

If assume $\Delta Q = \Delta K$ total $\rightarrow C_V = \frac{1}{n} \cdot \frac{\Delta K}{\Delta T} = \frac{3}{2} N_A k_B = \boxed{\frac{3}{2} R = C_V}$ ($\frac{3}{2} N k_B \Delta T$)

↳ predict: something that's not practical(?)

—//—

Oct 24



BEVATRON — Berkeley, CA

$C = (4\pi r + 4\,\text{gaps})$ $\quad r = \frac{m u \gamma(u)}{qB}$

at injection $k = 10\,\text{MeV}$
at end $k = 6.2\,\text{GeV} = 6200\,\text{MeV}$

$B = \frac{m\gamma(u)u}{qr} = \left(\frac{\gamma(u)u}{c}\right) \cdot \left(\frac{mc^2}{q}\right) \cdot \left(\frac{1}{rc}\right) \leftarrow 2.178 \times 10^{-10}\,\text{s}/\text{m}^2$

for $p^+ \rightarrow$ known $= \boxed{9.38 \times 10^8\,\text{V}}$

$\hookrightarrow B = \frac{\gamma(u)u}{c} \cdot \left(0.2053 \frac{\text{N}}{\text{cm/s}}\right)$ (T)

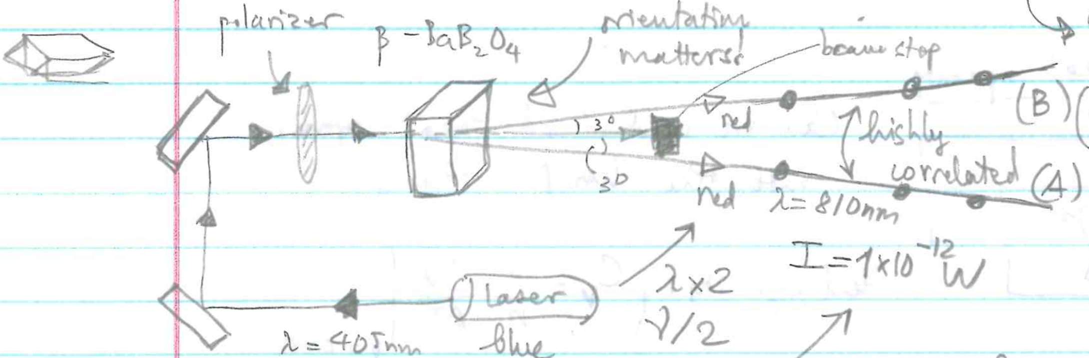
$\rightarrow \begin{cases} B_{10\,\text{MeV}} \simeq \boxed{3 \times 10^{-2}\,\text{T}} \\ B_{6200\,\text{MeV}} \simeq \boxed{1.55\,\text{T}} \end{cases}$

The Grangier Experiment

Classical Waves $\alpha_{2D} \geq 1$ → $\alpha_{2D} = 1.0 \pm 0.04$ → on the border of both theory

$\boxed{\text{Particles } 0 \leq \alpha_{2D} < 1}$

$\boxed{\text{Spontaneous Down Conversion}}$ (SPDC) → $\boxed{\text{Non linear Optics}}$ → different colors $i \neq 0$

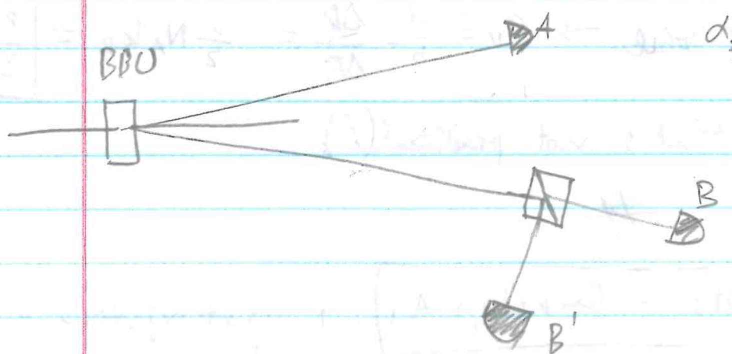


$I = 1 \times 10^{-12}$ W

P = 100mW → «efficient» $(1:10^{-11})$

$\alpha_{2D} \frac{R_{AB}}{R_A R_B \tau_c}$ >>> 1 → much more than random

Correlated — Photons might come @ random rates, but whenever A has photon, B has photon



$\alpha_{3D} = \frac{P_{ABB'}}{P_{AB} \cdot P_{AB'}}$ ← B & B' simultaneous conditioned on getting an event @ A

prob B/B' conditioned on A ~~total~~ space

$$\left\{ \begin{array}{l} P_{ABB'} = \frac{N_{ABB'}}{N_A} \\ P_{AB} = \frac{N_{AB}}{N_A} \quad \& \quad P_{AB'} = \frac{N_{AB'}}{N_A} \end{array} \right.$$

$\alpha_{3D} = \frac{N_{ABB'}}{N_{AB} \cdot N_{AB'}} \cdot N_A$ ← NO τ_c because N_A takes the place of $1/\tau_c$

→ Expect $\alpha_{3D} = 0$

for $R = 10^4$: $\boxed{1250}$; $\boxed{12.50}$

(4% eff)

$\boxed{\alpha_{2D} = \frac{1}{\tau_c R_{total}}}$

$\alpha_{2D} = \frac{R_{AB}}{R_A R_B \tau_c}$ → if perfect correlation $(R_{AB} = R_A = R_B)$

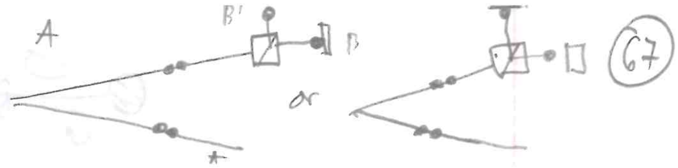
$\boxed{\text{Considerations on } \alpha_{2D} \text{ \& } \alpha_{3D}}$

$R_{total} = 25 R_A$ (4% eff)

α_{2D} (expect) $= \frac{1250}{25} = \boxed{50}$ $10^5 = R_A$

→ $\boxed{500}$ $10^4 = R_A$

Recall $R_{acc} = R_A R_B \tau_c$



$\boxed{\text{What about } (\alpha_{3D}\ acc)\ ?}$ → Accidentals "of the second kind"

↳ $\boxed{\alpha_{3D} > 0}$ take data for very long time

$$R^{3D}_{acc} = \tau_c \left(R_{AB} R_{B'} + R_{AB'} R_B \right)$$

→ Why 10,000 counts

↳ the lower the $R_{AB}, R_{B'}, R_{AB'}, R_B$ ⇒ the lower R^{3D}_{acc}

Oct 25, 2017

$\boxed{KMT}$ $\boxed{\text{KINETIC MOLECULAR THEORY}}$

Heat capacity @ constant pressure $C_p = \frac{1}{n} \frac{\Delta Q}{n \Delta T}\Big|_{p=const}$

$\boxed{\text{Model}}$ $\Delta Q = \Delta K_{total} + W$

$(P \cdot A)\Delta z = P\Delta V$

↳ $\boxed{\Delta Q = \Delta K_{total} + P\Delta V}$

↳ $C_p = \frac{1}{n} \frac{\Delta K}{\Delta T}\Big|_{p=const} + \frac{1}{n} \frac{P\Delta V}{\Delta T}\Big|_{p=const}$

$$\Rightarrow \quad C_p = C_V + \frac{P}{n}\left(\frac{\Delta V}{\Delta T}\Big|_{p=const} \right)$$

$\boxed{\text{volume expansion coefficient}}$

For solids $\frac{\Delta V}{\Delta T}\Big|_{p=const} \simeq 0 \longrightarrow \boxed{C_V \simeq C_p}$ (solid)

For ideal gas $\frac{\Delta V}{\Delta T}\Big|_{p=const} = \boxed{\frac{nR}{P}} \Rightarrow \boxed{C_p = C_V + R}$ (ideal gas)

KMT model $\boxed{C_V = \frac{3}{2}R} \neq \boxed{C_p - R}$

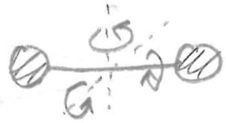
Why different?

(e) The Equipartition theorem and heat capacity

↳ $C_V = \frac{3}{2}R$ fails for polyatomic gases. Why?

Diatomic: $C_V \simeq 2.5R$ ↑ $5/2$

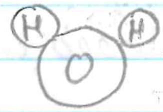
Triatomic $C_V \simeq 3R$ ↑ $6/2$



$\left\{ \begin{array}{l} \text{rotational} \\ \text{vibrational} \end{array} \right.$ + translational

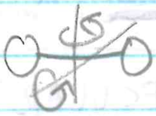
$\boxed{\text{Because}}$ Polyatomics have $\boxed{\text{other ways}}$ to hold kinetic energy!

Each "way"/mode of storing energy adds to specific heat $\frac{1}{2}R$



3 "axes" that can "store" energy

↳ $+\frac{3}{2}R \rightarrow \boxed{C_V = \frac{3}{2}R + \frac{3}{2}R} = 3R$



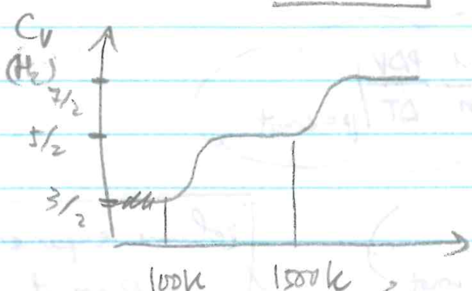
$\rightarrow \boxed{C_V = \frac{3}{2}R + \frac{2}{2}R = \frac{5}{2}R} = 2.5K$

↳ But model can be extended

↳ 2 more "ways" to store energy

$\left\{ \begin{array}{l} \text{vibrational KE, PE} \\ \text{↳} + \frac{1}{2}R \end{array} \right.$

→  $\Rightarrow + \boxed{2 \times \frac{1}{2}R}$



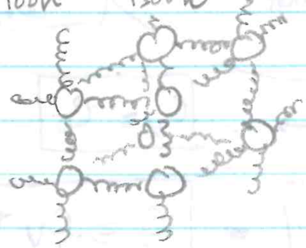
low energy (temp) → only translation

higher energy → translation + rotation

higher energy → translation + rotation + vibration

↳ Rotation has a quantized energy to start

$\boxed{\text{Solids}}$ → Model



each atom can store energy in 6 modes

3KE, 3PE

$\Rightarrow \boxed{C_p \simeq C_V = 3R}$

Empirical Observation Dulong - Petit (1819)

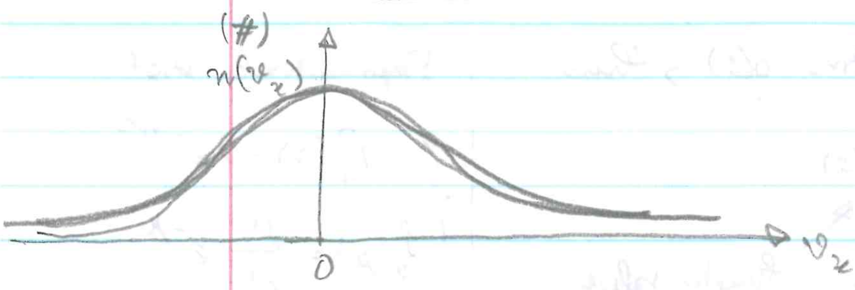
↳ Diamond is an outlier!

⇒ $\boxed{\text{EQUIPARTITION THEOREM}}$ → each mode of "energy storage"

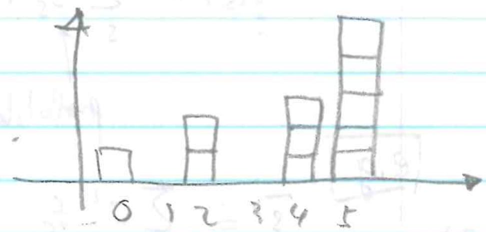
↳ degree of freedom can store $\boxed{\frac{1}{2}k_B T}$ of energy per molecule

↳ or $\boxed{\frac{1}{2}RT}$ of energy per mole

B. Probability Distribution



1) Discrete distribution



How to characterize dist? → $\{s_p\}$ → scores $\{0, 2, 2, 4, 4, 5, 5, 5, 5, 5\}$

→ $\{n_s\}$ → # ppl with same scores $\{1, 0, 2, 0, 3, 5\}$

$$\boxed{\sum_s n_s = N_p}$$

a) Normalization 2 Probability Dist

$\{f_s\}$: set of no. $f_s = \left\{\frac{n_s}{N_p}\right\}$ → fraction of ppl with value s,

$$\left\{\frac{1}{7}, \frac{0}{7}, \frac{2}{7}, \frac{0}{7}, \frac{3}{7}, \frac{5}{7}\right\}$$

$$\Rightarrow \sum_s f_s = \frac{\sum_s n_s}{\sum_s n_p} = \frac{N_p}{N_p} = 1$$ → prob. of getting a score $= 1$

$$\boxed{\text{if } \sum_s f_s = 1 \Rightarrow f_s \text{ is a normalized distribution}}$$

b.) Averages

$$\Rightarrow \bar{s} = \frac{1}{N_p}\sum_p s_p \rightarrow \frac{\text{average score}}{\text{total \#}} = \frac{1}{N_p} \cdot \sum_s n_s \cdot s$$

how many scores of each kind

$$\Rightarrow \boxed{\bar{s} = \sum_s f_s \cdot s}$$

Example: roll a single die $\begin{cases} s_{min} = 0 \\ s_{max} = 6 \end{cases}$ $f_s = 1/6$

$$\bar{s} = \sum_1^6 (1/6) \cdot s = \frac{1}{6} \cdot \sum_1^6 s = \frac{1}{6} \cdot \left(\frac{6 \cdot 7}{2}\right) = \frac{7}{2} = \boxed{3.5} = \bar{s}$$

(average roll of a die)

(E)

c. Average of a function of the value (expectation value)

If there's a function $g(s)$, then

$$\overline{g(s)} = \sum_s f_s \cdot g(s)$$

probability   function value

Exponential Dist f

$$\begin{cases} f_n(x) = Ae^{-nx} \\ f_n(\mu) = \dfrac{\mu^n}{n!} e^{-\mu} \end{cases}$$

$\boxed{\text{e.g.}}$

$(E^2) \rightarrow \quad \overline{s^2} = \sum_s s^2 \cdot f_s$

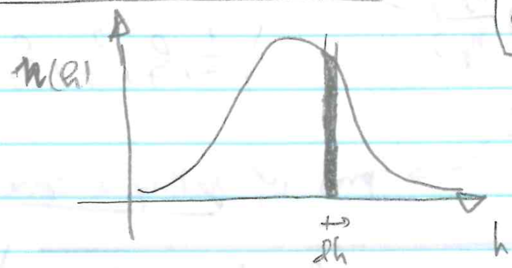
$\boxed{\text{Oct 27, 2017}}$

$\boxed{\text{2. CONTINUOUS DISTRIBUTION}}$

measurements can take on any real value.

Examples: heights (h) in a population...
- $n(h)$, $f(h)$
- x-velocities of atoms in a gas $n(v_x)$, $f(v_x)$

For continuous distribution



$\boxed{dN = n(h) \cdot dh}$ (# with height from h to $h + dh$

$\rightarrow \boxed{f(h) = \dfrac{n(h)}{N}} \rightarrow$ Unit: #/person

probability density

(a) Normalization $\boxed{\int f(h)\,dh = 1}$

(b) Average $\boxed{\int (f(h)\,dh) \cdot h = \bar{h}}$ average

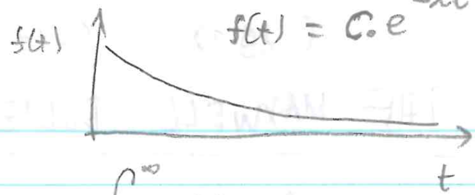
prob   respective h

(c) Expectation value in general

$$\boxed{\overline{g(h)} = \int g(h) f(h)\,dh}$$

$$\boxed{\int_0^\infty x^n e^{-at}\,dt = \frac{n!}{a^{n+1}}}$$ (important integrals #1)

$f(t) = C \cdot e^{-\lambda t}$



Example: Radioactive decay

Normalization $\int_0^\infty f(t)\,dt = 1 = \int_0^\infty C e^{-\lambda t}\,dt = 1 = \frac{C}{\lambda} \Rightarrow C = \lambda$

$$\Rightarrow \boxed{f_\lambda(t) = \lambda e^{-\lambda t}}$$

$$\bar{t} = \lambda \int_0^\infty t \cdot e^{-\lambda t} = \lambda \cdot \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda} = \bar{t}} \Rightarrow f(t) = \frac{1}{\bar{t}} e^{-t/\bar{t}}$$

(3) **Continuous distributions in more than 1 dimension?** (multivariable)

$n(v_x, v_y, v_z) \leftarrow$ number density of atoms with $(v_x, v_y, v_z) = \vec{v}$

velocity

$$\hookrightarrow \boxed{dN = n(v_x, v_y, v_z)\,(dv_x\,dv_y\,dv_z)}$$

$$\boxed{f(v_x, v_y, v_z) = \frac{n(v_x, v_y, v_z)}{N}}$$

Normalization

$$1 = \iiint_{\text{all } \vec{v}} f(v_x, v_y, v_z)\,dv_x\,dv_y\,dv_z$$

$$\Rightarrow \boxed{1 = \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z\, f(v_x, v_y, v_z)}$$

velocity can be (+)/(−)

Expectation value

$$\boxed{\overline{g(\vec{v})} = \iiint dv_x\,dv_y\,dv_z \cdot g(\vec{v})\, f(v_x, v_y, v_z)}$$

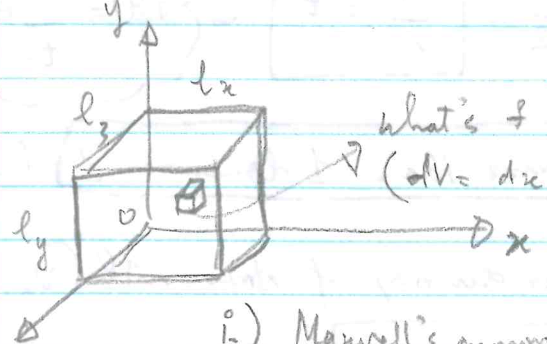
(1859) (1871)

C. THE MAXWELL – BOLTZMANN DISTRIBUTION FUNCTION

↳ Dist of velocities of atoms in a gas

1) Maxwell's ideal gas distribution

a) Maxwell's spacial distribution (atoms in a box)



what's f to find atom there?

$(dV = dx\,dy\,dz)$ $\quad (dN = n(x,y,z)\cdot dx\,dy\,dz)$

i.) Maxwell's symmetry argument → no place is special

$$f(x,y,z) = \overset{\text{const}}{A}$$

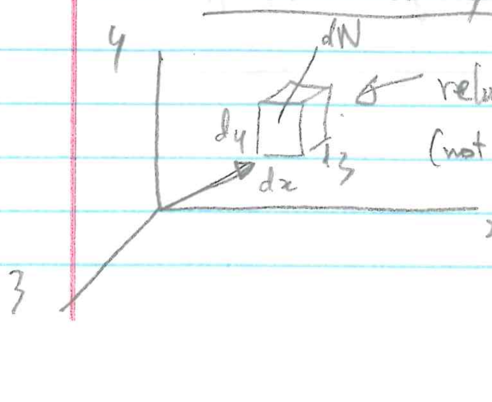
ii) Normalization $\quad \underset{\text{whole volume}}{\iiint} A\,dx\,dy\,dz = 1 \Rightarrow A\int_0^{l_x} dx \int_0^{l_y} dy \int_0^{l_z} dz = A\,l_x l_y l_z$

$$\rightarrow \boxed{A = \frac{1}{V}}$$

$$\Rightarrow \boxed{f(x,y,z) = \frac{1}{V}} \Rightarrow \boxed{n(x,y,z) = N f(x,y,z) = \frac{N}{V}}$$

$$\boxed{dN = n(x,y,z)\,dx\,dy\,dz = \frac{N}{V}\,(dx\cdot dy\cdot dz)}$$

b) Maxwell velocity dist



← velocity space (not physical space)

$$\boxed{dN = n(v_x, v_y, v_z)\,dv_x\,dv_y\,dv_z}$$

$$\boxed{f(v_x, v_y, v_z) = \frac{n(v_x, v_y, v_z)}{N}}$$

i) Maxwell's symmetry arguments

product

① v_x, v_y, v_z are uncorrelated $\Rightarrow$ $f(v_x, v_y, v_z) = f(v_x)\, g(v_y)\, h(v_z)$

same everywhere in space

② x, y, z are all the same $\Rightarrow$ $f(v_x, v_y, v_z) = f(v_x)\, f(v_y)\, f(v_z)$

all direction are equivalent

③ Dist must only depend on speed! $v^2 = v_x^2 + v_y^2 + v_z^2$

$$\boxed{F(v_x^2 + v_y^2 + v_z^2) = f(v_x) \cdot f(v_y) \cdot f(v_z)}$$

F(sum) = product F

↳ Exponential = solution!

$$\hookrightarrow f(v_x) = A e^{-b v_x^2}$$

ii) Normalization $1 = \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z \underbrace{A^3 e^{-b v_x^2} e^{-b v_y^2} e^{-b v_z^2}}_{f(v_x, v_y, v_z)}$

$$\rightarrow \quad A = \sqrt{\frac{b}{\pi}} \Rightarrow \boxed{f(v_x, v_y, v_z) = \left(\frac{b}{\pi}\right)^{3/2} \cdot e^{-b(v_x^2 + v_y^2 + v_z^2)}}$$

iii) Use equipartition theorem

↳ $\langle \frac{1}{2} m v^2 \rangle = \frac{3}{2} k_B T \Rightarrow$ determine (b)

$$\rightarrow \bar{k} = \iiint dv_x\, dv_y\, dv_z \left(\frac{1}{2} m v^2\right) f(v_x; v_y, v_z)$$

$$\rightarrow \bar{k} = \frac{3}{2} k_B T = \left(\frac{3}{4} m \cdot \frac{1}{b}\right) \Rightarrow \boxed{b = \frac{m}{2 k_B T}}$$

$\frac{-k}{k_B T}$

For ideal gas

$$\boxed{F(v_x, v_y, v_z) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{\frac{-m}{2 k_B T}(v_x^2 + v_y^2 + v_z^2)}}$$

② Other ideal gas distribution

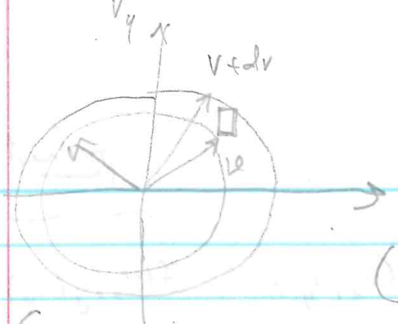
→ many velocities correspond to the same speed

a) Maxwell speed distribution

$dN = N g(v)\, dv$ ← # of molecules with speed between v & $v + dv$

2-d: $F_2(v_x, v_y) = \left(\frac{m}{2\pi k_B T}\right)^{1} e^{-\frac{m}{2k_BT}(v_x^2 + v_y^2)}$

$\pi(v+dv)^2 - \pi v^2 \simeq (2v\pi)dv$



$$dN_{ring} = dN_{square} \cdot \frac{A_{ring}}{A_{square}} \leftarrow dv_x dv_y$$

+1 dimension

$\hookrightarrow dN_{ring} =$

$$dN_{ring} = N \cdot \left(\frac{m}{2\pi k_B T}\right) \cdot e^{-\frac{K}{k_B T}}\, dv_x\, dv_y\, \frac{(2v\pi)\,dv}{dv_x\, dv_y}$$

$$\boxed{dN_{ring} = N \underbrace{\left(\frac{m}{k_B T}\right) e^{-\frac{1}{2}mv^2/k_BT} \cdot v}_{g(v)}\, dv}$$

$$g(v) = \left(\frac{m}{k_B T}\right) e^{-\frac{1}{2}mv^2/k_BT} \cdot v$$

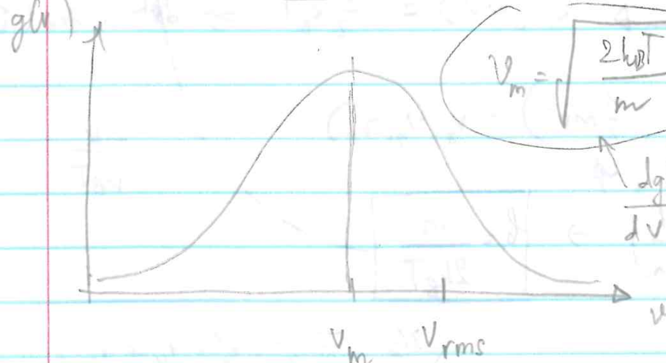
Area ...

+1 dimension

In 3-D, the ring is a spherical shell. $V_{shell} = (4\pi v^2)\,dv$

$$dN_{shell} = dN_{cube}\frac{V_{shell}}{V_{cube}} = N \cdot 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 e^{-\frac{1}{2}mv^2/k_BT}\, dv$$

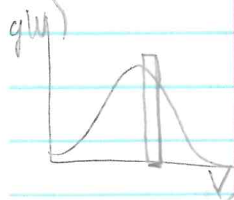
$$\Rightarrow \boxed{g(v) = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 e^{-\frac{1}{2}mv^2/k_BT}\, dv}$$



$$v_m = \sqrt{\frac{2k_BT}{m}} \leftarrow \frac{dg(v)}{dv} = 0 \qquad v_{rms} = \sqrt{\frac{3k_BT}{m}}$$

$$\boxed{\bar{v} = \sqrt{\frac{8}{\pi}\frac{k_BT}{m}} = \int_0^\infty v\, g(v)\, dv}$$

expected value for v.



6 Maxwell's kinetic energy distribution

$$dN = N g(v)\, dv = N f(k)\, dk$$

corresponding intervals $\left(dv = \frac{dv}{dk}dk,\ dk = dv \cdot \frac{dk}{dv}\right)$

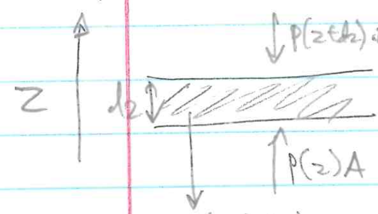
$$k = \frac{1}{2}mv^2 \Rightarrow \frac{dk}{dv} = mv = \sqrt{2mk}, \quad v = \sqrt{\frac{2k}{m}}$$

$$f(k)\,dk = g(v)\,dv = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2}\cdot\left(\frac{2k}{m}\right)e^{-k/k_BT}\cdot\left(\frac{\overset{dv}{dk}}{\sqrt{2mk}}\right)$$

$$\boxed{f(k)\,dk = \frac{2}{\sqrt{\pi}}\left(\frac{1}{k_BT}\right)^{3/2}\sqrt{k}\,e^{-k/k_BT}\,dk}$$

(3) ⇒ **Spacial Distribution** → what happens when the box gets tall?

a) The law of atmospheres:



slab of fluid of thickness dz and area A

$mg = \underbrace{(\rho A dz)}_{m}\cdot g$

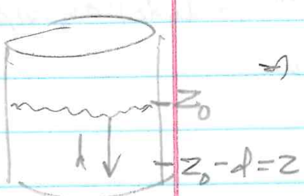
In equilibrium: $F_{net} = ma = 0 = -P(z+dz)A + P(z)A - g\rho A dz$

$\rightarrow -P(z+dz) + P(z) - \rho g\, dz = 0$

$\rightarrow \dfrac{P(z+dz) - P(z)}{dz} = -g\rho$

$$\Rightarrow \boxed{\frac{dP}{dz} = -g\rho}$$

Example 1 $\rho = \text{const}$ (incompressible fluid) $\rightarrow \displaystyle\int_{P_0}^{P} dP = \int_{z_0}^{z} -\rho g\, dz$



$\rightarrow (P - P_0) = -\rho g(z - z_0) = \rho g(z_0 - z)$

$\boxed{P = P_0 + \rho g d}$ $\quad P_0 = 10^5\,\frac{N}{m^2}$ $\quad g = 10\,m/s^2$

$d = 10m \rightarrow P' = 1\,atm$ $\quad \rho = 1000\,kg/m^3$

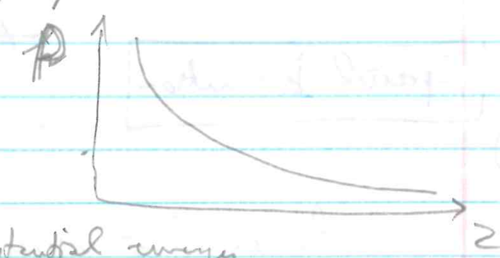
Example 2 Ideal gas: $\rho \neq \text{const} \rightarrow$ depends on P

$$\rho = \frac{mN}{V} = \frac{\mu n}{V} = \frac{\mu P}{RT} = \frac{mP}{k_BT} \Rightarrow \frac{dP}{dz} = -g\frac{mP}{k_BT}$$

$$\int_{P_0}^{P}\frac{1}{P}\,dP = \int_{z_0}^{z}\frac{-mg}{k_BT}\,dz \rightarrow \ln\left(\frac{P}{P_0}\right) = \frac{-mg}{k_BT}(z-z_0)$$

$$\Rightarrow \ln\left(\frac{P}{P_0}\right) = \frac{mg}{k_BT}(z_0-z)$$

$$\boxed{P = P_0\, e^{-\frac{mg}{k_BT}(z-z_0)}}$$



$$P = P_0\, e^{\frac{-U(z)}{k_BT}}$$ → potential energy

(b) Statistical Approach to Law of atmosphere

$$dN = N\,F(z)\,dz = ANe^{-U/k_BT}\,dz\,dx\,dy \quad (U = +mg(z-z_0))$$

$$\rho = m\frac{dN}{dx\,dy\,dz} \; ; \; P = \frac{k_BT}{m}\rho$$

(4) Maxwell-Boltzmann Distribution

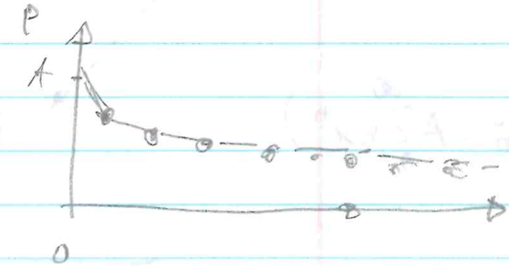
$$F(x,y,z,v_x,v_y,v_z)\,dx\,dy\,dz\,dv_x\,dv_y\,dv_z = A\,e^{-E_{total}/k_BT}(dx\,dy\,dz)(dv_x\,dv_y\,dv_z)$$

$\boxed{\text{Maxwell-Boltzmann Distribution}}$

$$\boxed{F(x,y,z,v_x,v_y,v_z)\,dx\,dy\,dz\,dv_x\,dv_y\,dv_z = A_{MB}\, e^{-\varepsilon/k_BT}\, d^3r\, d^3v}$$

Example

"moments" of a discrete distribution



$$f_n(\lambda) = Ae^{-\lambda n}$$

What is A? What is $\bar{n}$? What is σ_n^2?

A → A can be found by normalization $\sum_0^\infty f_n(\lambda) = 1$

$$\rightarrow 1 = A\sum_0^\infty e^{-\lambda n} = A\sum_{n=0}^{\infty}(e^{-\lambda})^n = A\sum_{n=0}^{\infty}(x)^n = 1 \quad (x = e^{-\lambda})$$

Binomial Expansion

$$(1+x)^n = \frac{1}{0!}x^0 + \frac{nx}{1!} + \frac{n(n-1)}{2!}x^2 + \dots$$

$$(1+(-x))^{-1} = \frac{1}{0!} + \frac{(-1)(-x)}{1!} + \frac{(-1)(-2)(-x)^2}{2!} + \frac{(-1)(-2)(-3)}{3!}(-x)^3 + \dots$$

$$= 1 + x^2 + x^3 + x^4 + x^5 + \dots$$

$$\Rightarrow \boxed{\sum_{n=0}^{\infty}(x)^n = \frac{1}{1-x} = \frac{1}{1-e^{-\lambda}}}$$

$$\rightarrow 1 = A\cdot\frac{1}{1-e^{-\lambda}} \Rightarrow \boxed{A = (1-e^{-\lambda})} \Rightarrow \boxed{f_n(\lambda) = (1-e^{-\lambda})\,e^{-\lambda n}}$$

$\bar{n}$ →

$$\bar{n} = \sum_{n=0}^{\infty} n\cdot f_n(\lambda) = \sum_{n=0}^{\infty} n\cdot A\cdot e^{-\lambda n} = A\sum_{n=0}^{\infty} n\cdot(x)^n \quad (x = e^{-\lambda})$$

$$= A\sum_{n=1}^{\infty} n(x)^n = ?$$

(1st term = 0)

let $n' = n-1$, $n = n'+1$

What is $\bar{n}$ ...? $\bar{n} = A\sum_{n'=0}^{\infty}(n'+1)x^{n'+1} = A\sum_{0}^{\infty}(n'+1)x^{n'}\cdot x$

$$\Rightarrow \bar{n} = Ax\sum_{0}^{\infty}(n'+1)x^{n'}$$

$$= \underbrace{Ax\sum_{0}^{\infty}n'x^{n'}} + \underbrace{Ax\sum_{0}^{\infty}x^{n'}}$$

$\begin{pmatrix} x = e^{-\lambda} \\ A = 1-x \end{pmatrix}$

$\left(\bar{n} = A\sum_{0}^{\infty}n'x^{n'}\right)$

$$\bar{n} = x\cdot(1)\cdot\bar{n} \qquad \frac{Ax}{1-x} = \cancel{\frac{1}{A}} = x$$

$(A = 1-x)$

$$\Rightarrow \boxed{\bar{n} = x\bar{n} + x} \Rightarrow \bar{n} = \frac{x}{1-x} = \boxed{\frac{e^{-\lambda}}{1-e^{-\lambda}} = \bar{n}}$$

σ_n $\rightarrow$ $(\sigma_n)^2 = \overline{(n-\bar{n})^2} = \sum_n (n^2 - 2n\cdot\bar{n} + \bar{n}^2)f_n(\lambda)$

$$= \underbrace{\sum_n n^2 f_n(\lambda)}_{\overline{n^2}} - 2\bar{n}\underbrace{\sum_n n f_n(\lambda)}_{\bar{n}} + \underbrace{\sum_n f_n(\lambda)}_{1}\cdot\bar{n}^2$$

$$\Rightarrow \sigma^2 = \overline{n^2} - 2\bar{n}^2 + \bar{n}^2 = \boxed{\overline{n^2} - \bar{n}^2 = \sigma^2}$$

Solution

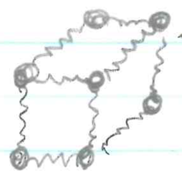
$$\boxed{\overline{n^2} = \frac{2x^2}{(1-x)^2} + \frac{x}{1-x} = \frac{2}{(e^{+\lambda}-1)^2} + \frac{1}{(e^{+\lambda}-1)}}$$

$\bar{n}^2 = \ldots$

$\Rightarrow \sigma^2 = \ldots$

Do yourself

Example #2 Moments of a continuous distribution



Consider an ensemble of 1-d simple harmonic oscillator @ temp T

A material / crystal with N atoms is an ensemble of 3N 1-d simple harmonic oscillators.

Using M-B dist

$\mathcal{E}$ = total energy $= \underbrace{\frac{1}{2}mv^2}_{KE} + \underbrace{\frac{1}{2}kx^2}_{spring}$

$$\boxed{F_{SHO}(x,v)\,dx\,dv = A_{mb}\, e^{-\mathcal{E}/k_BT}\, dx\, dv}$$

$$\mathcal{E} = \frac{1}{2}m\left(v^2 + \frac{k}{m}x^2\right)$$

$$\mathcal{E} = \frac{1}{2}m\left(v^2 + \omega^2 x^2\right)$$

What is A?

Normalization?

$$1 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv\, A_{mb}\cdot e^{-\mathcal{E}/k_BT}$$

$$\left(A_{mb}\cdot e^{-\mathcal{E}/k_BT} = A_{mb}\cdot e^{-\frac{m\omega^2x^2}{2k_BT}}\cdot e^{-\frac{mv^2}{2k_BT}}\right)$$

$$1 = A_{mb}\cdot \int_{-\infty}^{\infty} dx\cdot e^{-\frac{m\omega^2x^2}{2k_BT}}\cdot \int_{-\infty}^{\infty} e^{\frac{-mv^2}{2k_BT}} dv = A_{mb}\int_{-\infty}^{\infty} dx\cdot e^{-\frac{m\omega^2x^2}{2k_BT}}\cdot \int_{-\infty}^{\infty} dv\, e^{-\frac{mv^2}{2k_BT}}$$

$$\int_{-\infty}^{\infty} e^{-\lambda x^2} dx =$$

$$1 = A_{mb}\,\frac{\sqrt{\pi}}{\left(\frac{m\omega^2}{2k_BT}\right)^{1/2}}\cdot\frac{\sqrt{\pi}}{\left(\frac{m}{2k_BT}\right)^{1/2}} \Rightarrow \boxed{A = \frac{m\omega}{2k_BT(\pi)}}$$

What is $\bar{x}$?

$\bar{x} = 0$ (oscillator ...)

$-\infty \to 0 \to \infty$

$$\overline{x^2} = \frac{k_BT}{m\omega^2}$$

What is $\overline{x^2}$?

$$\overline{x^2} = A\int_{-\infty}^{\infty} dx\, x^2 e^{\frac{-m\omega^2x^2}{2k_BT}}\cdot \int_{-\infty}^{\infty} dv\, e^{-\frac{mv^2}{2k_BT}} = A\cdot\left[2\cdot\frac{1}{4}\cdot\frac{\sqrt{\pi}}{\left(\frac{m\omega^2}{2k_BT}\right)^{3/2}}\right]\cdot\left[\frac{\sqrt{\pi}}{\left(\frac{m}{2k_BT}\right)^{1/2}}\right]$$

$$\left(\int_{-\infty}^{\infty} x^2 e^{-\lambda x^2} dx = \ldots\right)$$

What is σ_x ? well ... $\sigma_x = \sqrt{\overline{x^2} - \overline{x}^2} = \sqrt{\overline{x^2}}$ ($\overline{x} = 0$)

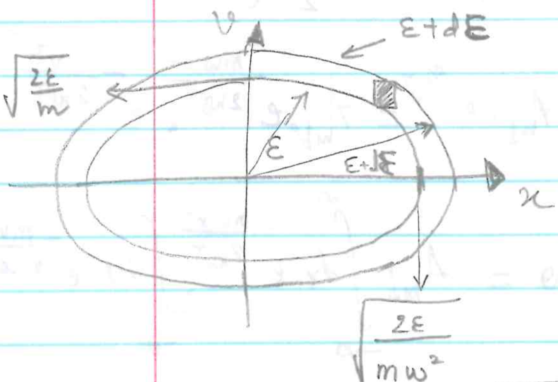
$$\Rightarrow \boxed{\sigma_x = \sqrt{\frac{k_B T}{m\omega^2}} = \sqrt{\frac{k_B T}{k}} = \sigma_x} \rightarrow \text{LIGO concern...}$$

($\omega^2 = \frac{k}{m}$)

What is Energy Distribution of an ensemble of 1-d simple harmonic oscillators at temperature T

$P(\varepsilon)\,d\varepsilon$, given $F(x,v)\,dx\,dv$

Well, $\varepsilon = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \Rightarrow v^2 + w^2x^2 = \left(\sqrt{\frac{2\varepsilon}{m}}\right)^2$ eqn for ellipse



$$\boxed{P(\varepsilon)\,d\varepsilon = [F(x,v)\,dx\,dv] \cdot \frac{A_{ring}}{A_{square}}}$$ (A_{square}: $dx\,dv$)

$A_{inner\ ring} = \boxed{\pi a b}$ (semi)

$$\Rightarrow A_{inner\ ring} = \pi \cdot \sqrt{\frac{2\varepsilon}{m\omega^2}} \cdot \sqrt{\frac{2\varepsilon}{m}} = \boxed{\frac{\pi 2\varepsilon}{m\omega}}$$

$$\boxed{A_{outer\ ring} = \frac{\pi 2(\varepsilon + d\varepsilon)}{m\omega}}$$

$$\Rightarrow A_{ring} = A_{outer\ ring} - A_{inner\ ring} = \boxed{\frac{2\pi\, d\varepsilon}{m\omega}}$$

$$\Rightarrow P(\varepsilon)d\varepsilon = F(x,v)\,dx\,dv\,\frac{A_{ring}}{dx\,dv} = (A_{mb})\,e^{-\varepsilon/k_BT} \cdot \left(\frac{2\pi\, d\varepsilon}{m\omega}\right)$$

$$= \left(\frac{m\omega}{2\pi k_B T}\right) \cdot \left(e^{-\varepsilon/k_BT}\right) \cdot \left(\frac{2\pi}{m\omega}\right) d\varepsilon$$

Energy distribution

$$\Rightarrow \boxed{P(\varepsilon)\,d\varepsilon = \frac{1}{k_B T}\, e^{-\varepsilon/k_BT}\, d\varepsilon}$$

, In fact, it must be normalized...

$$\Rightarrow \int_0^\infty P(\varepsilon)\,d\varepsilon = 1$$

$\boxed{\text{Distribution of "random" events that occur with probability } p}$

If I do an experiment N times & a success has probability (p)
(failure has probability $q = 1-p$)

then what's the distribution of successes?

Binomial Distribution $\quad f_{N,p}(n) = \begin{bmatrix} \text{probability of any} \\ \text{one combinations} \\ \text{with } n \text{ success} \end{bmatrix} \cdot C(N,n)$ ← # of combos with n successes

$$\rightarrow \boxed{f_{N,p}(n) = \underbrace{p^n(1-p)^{N-n}}_{\text{1 time}} \cdot \underbrace{\left(\frac{N!}{n!(N-n)!}\right)}_{\text{how many combos}}} \rightarrow \text{binomial coefficient}$$

$$\boxed{f_{N,p}(n) = p^n(1-p)^{N-n}\, C_N^n}$$

$f_{10,\frac{1}{2}}(5)$

Example: if flip coin 10 times, what is $P(5)$? $\rightarrow \left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5 \cdot C_{10}^5 = 0.246$

(1) Normalized $\quad \boxed{\sum_{n=0}^{\infty} f_{N,p}(n) = 1}$ (actually $= (p+q)^n$)

(2) Average $\quad \boxed{\bar{n} = Np}$

(3) St dev $\quad \boxed{\sigma^2 = Np(1-p)} \rightarrow \sigma = \sqrt{N}\sqrt{p(1-p)}$ ← $N\uparrow \rightarrow \sigma\uparrow$

But $\frac{\sigma}{\bar{n}} = \frac{1}{\sqrt{N}} \cdot \frac{\sqrt{p(1-p)}}{p} = \boxed{\frac{1}{\sqrt{N}} \cdot \sqrt{\frac{(1-p)}{p}} = \frac{\sigma}{\bar{n}}}$ ← fractional σ

?

If you do many ($N \gg 1$) experiments, each with small probability of success, such that
$\bar{n} = Np$ is "reasonable"

We need to approximate!

$$\hookrightarrow f_{N,p}(n) \simeq \frac{(Np)^n e^{-Np}}{n!} = \frac{\bar{n}^n e^{-\bar{n}}}{n!} = \frac{\mu^n e^{-\mu}}{n!}$$

$\Rightarrow$ POISSON DISTRIBUTION: $\boxed{f_{N,p}(n) = \frac{\mu^n e^{-\mu}}{n!}}$

You need to show $\left\{ \begin{array}{l} \bar{n} = \mu \\ \sigma = \sqrt{\mu} \end{array} \right\} \rightarrow \boxed{\frac{\sigma}{\bar{n}} = \frac{1}{\sqrt{\mu}}}$ Fractional uncertainty $\downarrow$ when $\mu \uparrow$

By knowing exp is Poissonian $\Rightarrow$ No need to do exp many times

$$\rightarrow \sigma = \frac{1}{\sqrt{\mu}} \cdot \bar{n}$$

Nov 1, 2017

D. BLACK BODY RADIATION

$\rightarrow$ the beginning of quantum mechanics

$\triangleright$ What happens when objects are warm/hot? $\Rightarrow$ they emit light "self lumino[us]"

$\Rightarrow \left\{ \begin{array}{l} \text{electromagnetism} \\ \text{thermodynamics} \end{array} \right.$

[Can we use the same ideas about thermal equilibrium to calculate the properties of self-luminous objects?]

(1) Thermal radiation

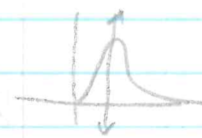
Empirical Observation: $\rightarrow$ Quality (Color)
$\rightarrow$ Quantity (intensity / temperature)

(a) Black bodies $\rightarrow$ a non-reflective object $\Rightarrow$ the only light emitted is from thermal radiation if an object is a black body, then its thermal radiation follows a universal set of rules

b) Stefan's Law

$$\boxed{R_T = \sigma T^4}$$

radiancy (W/m²) empirical constant $\left(5.67\times10^{-8}\ \frac{W}{m^2 K^4}\right)$



c) Wein's displacement law → What's the most intense color?

1) $\boxed{\lambda_{max} \cdot T = b} = 2.898\times10^{-3}\ m.K \Rightarrow \boxed{\lambda_{peak} = \frac{b}{T}}$

(peak)

wave length of max intensity

2) $\boxed{\nu_{max}\cdot \frac{1}{T} = b'} = 5.879\times10^{10}\ \frac{Hz}{K} \Rightarrow \boxed{\nu_{peak} = b'T}$

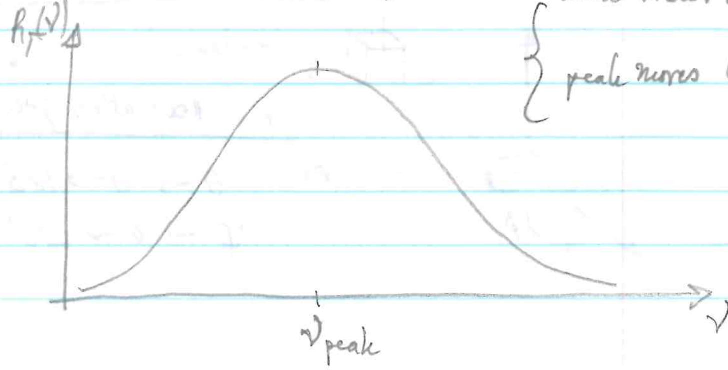
d) Spectral Radiancy → there's really a distribution of colors.

radiancy dist function → $\boxed{R_T(\nu)}$ (universal for black bodies ... like KE for gas)

$\frac{W}{m^2 Hz}$ ⇒ $\boxed{R_T(\nu)\,d\nu}$ → the radiancy of light in the frequency interval from ν to $\nu + d\nu$

$$\boxed{\int_0^\infty R_T(\nu)\,d\nu = \underset{R_T}{\underset{\shortparallel}{\sigma T^4}}}$$

← normalized in a different way



{ area under curve ∝ T
{ peak moves left ∝ T

Wein's exponential Law

$$R_T(\nu) = c_1 \nu^3 e^{-c_2\nu/k_BT}$$

empirical const

What is $R(\lambda)\,d\lambda$?

$\lambda\uparrow \to \nu\downarrow$

$$R(\lambda)\,d\lambda = -R(\nu)\,d\nu$$

corresponding intervals

$\nu\lambda = c \Rightarrow \nu = \frac{c}{\lambda}$

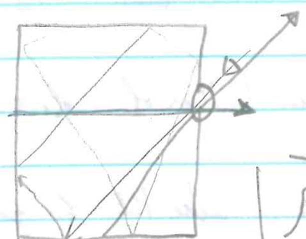
$\Rightarrow \frac{d\nu}{d\lambda} = \frac{-c}{\lambda^2} \to \boxed{d\nu = \frac{-c}{\lambda^2}\,d\lambda}$

$$R_T(\nu)\,d\nu = R_T\left(\nu = \frac{c}{\lambda}\right)d\nu$$

$$-R_T(\lambda)\,d\lambda = -c_1\left(\frac{c}{\lambda}\right)^3 e^{-c_2\left(\frac{c}{\lambda}\right)\frac{1}{k_B T}}\,d\lambda \cdot \left(\frac{+c}{\lambda^2}\right)$$

$$\hookrightarrow \boxed{R_T(\lambda)\,d\lambda = c_1 \cdot \frac{c^4}{\lambda^5} \cdot e^{-c_2 \cdot \left(\frac{c}{k_B T}\right)\lambda^{-1}}\,d\lambda}$$

(e) "Cavity radiators": perfect black bodies



The aperture, with area A has the spectrum radiancy of a "perfect" blackbody

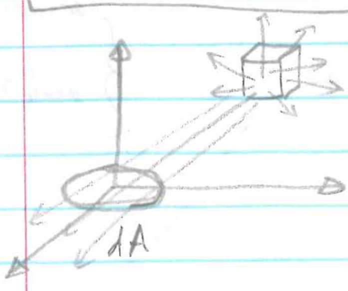
$\boxed{\rho_T(\nu)\,d\nu}$ ← energy density in the box in the interval ν to $\nu + d\nu$

$\downarrow$ J/m^3

$\rho_T(\nu)$: $\frac{J}{m^3 Hz}$

$$\rho_T(\nu)\,d\nu = -\rho_T(\lambda)\,d\lambda = -\frac{c}{\lambda^2}\rho_T\left(\nu = \frac{c}{\lambda}\right)$$

What's the relationship between $\rho_T(\nu)$ & $R_T(\nu)$?



(Radiate)

Some radiation goes thru the hole ... at some time ...

$\theta \to 0 \to 180^\circ$ flux $\propto \cos\alpha$

$\varphi \to 0 \to 180^\circ$

$R_T(\nu)d\nu = C_1 \nu^3 \cdot e^{-C_2\nu/k_B T} d\nu$

energy density — power/unit area

$$\boxed{\rho_T(\nu)d\nu = \frac{4}{c} R_T(\nu)d\nu}$$

J/m^3 $\qquad \frac{1}{m/s}\cdot\frac{W}{m^2} = \frac{W\cdot s}{m^3}$ //

Nov 3, 2017

Max Planck – 1900

$$\underbrace{\rho_T(\nu)d\nu}_{\text{unit of energy}} = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/k_BT}-1}d\nu$$

$$\rho_T(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5}\cdot\frac{1}{e^{hc/\lambda k_BT}-1}d\lambda$$

but this proof was long, complex & self-contradictory

$(c = \lambda\nu)$

$d\nu = \frac{-c}{\lambda^2}d\lambda$

h: fitting parameters ... (quantization

2 Theory of Cavity Radiation

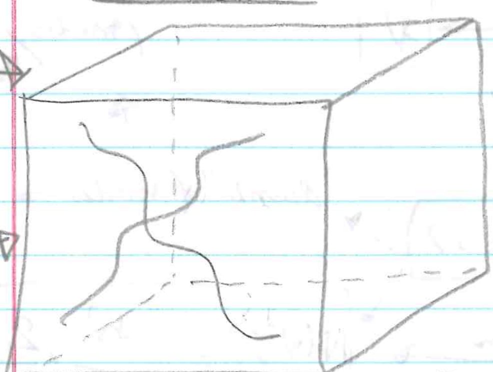
following the approach of Raleigh we will consider modes of the EM field

standing waves of the EM field

a) Light modes

empty box

Conducting box



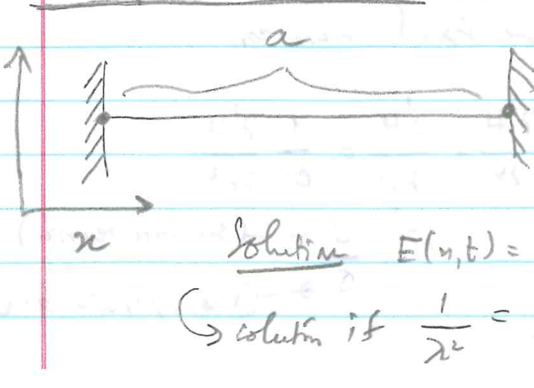
In the box, these 2 eq. has to be satisfied

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} \quad \text{(wave eq.)}$$

$$\oint \vec{E}\cdot d\vec{A} = 0 \quad \text{(Gauss' Law – empty box)}$$

Boundary condition: $E_{\parallel} = 0 \rightarrow \vec{E}_{\text{surface}}$ = "normal"

Examples in 1 dimension (Waves on a cello string)

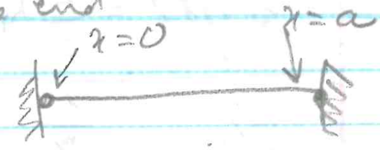


Wave eqn $\boxed{\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}}$ $\left(c^2 = \frac{T}{\mu}\right)$ — tension; mass/length

Solution $E(x,t) = E_m \cdot \sin\left(\frac{2\pi}{\lambda}x + \varphi_x\right)\cdot \sin\left(2\pi\nu t + \varphi_t\right)$

↳ solution if $\frac{1}{\lambda^2} = \frac{\nu^2}{c^2} \Rightarrow \boxed{c = \nu\lambda}$

Boundary conditions: String is fixed @ the end



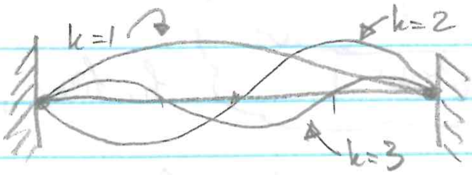
$x=0 \rightarrow y(x=0,t)=0$

$\Rightarrow \boxed{\varphi_x = 0}$

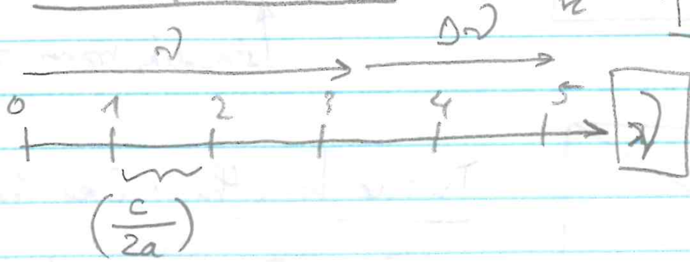
$x=a \rightarrow y(x=a,t)=0 \Rightarrow \sin\left(\frac{2\pi}{\lambda}\cdot a\right)=0 \Rightarrow \frac{2\pi}{\lambda}\cdot a = k\pi$

$\Rightarrow \boxed{\lambda = \frac{2a}{k}}$ $(k=1,2,3,\dots)$ (modes)

$k=1,2$



$\rightarrow$ Modes of oscillation $\lambda = \frac{2a}{k}$, $\boxed{\nu = \frac{c}{\lambda} = \frac{ck}{2a}}$



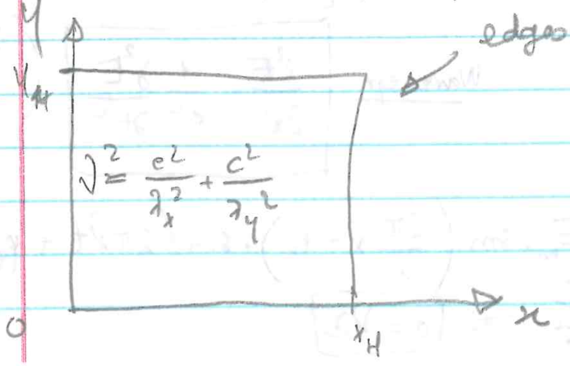
(modes/Hz)

$\boxed{\text{How many modes between } \nu \text{ \& } \nu + d\nu}$ $\rightarrow$ "density" of modes $= \frac{1}{c/2a} = \boxed{\frac{2a}{c}}$

# of modes in $d\nu$ $\rightarrow$ $\boxed{N(\nu)d\nu = \frac{d\nu}{\cancel{\ }}\cdot\frac{2a}{c}}$ (1-d)

Example in 2-d

Modes of a square drum head



edges are fixed

wave eqn

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 H}{\partial t^2}$$

$c^2 = \frac{S}{\sigma}$ ← (surface tension) ← (mass / unit area)

$x=0, y=0 \rightarrow H=0$

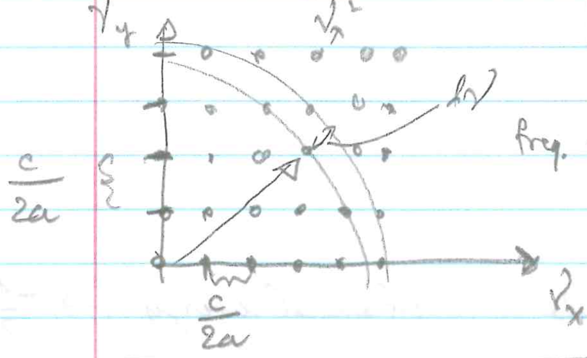
$$H(x,y,t) = H_m \cdot \sin\left(\frac{2\pi}{\lambda_x} x\right) \cdot \sin\left(\frac{2\pi}{\lambda_y} \cdot y\right) \cdot \sin\left(2\pi\nu t + \varphi t\right)$$

@ $\underline{x=a,\ y=a,\ H=0}$

$$\Rightarrow \lambda_x = \frac{2a}{k_x} \quad , \quad \lambda_y = \frac{2a}{k_y}$$

Modes of oscillation

$$\nu^2 = \underbrace{\frac{c^2}{4a^2} n_x^2}_{\nu_x^2} + \frac{c^2}{4a^2} n_y^2 = \nu_x^2 + \nu_y^2 \Rightarrow \boxed{\nu = \sqrt{\nu_x^2 + \nu_y^2}}$$



freq. = length to a dot

$dA = 2\pi\nu\, d\nu$

$\boxed{\text{How many modes between } \nu, \text{ to } \nu + d\nu}$

mode / Hz^2

$$N(\nu)d\nu = [\text{area of arc } (\nu \rightarrow \nu + d\nu)] \cdot [\text{density of dots}]$$

$$\Rightarrow N(\nu)d\nu = \frac{1}{4} \cdot (2\pi\nu\, d\nu) \cdot \frac{1}{(c/2a)^2} \Rightarrow$$

quarter circle

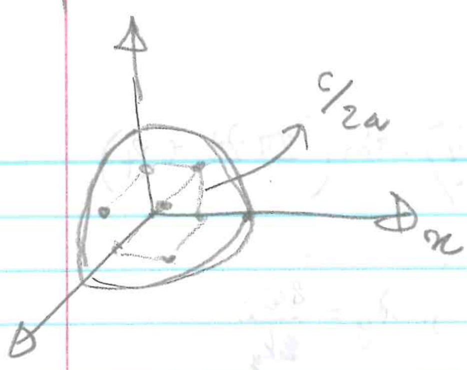
(assumption: $d\nu \gg \frac{c}{2a}$)

$$\rightarrow \boxed{N(\nu)d\nu = \frac{2\pi a^2}{c^2} \cdot \nu\, d\nu} \leftarrow \text{increase by } \left(\frac{\pi a^2}{c^2}\nu\right)$$

$\boxed{\text{Raleigh's model (3-D)}}$

$$\lambda_x = \frac{2a}{k_x}, \ \lambda_y = \frac{2a}{k_y}, \ \lambda_z = \frac{2a}{k_z}, \quad \nu^2 = \frac{c^2}{\lambda_x^2} + \frac{c^2}{\lambda_y^2} + \frac{c^2}{\lambda_z^2}$$

$$\rightarrow \nu^2 = \frac{c^2}{4a^2}\left(n_x^2 + n_y^2 + n_z^2\right) = \nu_x^2 + \nu_y^2 + \nu_z^2$$



$$N(\nu)d\nu = \underbrace{[\text{volume in shell}]}_{1/8} \cdot \underbrace{[\#\text{modes in vol}]}_{\text{density of modes}}$$

$$= \frac{1}{8}\left[4\pi\nu^2 \cdot d\nu\right] \cdot \left[\frac{1}{(c/2a)^3}\right] \quad \text{modes}/Hz^3$$

$dV = 4\pi\nu^2 d\nu$

$$\rightarrow N(\nu)d\nu = 2\left[\frac{1}{8}\left[4\pi\nu^2 d\nu\right] \cdot \frac{1}{(c/2a)^3}\right] \leftarrow \text{2 modes of polarization}$$

$$\rightarrow \boxed{N(\nu)d\nu = \frac{8\pi a^3}{c^3}\nu^2 d\nu} \quad (a^3 = \text{Volume of the box})$$

$$\boxed{N(\nu)d\nu = \frac{8\pi V}{c^3}\nu^2 d\nu}$$

November 6, 2017

I) Equipartition and Electromagnetism → every mode of mechanical energy ($\rightarrow +\frac{1}{2}k_BT$/mode)

→ idea: apply equipartition theorem to light (EM degree of freedom) just like for mechanical d.f.

Analogy

light	1-d simple harmonic oscillator
$\frac{d^2E_x}{dx^2} = -\omega_0^2 E_x$	$\frac{d^2x}{dt^2} = -\omega_0^2 x$
$E = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\mu_0 B^2$	$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$

Hypothesis $\left(P(E) = \frac{1}{k_BT}e^{-E/k_BT}\right) \leftarrow$ same form for SHO continuous $\overline{\varepsilon}$ does not work here ...

ρ_T grows forever

c) The Raleigh - Jeans formula

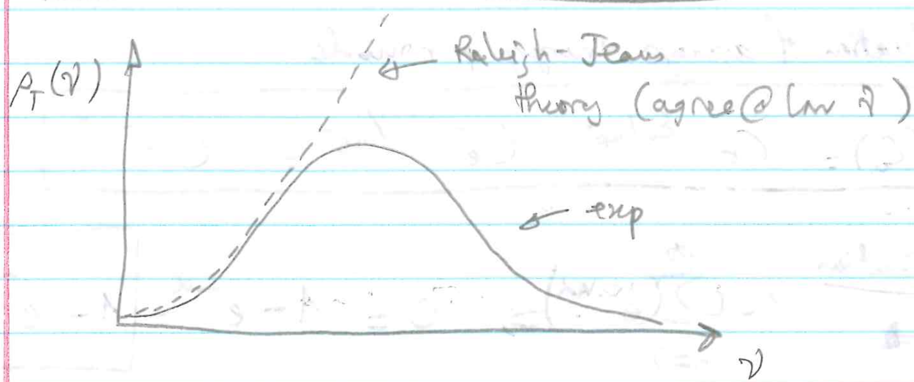
$$\boxed{\rho_T(\nu) = \frac{N(\nu)\,d\nu}{V} \cdot \overline{\varepsilon} = \frac{8\pi}{c^3}(k_B T)\nu^2 d\nu}$$

(this is wrong, btw) ...

volume

problem: using Eqpt theorem ...

d) Comparison of theory & experiment (ULTRA VIOLET Catastrophe)



E) Planck's theory of cavity radiation

Is there a way to avoid the ultraviolet catastrophe?

1) Planck's approach

(a) Maintain $N(\nu)d\nu = \frac{8\pi V}{c^3}\nu^2 d\nu$

(b) Maintain the Maxwell-Boltzmann dist, but assert that ε can only come in discrete quantities (ε_n)

(continuous $\longrightarrow$ discrete)

$$\boxed{\varepsilon_n = nh\nu}$$

Planck's constant

0, 1, 2, 3, ...

normalization constant

$$\rightarrow \boxed{P(\mathcal{E}_n) = C \cdot e^{-\mathcal{E}_n/k_BT} = C \cdot e^{-nh\nu/k_BT}}$$

↳ how does quantization work?

as $n\uparrow$ → mode 0 has almost all of the energy (because the area has to be 1)

↳ ~~at higher mode, the energ~~

at high energies → unlikely that there are modes...

(2) Evaluation of average energy per mode

$$\boxed{P(\mathcal{E}_n) = Ce^{-\mathcal{E}_n/k_BT} = Ce^{-nh\nu/k_BT} = Ce^{-n\alpha}}$$

Normalization

$\sum_0^\infty e^{-n\alpha} = \frac{1}{1-e^{-\alpha}}$ ⟹ $1 = C\sum_{n=0}^{\infty}(e^{-n\alpha}) \Rightarrow C = 1 - e^{-\alpha} = \boxed{1 - e^{-h\nu/k_BT}}$

Average energy

$$\Rightarrow \bar{\mathcal{E}} = C\sum_0^\infty \mathcal{E}_n \cdot e^{-nh\nu/k_BT} = h\nu \cdot \frac{e^{-h\nu/k_BT}}{1 - e^{-h\nu/k_BT}} = \boxed{h\nu \cdot \frac{1}{e^{+h\nu/k_BT} - 1}}$$

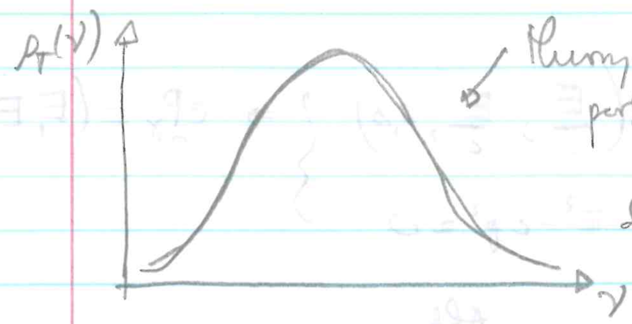
For $\nu h \ll k_BT \Rightarrow e^{h\nu/k_BT} \simeq 1 + \frac{h\nu}{k_BT}$

$$\rightarrow \boxed{\bar{\mathcal{E}} \simeq k_BT} \quad \left(= \bar{\mathcal{E}} = h\nu \cdot \frac{1}{1 - 1 + \frac{h\nu}{k_BT}} = k_BT \right)$$

For $h\nu \gg k_BT$

↳ $\boxed{\bar{\mathcal{E}} = 0}$

3. Comparison Theory – Experiment



Theory & exp matches perfectly

$\bar{\varepsilon}$ (discrete)

$$d\nu\,\rho(\nu) = \underbrace{\left(\frac{8\pi V}{c^3}\,d\nu\,.\,\nu^2\right)}_{N(\nu)\,d\nu}\overbrace{(h\nu)\left(\frac{1}{e^{h\nu/k_BT}-1}\right)}^{\bar{\varepsilon}\text{ (discrete)}}$$

(★) Difference between $\bar{\varepsilon}$ & $\bar{\varepsilon}_n$ $\Rightarrow$ $\begin{cases}\bar{\varepsilon} = k_BT \ \forall\ \nu \\ \bar{\varepsilon}_n = k_BT \text{ for } h\nu \ll k_BT \\ \bar{\varepsilon}_n = 0 \text{ for } h\nu \gg k_BT\end{cases}$

IV. PARTICLE NATURE OF LIGHT AND MATTER

A. Einstein's conception of Planck's equation

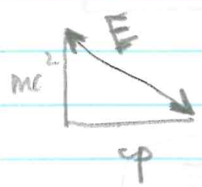
★ $\left.\rho(\nu)\,d\nu = \left(\frac{8\pi h}{c^3}\right)\nu^3\left(\frac{1}{e^{h\nu/k_BT}-1}\right)\right\} \Rightarrow$ distribution function for particles with zero mass

$\boxed{\varepsilon = nh\nu}$ $\Rightarrow$ there can be 0, 1, 2, 3, ... particles with energy $\boxed{\varepsilon_\gamma = h\nu}$

We know: $E^2 = c^2p^2 + \underset{0}{m^2c^4}$ $\Rightarrow$ $\boxed{E = cp \Rightarrow p = \frac{h\nu}{c} = \frac{h}{\lambda}}$

Nov 7, 2017

Collisions involving Photons (massless particles)



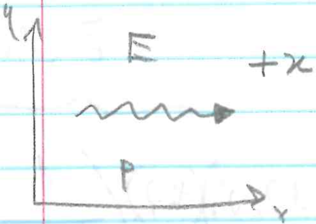
$E_\gamma = h\nu = \frac{hc}{\lambda}$ ($h = 6.626\times10^{-34}$ Js)

$E = cp$ $\rightarrow$ $\boxed{p = \frac{h\nu}{c} = \frac{h}{\lambda}}$

All relativistic particles follow rule: $E^2 = c^2p^2 + \underset{0}{m^2c^4}$

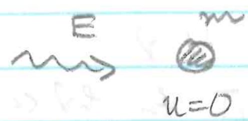
$$\underset{\sim}{P}_\gamma = \left(\frac{E}{c}, p_x, p_y, p_z\right)$$

(x, y)

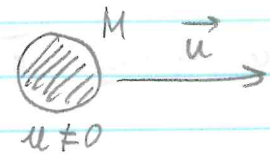


$$\Rightarrow \underset{\sim}{P}_\gamma = \left(\frac{E}{c}, \frac{E}{c}, 0, 0\right) \hookrightarrow E^2 - c^2p^2 = 0 \quad \Rightarrow c\underset{\sim}{P}_\gamma = (E, E, 0, 0)$$

Before



After



$$c\underset{\sim}{P}_\gamma = (E, E, 0, 0)$$
$$c\underset{\sim}{P}_m = (mc^2, 0, 0, 0)$$
$$c\underset{\sim}{P}_M = \left(\gamma(u)Mc^2, \gamma(u)M\frac{u}{c}c^2, 0, 0\right)$$

$$c\underset{\sim}{P}_\gamma + c\underset{\sim}{P}_m = c\underset{\sim}{P}_M \rightarrow \begin{cases} E + mc^2 = \gamma(u)Mc^2 \\ E + 0 = \gamma(u)M\frac{u}{c}c^2 \\ 0 = 0, \; 0 = 0 \end{cases}$$

square & subtract gambit

$$\begin{cases} E^2 + 2mc^2E + m^2c^4 = \gamma^2(u)M^2c^4 \\ E^2 = \left(\gamma(u)\frac{u}{c}\right)^2 \cdot M^2c^4 \end{cases}$$

$$(2mc^2E) + m^2c^4 = (Mc^2)^2 \cdot \underbrace{\gamma^2(u)\left(1 - \frac{u^2}{c^2}\right)}_{1}$$

$$\Rightarrow \boxed{Mc^2 = \sqrt{mc^2(2E + mc^2)}}$$

$$\Rightarrow \boxed{\gamma(u) = \frac{E + mc^2}{\sqrt{mc^2(2E + mc^2)}}}$$

what is u? $\rightarrow \boxed{u = \frac{Ec}{E + mc^2}}$

$$\begin{aligned} cp &= \gamma(u)muc \\ E &= \gamma(u)mc^2 \end{aligned}$$

$$\frac{cp}{E} = \frac{u}{c} \rightarrow \frac{u}{c} = \frac{E}{E + mc^2}$$

$\lambda = 770\,nm$ → m ; ${}^{19}K \xrightarrow{\text{absorb}} K^*$

$m = 39\,amu$

$mc^2 = 39.(938\,MeV)$

$h = 6.626 \times 10^{-34}\,Js = 4.136 \times 10^{-15}\,eVs$

$\begin{pmatrix} E = h\nu \\ cp = h\nu \end{pmatrix} = \frac{hc}{\lambda} \to \boxed{E_\gamma = 1.614\ eV}$

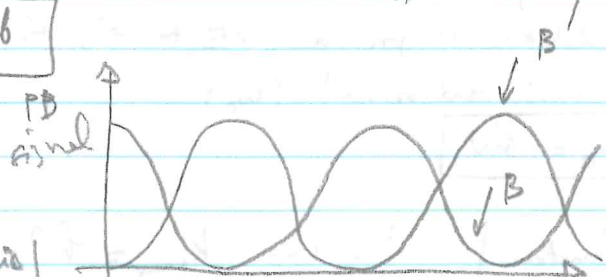
$\to \frac{u}{c} = \frac{E}{E+mc^2} = \frac{1.614\,eV}{1.614\,eV + (39.938)\times 10^6\,eV} \simeq 4.4\times10^{-11}$

$\to \boxed{u \simeq 0.013\ m/s = 1.3\,cm/s}$ → can be used to slow atoms down (close to 0K)

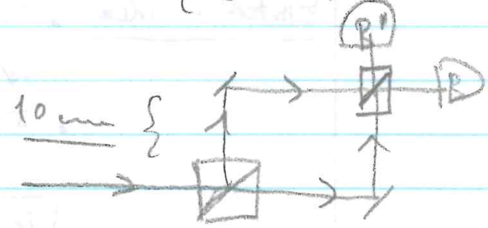
Recall $V_{rms} = \sqrt{\frac{3k_BT}{m}} \simeq 440\ m/s$

—//—

lab



this time

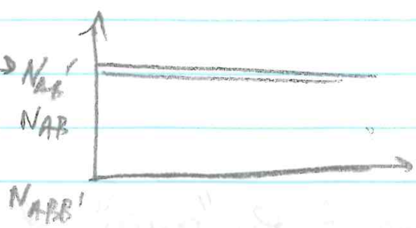


(unfolded Michelson)

$\tau_c = 8.0\,ns$ if $\alpha_{3D} \ll 1$, then 1 photon enters system at a time

Last week: photon either goes straight or reflected

If data taken at low rate ⇒ can we see interference?

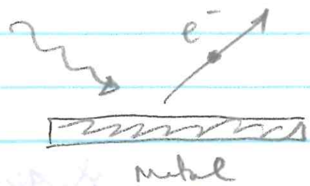


→ "Interference is not a multi-particle effect"

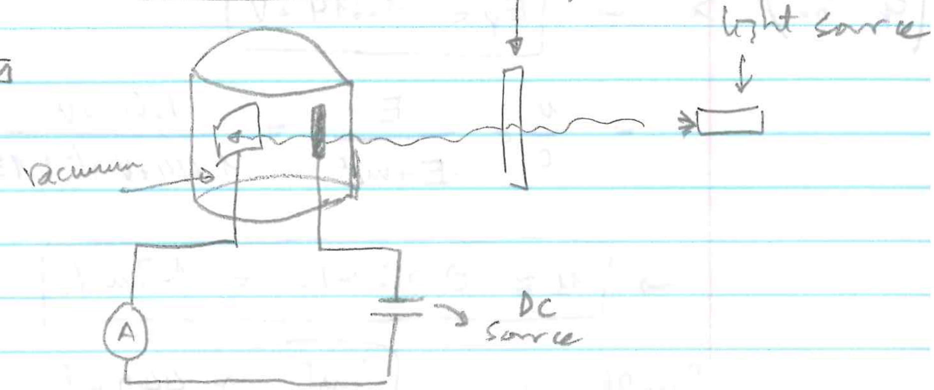
↳ Idea: interference of 1 particle

Nov 8, 2017

Photons & the Photoelectric Effect



Experimentally



Einstein's idea

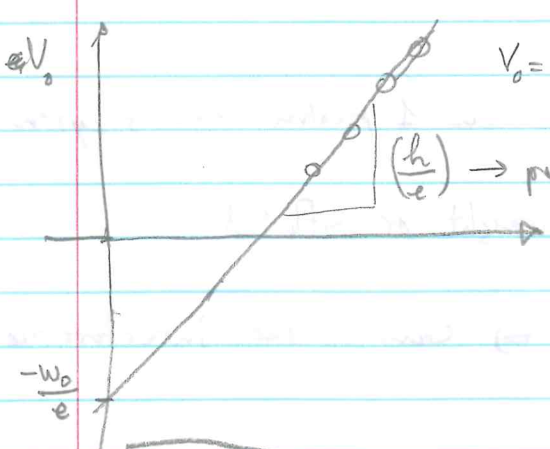
- Every electron ejected was due to a single photon colliding w/ a single electron.
- In addition to providing KE to e^-, the photon energy also must do "work" (w_0)

→ $\boxed{K_{max} + W_0 = h\nu}$

where $\boxed{W_0 : \text{work function}}$

Prediction $K_{max} = h\nu - w_0$

eV_0 ← stopping potential



$$V_0 = \frac{h\nu}{e} - \frac{w_0}{e}$$

$\left(\frac{h}{e}\right)$ → predicted $4.136 \times 10^{-15}\ Vs$

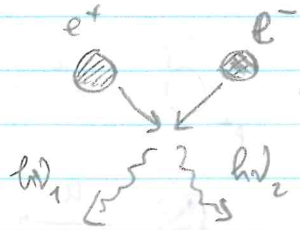
Photon collisions

↳ other compelling evidence for the "reality" of photons is the observation of collisions involving light

$$c\underset{\sim}{P}_\gamma = \left(E, c\vec{P}_\gamma \right)$$

1. Positron annihilation & production

a



$e^+ + e^- \rightarrow h\nu_1 + h\nu_2$

→ there has to be 2 photons + because there's a frame (com) in which $P_0 = 0$ → can't be 1 photon (cP)

In the rest frame of the "positronium"

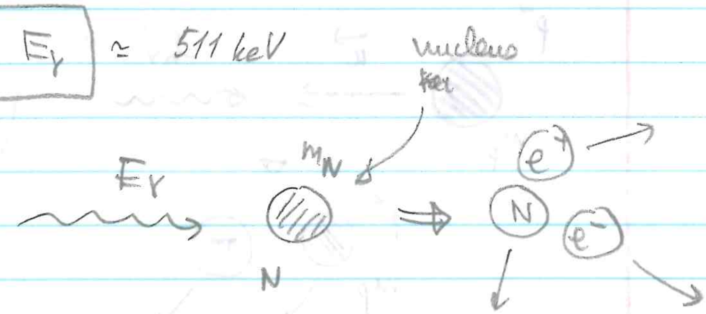
$$\begin{cases} E_i = 2m_e c^2 \quad \vec{P}_i = 0 \quad \Rightarrow \quad cP_i = (E_i, 0, 0, 0) \\ cP_f = (h\nu_1 + h\nu_2, \underbrace{h\nu_1 \hat{n}_1 + h\nu_2 \hat{n}_2}_{\vec{0}}) \end{cases}$$

$\boxed{\vec{P_1} = -\vec{P_2}} \Rightarrow h\nu_1 = h\nu_2 = h\nu$

$\Rightarrow \quad 2m_e c^2 = h\nu_1 + h\nu_2 = 2h\nu$

$\hookrightarrow \boxed{h\nu = m_e c^2 = E_\gamma} \simeq 511 \text{ keV}$

b Pair production



What is the threshold energy for pair production?

all @ rest (same v in lab frame)

$cP_i = (E_\gamma + m_N c^2, E_\gamma, 0, 0)$

$cP_f = \left((m_N c^2 + 2m_e c^2)\gamma(u), (m_N c^2 + 2m_e c^2)\cdot\gamma(u)\cdot\frac{u}{c}, 0, 0\right)$

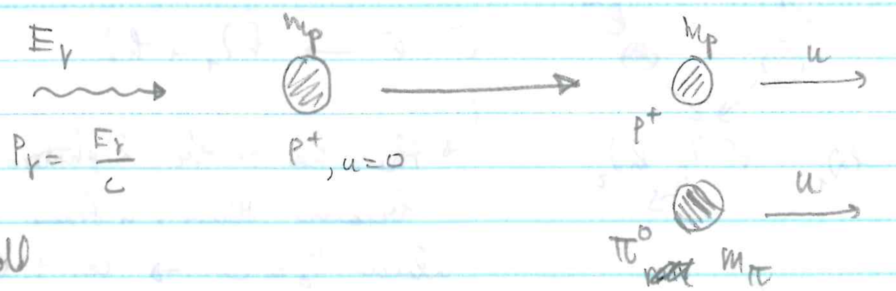
use square & subtract gambit

$$\Rightarrow \begin{cases} E_\gamma + m_N c^2 = (m_N c^2 + 2m_e c^2)\gamma(u) \\ E_\gamma = (2m_e c^2 + m_N c^2)\cdot\gamma(u)\cdot\frac{u}{c} \end{cases}$$

energy to create e^+, e^- & create momentum

$\Rightarrow \boxed{E_\gamma = 2m_e c^2 \left[1 + \frac{m_e}{m_N}\right]}$

$\boxed{4}$ <u>Production of other particles</u> $\gamma + p^+ \to p^+ + \pi^0$

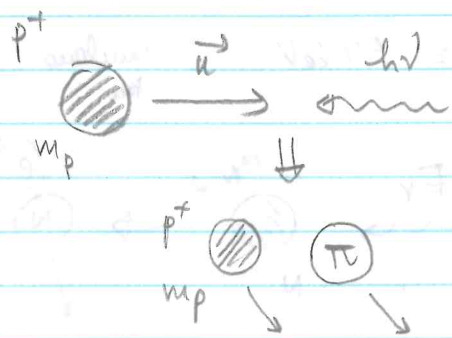


At threshold

$$\underset{\sim}{cP_i} = \left(E_\gamma + m_p c^2, E_\gamma, 0, 0\right)$$
$$\underset{\sim}{cP_f} = \left(\gamma(u)(m_p c^2 + m_\pi c^2), \frac{u}{c}\gamma(u)(m_p c^2 + m_\pi c^2), 0, 0\right)$$

Same gambit $\to$ $\underset{\text{@ threshold}}{E_\gamma} = \boxed{m_\pi c^2\left(1 + \frac{m_\pi}{m_p}\right) = E}$

$\boxed{\text{GZK suppression}}$



if $E_\gamma = 1.15\,\text{meV}$ (Cosmic microwave background photon)

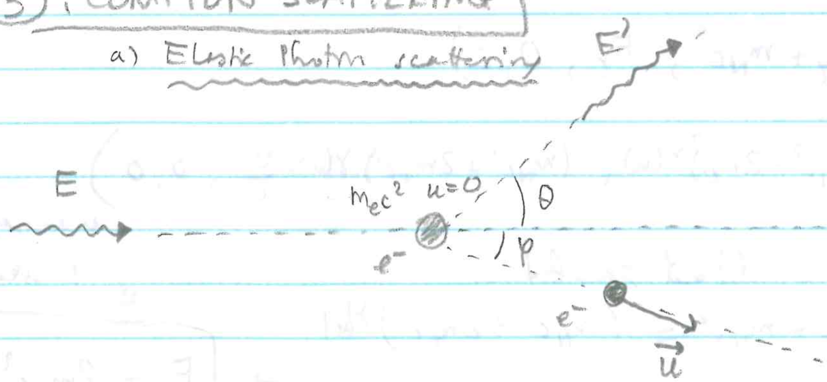
$\hookrightarrow$ if p^+ moves fast enough $\to$ the $h\nu$ is <u>Doppler shifted</u>.

$\hookrightarrow$ What is $\boxed{p^+}$ energy?

$\boxed{\text{Nov 10, 2017}}$

③. COMPTON SCATTERING

a) <u>Elastic Photon scattering</u>



Nov 10, 2017

b) Compton's theory: Goal: eliminate u', φ & solve for $E'(E,\theta)$

Before $\underset{\sim}{P_i} = \left(\frac{E}{c} + m_e c, \frac{E}{c}, 0, 0\right)$

After $\underset{\sim}{P_f} = \left(\frac{E'}{c} + \gamma(u')m_e c, \frac{E'}{c}\cdot\cos\theta + \gamma(u')m_e u'\cdot\cos\varphi, \frac{E'}{c}\sin\theta - \gamma(u')m_e u'\sin\varphi, 0\right)$

Equate term by term:

- $\left\{\frac{E}{c} + m_e c = \frac{E'}{c} + \gamma(u')m_e c\right\}\frac{1}{m_e c}$
- $\left\{\frac{E}{c} = \frac{E'}{c}\cos\theta + \gamma(u')m_e u'\cos\varphi\right\}\frac{1}{m_e c}$
- $\left\{0 = \frac{E'}{c}\sin\theta - \gamma(u')u'm_e\sin\varphi\right\}\frac{1}{m_e c}$

$\boxed{\text{Define}}$ $\varepsilon = \frac{E}{m_e c^2}$, $\varepsilon' = \frac{E'}{m_e c^2}$

(unitless) $\Rightarrow \begin{cases} \varepsilon + 1 = \gamma(u') + \varepsilon' \\ \varepsilon = \varepsilon'\cos\theta + \gamma(u')\cos\varphi\cdot\frac{u'}{c} \\ \varepsilon'\sin\theta = \gamma(u')\frac{u'}{c}\sin\varphi \end{cases} \Rightarrow \begin{cases} \varepsilon + 1 - \varepsilon' = \gamma(u') & (P_0) \\ \varepsilon - \varepsilon'\cos\theta = \gamma(u')\cos\varphi\cdot\frac{u'}{c} & (P_1) \\ \varepsilon'\sin\theta = \gamma(u')\frac{u'}{c}\sin\varphi & (P_2) \end{cases}$

(1) Square & add (P_1), (P_2)

(2) Square & subtract $\rightarrow \gamma(u') - \gamma(u')\frac{u^2}{c^2} = 1$

Result: $\boxed{\frac{1}{\varepsilon'} - \frac{1}{\varepsilon} = 1 - \cos\theta}$ $(*)$

$\varepsilon' = \frac{h\nu'}{m_e c^2} = \frac{hc}{m_e c^2}\cdot\frac{1}{\lambda'}$, $\varepsilon = \frac{hc}{m_e c^2}\cdot\frac{1}{\lambda}$

$\Rightarrow (*) \rightarrow \boxed{(\lambda' - \lambda)\frac{m_e c^2}{hc} = 1 - \cos\theta} \Rightarrow \boxed{\Delta\lambda = \frac{hc}{m_e c^2}(1 - \cos\theta)}$

$\left.\begin{matrix} m_e c^2 = 0.511\ \text{MeV} \\ hc = 1240\ \text{eV Å} \end{matrix}\right\} \Rightarrow \boxed{\frac{hc}{m_e c^2}} = \lambda_c = 0.0243\ \text{Å} = \underline{2.43\times10^{-12}\ \text{m}}$

Compton wavelength

c. Compton's experiment

$$\boxed{\Delta\lambda = \lambda_c (1-\cos\theta)}$$

0.024Å

$\lambda_{visible} \simeq 5000\text{Å} \rightarrow \sim 2eV$

→ very hard to see $\Delta\lambda$

→ Compton used X-rays (are composed of photons with $E \simeq 10{,}000\,eV$)

→ $\lambda_x \simeq 1\text{Å}$

Solves two problems

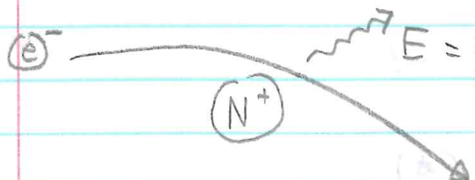
→ $\lambda_c \simeq\simeq \lambda_x$

→ $E_{x\text{-}ray} >> E_{binding}$

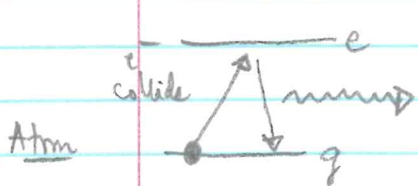
ii) X-ray production → bash electrons into materials (historically)

① Bremsstrahlung radiation (means slow down)

→ $\boxed{\text{Acceleration causes charges to radiate}}$

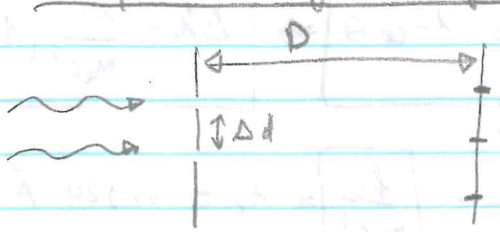


$E = \frac{hc}{\lambda}$ $E_{max} = K_{e^-}$



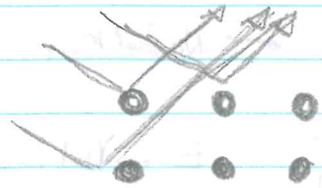
$h\nu$ (with characteristic color depending on the atom)

iii) Analysis of X-ray wavelength

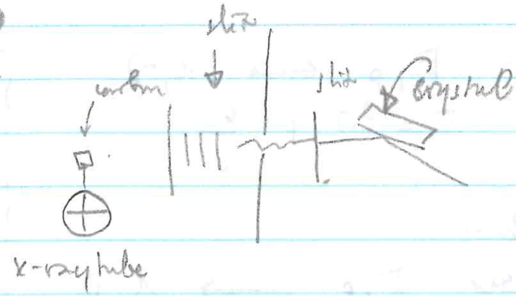


$\Delta y = \frac{\lambda D}{d}$

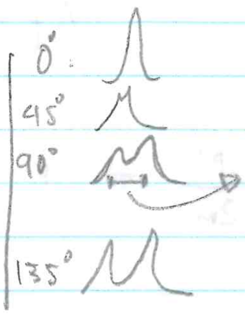
Bragg Scattering / diffraction from atoms in a crystal



Compton's setup



Data:



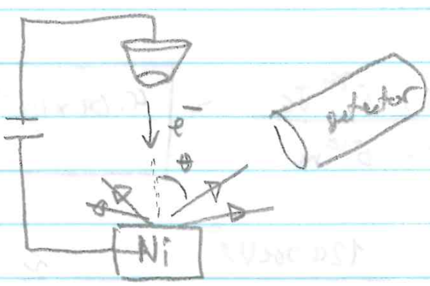
$\Rightarrow$ photon exists.

$\Delta\lambda = \frac{hc}{m_e c^2}$

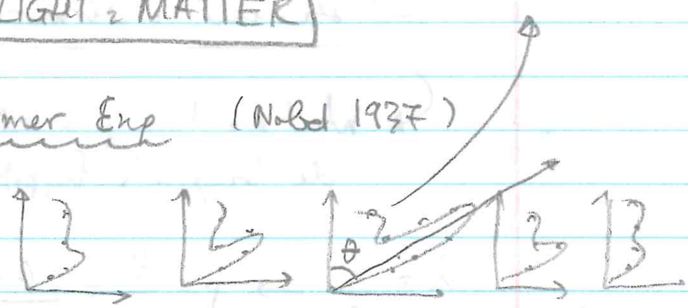
V. THE WAVE NATURE OF LIGHT & MATTER

A. The Davisson and Germer Exp (Nobel 1937)

① Exp

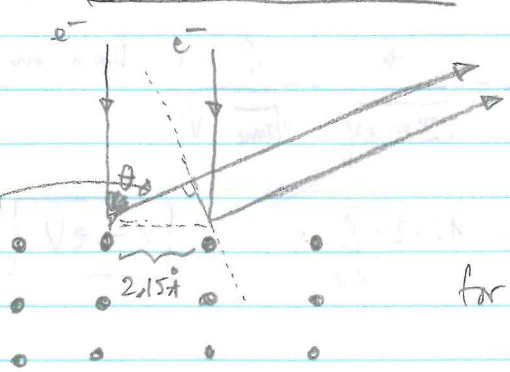


max → 50°



different peaks at different voltages

② Electron Interference ← wave properties



$\Delta p = a \sin\theta$

If electrons exhibit wave-like properties you can calculate

$\boxed{\Delta p = a\sin\theta = n\lambda}$ → constructive int.

for 2.15Å, $\theta = 50°$ $\Rightarrow$ $\lambda = \frac{1.65\text{Å}}{n}$ → $\boxed{\lambda_e = 1.65\text{Å}}$

B. de Broglie's hypothesis and wave-particle duality

1) de Broglie's hypothesis (1924) ← Nobel 1929

Light: $E, p \longleftrightarrow \lambda, \nu$

$E = h\nu \quad p = \frac{h}{\lambda}$

$E = \gamma(u) mc^2$
$p = \gamma(u) mu$

Hypothesis:
matter $E, p \longleftrightarrow \lambda, \nu$

$u \ll c \Rightarrow \begin{cases} E = mc^2 + \frac{1}{2}mu^2 \\ p = mu \end{cases}$

In non-relativistic: $KE = \frac{1}{2}mu^2 = \frac{p^2}{2m} \rightarrow p = \sqrt{mK \cdot 2}$

$$\boxed{\nu = \frac{E}{h} \quad \lambda = \frac{h}{p} = \frac{h}{\sqrt{2mK}}}$$

2) Davisson & Germer's confirmation

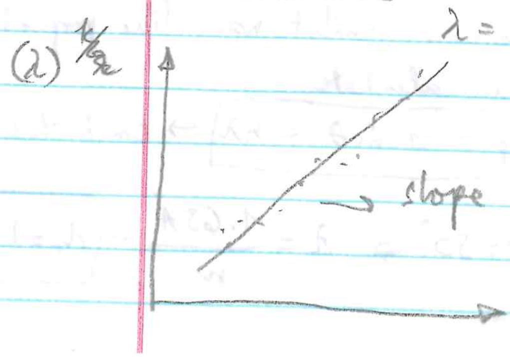
What are the p & KE of electron of $\lambda_e = 1.65\text{Å}$ according to de Broglie's hypothesis?

$\lambda = \frac{h}{p} \rightarrow p = \frac{h}{\lambda} = \frac{6.626\times10^{-34}\,\text{Js}}{1.65\times10^{-10}\,\text{m}} = \boxed{4.02\times10^{-24}\,\frac{\text{Js}}{\text{m}}}$ (p)

$k = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} = \frac{h^2c^2}{2m_ec^2\lambda^2} = \frac{12400\,\text{eVÅ}}{2(511{,}000\,\text{eV})\cdot 1.65\text{Å}} \approx \boxed{55.3\,\text{eV}}$

Davisson get 54V

Further measurements



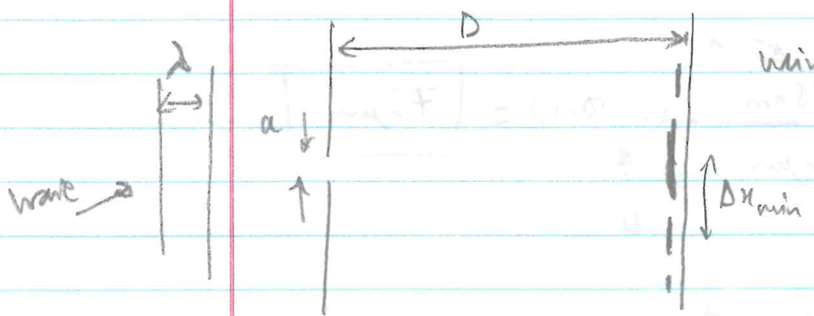
$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mK}} = \frac{h}{\sqrt{2meV}} = \frac{h}{\sqrt{2me}}\cdot\frac{1}{\sqrt{V}}$

$\boxed{k = eV}$

$\rightarrow$ slope $= \frac{h}{\sqrt{2me}} = 12.27\,\frac{\text{Å}}{\sqrt{V}}$

2. Modern matter-wave interference experiments

a) Single-slit diffraction



minima @ $\frac{\cancel{\pi}}{\lambda} a \sin\theta = n\cancel{\pi}$

minima when $\frac{\Delta x}{D} \simeq \sin\theta$

$$\boxed{x_{min} = n \frac{\lambda}{a} D}$$

any non-zero int

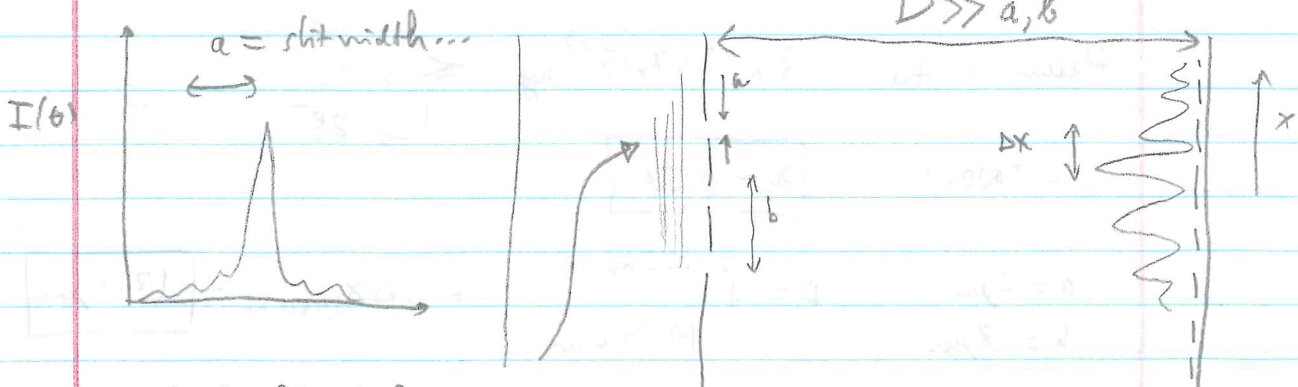
$$\boxed{I(\theta) = \frac{I_0 \cdot \sin^2\left(\frac{\pi a}{\lambda}\sin\theta\right)}{\left(\frac{\pi a}{\lambda}\sin\theta\right)^2}}$$ ← intensity

(Anton Zeilinger)

b) Neutron experiment (1988 – 1990) (Neutron)

$m = 1.67 \times 10^{-27}$ kg
$V = 214$ m/s

$\Rightarrow \lambda = \frac{h}{mV} = 1.85 \times 10^{-9}$ m $= \boxed{18.5 \text{Å}}$



c) N-slit interference

$I(\theta) = (\text{single slit diffraction}) \times (n\text{-slit-interference})$

$$\boxed{\Delta x = \frac{\lambda}{b} D}$$

n-slit-interference pattern has max @ $\boxed{x_n = n \cdot \frac{\lambda}{b} D}$

d. Two-slit experiments with neutron

$m = 1.67 \times 10^{-27}$ kg
$v = 214$ m/s
$\Big\} \to \lambda = \frac{h}{mv} = 1.85$ nm

$$\Delta x = \frac{\lambda}{\theta} D = \frac{1.85\,\text{nm}}{126\,\mu\text{m}} \cdot (5.00\,\text{m}) = \boxed{73\,\mu\text{m}}$$

λ, b, D

e. Two-slit electron experiment

2012 – Herman Batelaan $\to$ Experimental parameters

$$k = 600\,\text{eV} \longrightarrow \lambda = \frac{h}{\sqrt{2mk}} = \frac{hc}{\sqrt{2mc^2k}} = \frac{12400\,\text{eV}\,\text{Å}}{\sqrt{2(511{,}000\,\text{eV})(600\,\text{eV})}} = \boxed{0.5\,\text{Å}}$$

Two slits $a = 62$ nm
$b = 272$ nm

f. Two-slit atom interference
$\to$ complex composite system $({}^{4}_{2}He)$

Helium $= 4u = 4 \times 1.67 \times 10^{-27}$ kg $\to$ $2p^+$, $2n$, $2e^-$

$v \simeq 2000$ m/s $\to \boxed{\lambda \simeq 0.5\,\text{Å}}$

$a = 1\,\mu$m
$b = 8\,\mu$m
$D = \begin{cases} 145\,\text{nm} \\ 1950\,\text{nm} \end{cases} \Rightarrow \Delta x_{1950\,\text{m}} = \boxed{12.4\,\mu\text{m}}$

3. WAVES OR PARTICLES

calculations ⟷ measurements.

(★) Compton Effect } → collision of particles
changing energy

→ $\Delta\lambda = \frac{h}{mc}(1-\cos\theta)$

(★) N-slit interference (matter or light) } → wave superposition → "intensity"

→ arrival of particles

(★) PH241 lab (light) } → wave superposition in an interferometer.

→ # of coincidences of detector clicks ...

ⓐ Bohr's principle of complementary → you can never do an Exp when the complementary descriptions lead to a paradox

Is light a wave or a "shower of photons"?
→ Neither ... or both.

→ Two "contradictory" pictures, but you need Both

C. THE UNCERTAINTY PRINCIPLE

indeterminacy

$\hbar = \frac{h}{2\pi}$ ← rationalized Planck const

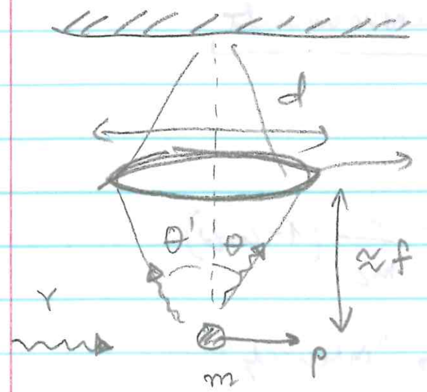
(★) A way to understand how a combination of wave & particle properties can be interpreted

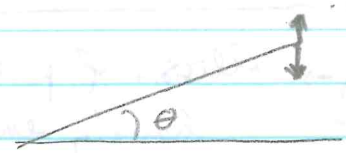
① Statement of the principle { Heisenberg's uncertainty principle

→ We cannot simultaneously measure the exact values of x & p_x.
(x : location) (p_x : momentum along that same axis)
Instead, the precision of the 2 measurements is limited by $\boxed{\Delta x \Delta p_x \geq \frac{\hbar}{2}}$

(thought)

(2) Bohr's gedanken experiment: (the Heisenberg microscope)





$$\sin\theta' \approx \frac{d/2}{f} \left(\approx \tan\theta'\right)$$

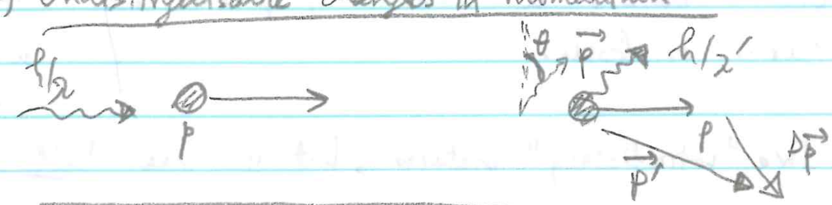
(1) Spatial resolution of a microscope Raleigh's criterion

$$\Delta\theta_R \simeq \frac{\lambda}{d} \simeq \sin\theta_R = \frac{\Delta x}{f} \Rightarrow \boxed{\Delta x \approx \frac{1}{2}\frac{\lambda}{\sin\theta'}}$$

diameter of lens

resolution effected by diffraction

(2) Undistinguisable changes in momentum



$$\boxed{P_x' = P_x + \frac{h}{\lambda} - \frac{h}{\lambda'}\sin\theta}$$

$$\Delta x \updownarrow \Delta p \updownarrow$$

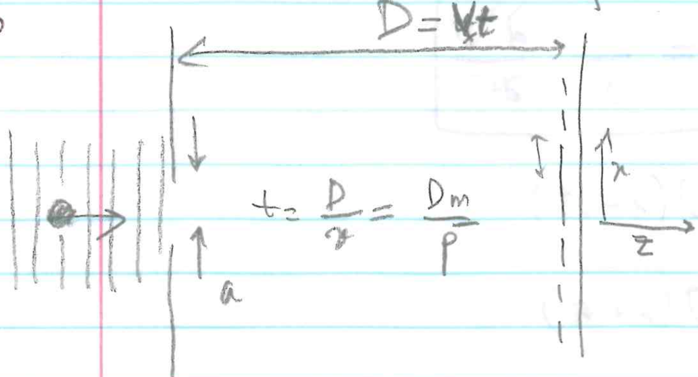
(☆) $\begin{cases} (P_x' \max) \Leftrightarrow \theta = -\theta' \Rightarrow P_x' = P_x + \dfrac{h}{\lambda} + \dfrac{h}{\lambda'}\sin\theta' \\ (P_x' \min) \Rightarrow \theta = \theta' \Rightarrow P_x' = P_x + \dfrac{h}{\lambda} - \dfrac{h}{\lambda'}\sin\theta' \end{cases}$

$$\Rightarrow \boxed{\Delta P = P_{max} - P_{min} = \frac{2h}{\lambda'}\sin\theta'}$$

$$\hookrightarrow \Delta P_x \Delta x = \frac{2h}{\lambda'}\cdot\sin\theta'\cdot\frac{\lambda'}{\sin\theta'} = 2h > \frac{h}{4\pi} = \frac{\hbar}{2}$$

3. Some consequences of the uncertainty principle

Nov 20



$x_{min} = n \cdot \frac{\lambda}{a} D$ ← wave picture (diffraction)

if $\frac{\lambda}{a} <<<< 1$

↳ can't really see diffraction.

Particle picture → going through the slit → know $\Delta x = a$

$$\rightarrow \Delta p_x \geq \frac{h}{4\pi \Delta x} \Rightarrow \Delta v_x = \frac{\Delta p_x}{m}$$

At the screen

$$\hookrightarrow \Delta x_{screen} = \Delta v_x \cdot t = \Delta v_x \cdot \frac{Dm}{p} = \Delta v_x \cdot \frac{D}{v}$$

"spread"

$$\Rightarrow \Delta x_{screen} = \frac{\Delta p_x}{\cancel{m}} \cdot \frac{D}{p_z} \cdot \cancel{m} = \frac{\Delta p_x \cdot D}{p_z} \Rightarrow \boxed{\Delta x_{screen} = \frac{h}{4\pi a} \cdot \frac{D}{p_z}}$$

Δp_x: $\frac{h}{4\pi \Delta x}$ → a

p_z: $\frac{h}{\lambda_z}$

$$\Rightarrow \boxed{\Delta x_{screen} = \frac{\lambda}{4\pi a} \cdot D}$$

D. WAVE FUNCTIONS

→ relationship between wave particle duality & uncertainty principle.

1) Born's interpretation

Light

$\vec{E} = \vec{E}(x,t)$

$I = \frac{1}{2} c \varepsilon_0 \vec{E}^2 = nh\nu$

Matter

$\Psi(x,t)$

$|\Psi(x,t)|^2$ → probability density ← dist. function

↳ "absolute square" (× by complex conj)

2. CLASSICAL WAVE

Wave equation $\boxed{\dfrac{\partial^2 \psi}{\partial x^2} = \dfrac{1}{v^2} \cdot \dfrac{\partial^2 \psi}{\partial t^2}}$

solutions: $\begin{cases} \psi_+ = \psi(x - vt) \\ \psi_- = \psi(x + vt) \end{cases}$

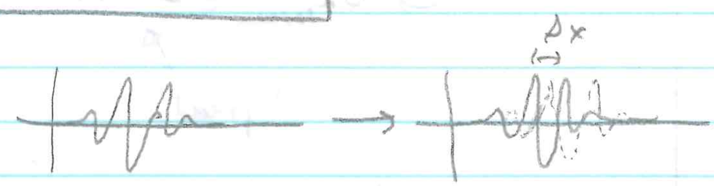
a) The principle of superposition

linear operation

Any two solutions added together are also solutions

$\boxed{\psi_1(x,t) + \psi_2(x,t) = \psi(x,t)}$ ← why interference works

b) localized wave



required to represent particle

c) Harmonic waves (unlocalized) $\psi_0(x,t) = \psi_0 \cos\left(\frac{2\pi}{\lambda}(x - vt)\right) = \psi_0 \cos(kx - wt)$

$\frac{2\pi}{\lambda}$ $\frac{2\pi}{T}$

$\lambda, \nu, v \longrightarrow k, \omega, v$

$(\lambda\nu = v)$ $\left(v = \frac{w}{k}\right) = \frac{\lambda}{T}$

d) Complex number $z = x + iy = |z|(\cos\phi + i\sin\phi)$

real img

$\boxed{z^* = x - iy = |z| e^{-i\phi}}$ $\boxed{z = |z| e^{i\phi}}$

complex conjugate

$\boxed{iz = ix - y = |z| e^{-i(\theta + \pi/2)}}$

Harmonic waves using de Broglie

$$E = h\nu = \hbar\omega$$
$$p = \frac{h}{\lambda} = \hbar k$$

$$\boxed{\begin{aligned}\psi(x,t) &= \psi_0 \cos(kx - \omega t) = \psi_0 \cos\left(\frac{p}{\hbar}x - \frac{E}{\hbar}t\right) \\ &= \mathrm{Re}\left[\psi_0 e^{i/\hbar(px - Et)}\right]\end{aligned}}$$

unlocalized

Harmonic waves have perfectly defined energy & momentum

d) **Wave packets** $\longrightarrow$ a superposition of harmonic waves that have some localization.

i) Simple example

2 harmonic waves addition

$$\boxed{\Psi(x,t) = \psi_0 \cdot \mathrm{Re}\left[e^{i(k_1 x - \omega_1 t)} + e^{i(k_2 x - \omega_2 t)}\right]}$$

Express in terms of $\bar{k} = \frac{k_1 + k_2}{2}$; $\bar{\omega} = \frac{\omega_1 + \omega_2}{2}$

$\Delta k = k_2 - k_1$; $\Delta\omega = \omega_2 - \omega_1$

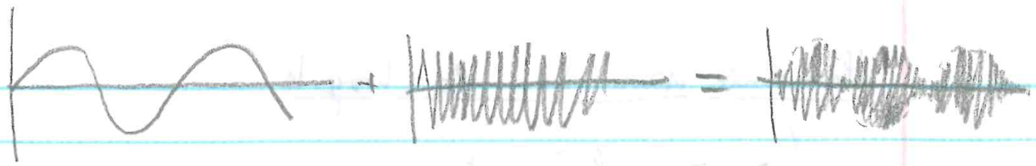
$$\Longrightarrow \psi(x,t) = \psi_0 \cdot \mathrm{Re}\left[e^{i(\bar{k}x - \bar{\omega}t)} \underbrace{\left\{e^{i/2(\Delta k x - \Delta\omega t)} + e^{-i/2(\Delta k x - \Delta\omega t)}\right\}}_{\text{sum of } z \,\&\, z^*}\right]$$

$$= 2\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$$

$$\boxed{\psi(x,t) = 2\psi_0 \cdot \underbrace{\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)}_{\text{group}} \cdot \underbrace{\cos(\bar{k}x - \bar{\omega}t)}_{\text{phase}}}$$

$v = \cancel{\lambda} \nu =$

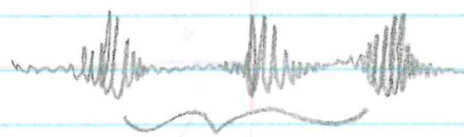
Exp :



$\boxed{\text{Phase velocity}}$ $v_p = W = \dfrac{\omega}{k}$

$\boxed{\text{Group velocity}}$ $v_g = V = \dfrac{\Delta\omega}{\Delta k} \longrightarrow \dfrac{d\omega}{dk}$

ii) → $\boxed{\text{More waves = more localization}}$



still repetition

$\boxed{\text{iii) Isolated Wave Packet}}$ → eliminate repetition

$\left(k = \dfrac{2\pi}{\lambda}\right)$

→ $\Delta x_{rep} \Rightarrow \infty \Rightarrow \Delta k \to 0 \to \cancel{\Delta \frac{2\pi}{\lambda} \to 0} \to$ ~~close~~

(space between distributions)

$\left(\lambda = \dfrac{h}{p}\right)$

Requires a continuum of waves

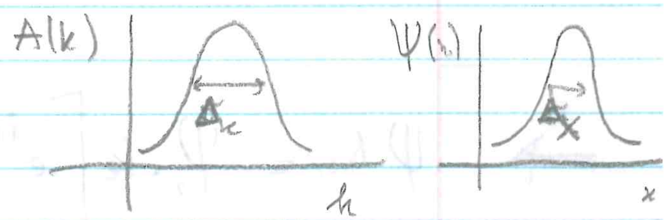
$$\boxed{\psi(x,t) = \mathrm{Re}\int_{-\infty}^{\infty} dk\, A(k) \cdot e^{i(kx - \omega t)}}$$

(k related to momentum)

$k = \dfrac{p}{\hbar} \rightarrow p = \hbar k$

probability of find a particle

$\boxed{\text{Fourier Integral}}$



$\boxed{\text{e) Fourier's theorem: uncertainty}}$

→ $\Delta x \Delta k \geq \dfrac{1}{2}$ → too make very localized wave packet
→ need a lot of wave number

→ $\dfrac{\Delta p}{\hbar}$

⇒ To express a limited wave ⇒ need a distribution of momentum

$\Rightarrow \boxed{\Delta x \cdot \Delta p \geq \dfrac{\hbar}{2}}$ (WTF? Heisenberg ...)

V Early Atomic Theory → Parallel Development to wave - particle duality.

A. Thomson's atomic model: "plum pudding"

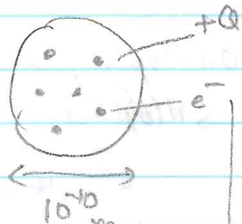
1. Subatomic particles: — electrons → "discovered" 1895 (J.J. Thomson)

$\hookrightarrow \frac{e}{m_e} = 1.76 \times 10^{11}\ C/kg$

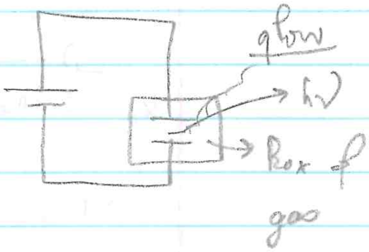
$e = 1.6 \times 10^{-19} C$

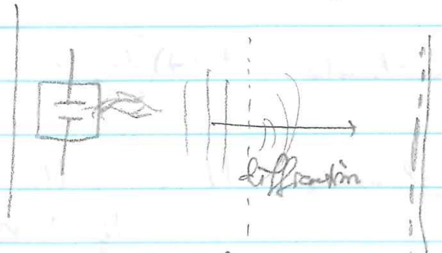
$m_e = 9.1 \times 10^{-31} kg << m_{atom}$

2. The Thomson's atomic model

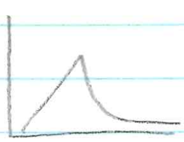


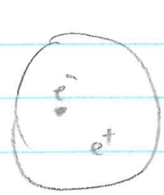
a) Atomic fluorescence spectroscopy





b) Problem with Thomson's model (hydrogen) → seems to have only $1 e^-$





$$F_{e^-} = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} & (r>R) \\ \frac{1}{4\pi\epsilon_0}\frac{e^2}{R^3} r & (r<R) \end{cases}$$

$\hookrightarrow$ For an "excited" electron $F = -kr$, $k = \frac{1}{4\pi\epsilon_0}\frac{e^2}{R^3}$ $\omega = \sqrt{\frac{k}{m_e}}$, $\nu = \frac{\omega}{2\pi}$

$\Rightarrow$ if $R \approx 1 Å \rightarrow \nu \approx 2.5 \times 10^{15} s^{-1} \rightarrow \lambda \approx 1200 Å$ (only 1 color)

BUT H has a rich spectrum

1885 — Johannes Balmer 4 wavelengths in H $\lambda = 3646 Å \cdot \left(\frac{n^2}{n^2-4}\right)$

$\hookrightarrow$ $\boxed{\nu = \frac{c}{\lambda} = c \cdot (1.097 \times 10^7 m^{-1}) \cdot \left(\frac{1}{2^2} - \frac{1}{n^2}\right)}$ $\hookrightarrow n = 3, 4, 5, 6$

(1.097... Rydberg const)

1890 — Johannes Rydberg

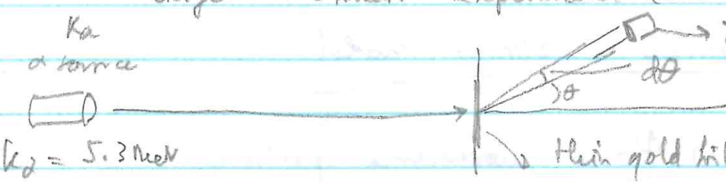
Rydberg const

$$\boxed{\nu = c \cdot \underbrace{(1.097 \times 10^{7}\, m^{-1})}_{R} \left(\frac{1}{m^2} - \frac{1}{n^2}\right)}$$

$\boxed{h \cdot cR = 13.6\ eV}$ → I_e of H.

B. Rutherford scattering and the nuclear model of atoms

→ The Geiger - Marsden Experiment (1912-1913) : Rutherford scattering



→ Measure distribution function $N(\theta)\, d\theta$ → $N(\theta)\, d\Omega$

efficiency

Clever solution to inefficiency N , N_G , N_M

$N_G = N \cdot \eta_g$ $\quad N_m = N \cdot \eta_m$

$N_{GM} = N \cdot \eta_G \cdot \eta_M \rightarrow \boxed{\frac{N_G \cdot N_M}{N_{GM}} = N}$

a) Experimental Result → $\Theta_{rms} \simeq 1^\circ$

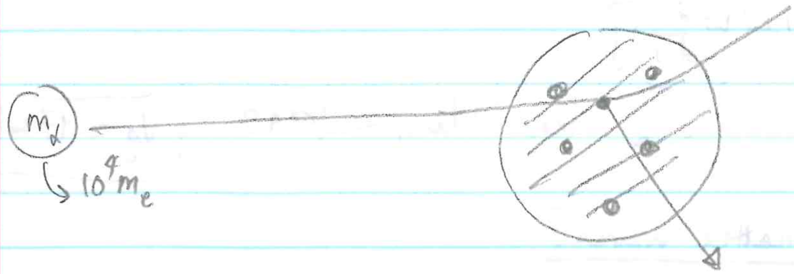
→ 99% scattered less than 3°

→ A few α's scattered by more than 90° (0.0001% – 0.01%)

Remarkable observation by Marsden!

→ The number of large-angle scattered α is linearly proportional to the foil thickness

(2) Thomson model predictions for Rutherford scattering ...



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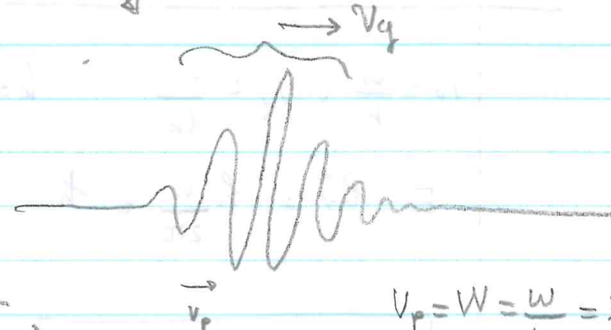
Nov 28, 2017

Group velocity & phase velocity



$$\psi = 2\psi_0 \underbrace{\cos\left(\frac{\Delta k x}{2} - \frac{\Delta \omega t}{2}\right)}_{\text{group}} \cdot \underbrace{\cos\left(\bar{k}x - \bar{\omega}t\right)}_{\text{phase}}$$

$$V_p = W = \frac{\omega}{k} = \lambda\nu$$

$$V_g = \frac{d\omega}{dk}$$

Example group & phase velocity for light in a medium

$$\lambda\nu = \frac{c}{n}$$ ← index of refraction

$$\lambda = \frac{2\pi}{k}, \quad \nu = \frac{\omega}{2\pi} \Rightarrow \boxed{\nu\lambda = \frac{\omega}{k} = \frac{c}{n} = V_p} \Rightarrow \boxed{\frac{ck}{n} = \omega}$$

Also: $\boxed{\frac{d\omega}{dk} = \frac{c}{n}}$ $\Rightarrow V_g = V_p$ true if n is a constant

So... what happens if $\boxed{n(\omega)\,?}$ $\quad n = n(\omega) = n(\omega(k))$

$$\Rightarrow \frac{d\omega}{dk} = \frac{d}{dk}\left[\left(\frac{c}{n(\omega)}\right)\cdot k\right] = c\cdot\left[\frac{1}{n(\omega)} + k\cdot\frac{d}{dk}\left(\frac{1}{n(\omega)}\right)\right]$$

$$-\frac{1}{n^2(\omega)}\cdot\frac{dn}{dk} = \boxed{-\frac{1}{n^2(\omega)}\cdot\frac{dn}{d\omega}\cdot\frac{d\omega}{dk}}$$

$$\frac{d\omega}{dk} = \frac{c}{n(\omega)} + \frac{(-1)}{n^2(\omega)}\cdot ck\cdot\frac{dn}{d\omega}\cdot\frac{d\omega}{dk}$$

$$\boxed{\frac{d\omega}{dk} = V_g = \frac{c}{n(\omega)\left[1 + \frac{c\omega}{n(\omega)}\cdot\frac{dn}{d\omega}\right]}} \; ; V_p \neq V_g$$

$$\hookrightarrow \frac{d\omega}{dk} = \frac{c}{\left[n(\omega) + \omega \frac{dn}{d\omega}\right]} = V_g$$

Lene Hau – 1999 $\boxed{V_g = 17\,m/s}$

Ok… what about matter waves?

$$V_p = \frac{\omega}{k}, \quad V_g = \frac{d\omega}{dk}, \quad k = \frac{2\pi}{\lambda}; \quad p = \frac{h}{\lambda} \Rightarrow p = \frac{h}{2\pi}k = \hbar k$$

$$E = h\nu = h\frac{\omega}{2\pi} = \hbar\omega$$

$$\Rightarrow \omega = \frac{E}{\hbar}, \quad k = \frac{p}{\hbar}$$

$$\Rightarrow \boxed{V_p = \frac{\omega}{k} = \frac{E/\hbar}{p/\hbar} = \frac{E}{p}} \qquad \boxed{V_g = \frac{d\omega}{dk} = \frac{dE}{dp}}$$

Nonrelativistic particle $\boxed{E = K = \frac{1}{2}mv^2 = \frac{p^2}{2m}}$

$$\boxed{V_p = \frac{E}{p} = \frac{p}{2m} = \frac{mv}{2m} = \frac{v}{2}}$$

V_p does NOT represent the particles' velocity

$$\boxed{V_g = \frac{dE}{dp} = \frac{2p}{2m} = \frac{p}{m} = v}$$

V_g represents the particle's speed.

Relativistic Particle $E = \sqrt{c^2p^2 + E_{rest}^2} = \gamma(u)mc^2 = \sqrt{c^2p^2 + (mc^2)^2}$

$$\boxed{E = (mc^2)\cdot\sqrt{1 + \left(\frac{cp}{mc^2}\right)^2}}$$

$$\boxed{E = cp\sqrt{1 + \left(\frac{mc^2}{cp}\right)^2}}$$

$V_p \geq c$

$$\boxed{V_p = \frac{E}{p} = c\sqrt{1+\left(\frac{mc^2}{cp}\right)^2}} \to c \quad \text{if } mc^2 \ll cp \text{ (massless...)}$$

$$\to \infty \quad \text{if } cp \ll mc^2 \text{ (massive...)}$$

↳ again, phase velocity cannot represent particle's velocity.

$$V_g = \frac{dE}{dp} = \frac{1}{2}\left(c^2p^2 + (mc^2)^2\right)^{-1/2} \cdot 2c^2p = \frac{c^2p}{E} = \boxed{c \cdot \left(\frac{cp}{E}\right) = V_g}$$

$$\boxed{V_g = \frac{c}{\sqrt{1+\left(\frac{mc^2}{cp}\right)^2}}} \to c \quad \text{if } m \to 0 \ (mc^2 \ll cp)$$

$$\to \approx$$

Also, ...

$$\boxed{V_g = \frac{\cancel{c^2}p}{\cancel{mc^2}\sqrt{1+\left(\frac{cp}{mc^2}\right)^2}} = \frac{p/m}{\sqrt{1+\left(\frac{cp}{mc^2}\right)^2}}} \to \left[\frac{p}{m}\right] \quad \text{if } (mc^2 \gg cp)$$

If one express $p = \frac{h}{\lambda}$, then we can write $\boxed{V_g = \frac{c}{\sqrt{1+\left(\frac{mc^2}{ch}\lambda\right)^2}}}$

So... imagine a photon is a particle with mass m_γ

↳ It would travel with speed $\boxed{V_g \lesssim c}$ but the speed would be wavelength dependent.

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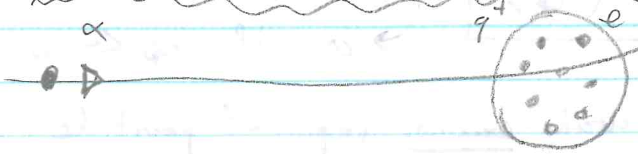
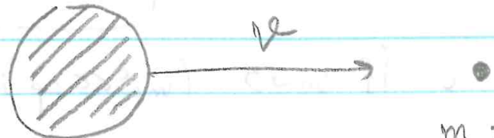
Nov 29, 2017 Thomson's model predictions for Rutherford scattering (cont)

or Why was Rutherford so surprized?

a) Estimate of deflection angle due to a collision



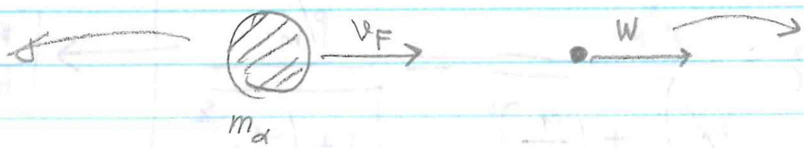
i) Elastic scattering from an electron



$m_\alpha = 4m_p$ $\qquad$ $m_e = \frac{1}{1836} m_p$

$\Downarrow$ assume head-on collision ...

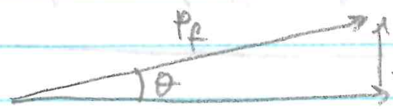
$v_F = \underbrace{\frac{m_\alpha}{m_\alpha + m_e}}_{\approx 1} v$



$w = \frac{2m_\alpha}{m_\alpha + m_e} v \approx 2v$

$\boxed{v_F \approx v}$

Estimate that



$\tan\theta = \frac{\Delta p}{p} = \frac{2m_e V}{m_\alpha V} \approx \boxed{2.7\times10^{-4}} = \tan\theta$ ← biggest angle possible

For $\tan\theta \ll 1 \longrightarrow \tan\theta = 2.7\times10^{-4} \simeq \theta$

$\boxed{\theta \approx 0.02^\circ}$ ⇒ NOT explained by Rutherford's exp!

ii) Estimate from a massive ($\approx$ stationary) positive charge

$m_{Au} = 197\, m_p$

$q_{Au} = +79e = Z_{Au}\, e$

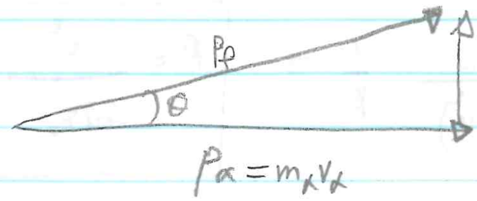
$q_\alpha = +2e = +Z_\alpha e$

$$F_e = \begin{cases} \dfrac{Z_1 Z_2 e^2}{4\pi\epsilon_0} \cdot \dfrac{r}{R^3} & r < R \\[2ex] \dfrac{Z_1 Z_2 e^2}{4\pi\epsilon_0} \cdot \dfrac{1}{r^2} & r > R \end{cases}$$

$$F_{max} = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0} \cdot \frac{1}{R^2}$$

Estimate : f_{max} acts for time : $\Delta t = \frac{2R}{v_\alpha}$ (across atom)

$$\Delta p = f \Delta t$$



$$\Delta p = f_{max} \cdot \Delta t = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0} \frac{2R}{R^2 v_\alpha}$$

$$= \cancel{Z_1 Z_2}$$

$$\boxed{\frac{\Delta p}{p} = \tan\theta = \frac{e^2}{4\pi\varepsilon_0} \cdot \frac{Z_1 Z_2}{R} \cdot \frac{1}{\left(\frac{1}{2} m_\alpha v_\alpha^2\right)}}$$

plug in the numbers... $R = 1.0\,\text{Å}$

$\tan\theta \simeq 4.6 \times 10^{-4} \approx \theta \Rightarrow \boxed{\theta \approx 0.03^\circ}$, again, NOT even close to 90°...

Why?

b) Multiple scattering... If you pass N electrons (or Atoms) and are scattered from each, what happens to θ?

(random walk...)

c) Theory vs. Exp

In a random walk, N steps, then $\Theta_{rms} = \sqrt{N} \cdot \Theta_{rms}$

$$\frac{N(\Theta)\,d\Theta}{I} = \frac{2\Theta}{\overline{\Theta^2}} e^{-\Theta^2/\overline{\Theta^2}} d\Theta \ldots$$

incident

predictions : $\theta_{rms} \approx 1^\circ \longrightarrow$ Agree! $\leftarrow$ small angle scat is by electrons...

$f(<3^\circ) > 99\% \longrightarrow$ agrees w/ exp...

$f(>90^\circ) \approx 10^{-3500} \ldots \longrightarrow$ does NOT agree w/ exp...

$f(>90^\circ) \propto \sqrt{\text{thickness}} \longrightarrow$ NOT agree w/ exp...

d) **Rutherford's Idea:** $\frac{\Delta p}{p} = \left[\frac{e^2}{4\pi\epsilon_0}\underbrace{\frac{z_1 z_2}{\left(\frac{1}{2}m_\alpha v_\alpha^2\right)}}_{\text{const}}\right]\frac{1}{R}$ — change R to what θ?

→ If $R \ll 1\mathring{A}$ then $\tan\theta \approx 1$

What R is required to get $\tan\theta \approx 1$?

→ $\frac{\Delta p}{p} = 1 = \frac{e^2}{4\pi\epsilon_0}\cdot\frac{z_1 z_2}{\left(\frac{1}{2}m_\alpha v_\alpha^2\right)}\cdot\frac{1}{R} \Rightarrow \boxed{R = \frac{e^2}{4\pi\epsilon_0}\cdot\frac{z_1 z_2}{\left(\frac{1}{2}m_\alpha v_\alpha^2\right)}}$

→ $\boxed{R \approx 4.6\times10^{-4}\mathring{A}}$

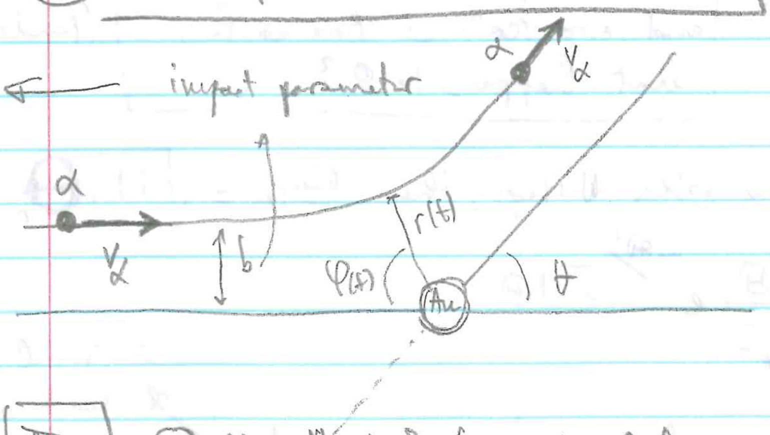
→ Idea → All the mass & positive charge are in a tiny volume at the center of an atom

→ "nuclear" atom

Detailed Calculation...

(3) Rutherford's Detailed Calculations → Assumptions:

- ignore e^- scattering
- nonrelativistic
- scattering from heavy atom → ignore recoil
  - → $v_i = v_f$ (α)
- α particles never enters nucleus
- α are uniformly distributed across the foil



$\boxed{\text{Idea}}$ (1) Use Newton's laws to find $\boxed{\theta(b)}$

(2) Determine $N(b)\,db = dN$

→ Determine $N(\theta)\,d\theta$...

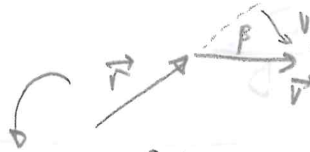
Solution for a single collision

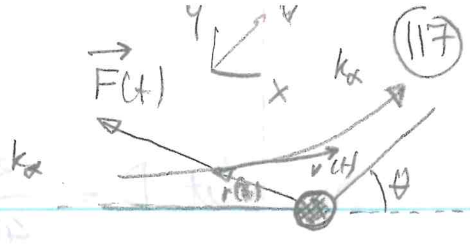
→ (1) Conserv. of E → $v_\alpha = v_f$

→ (2) Conserv. of L → $\tau = \vec{0}$ for radial force

$L = mvr_\perp = mv_\perp r$

→ (3) $F = ma$





Dec 1, 2017

6) (cont) Solution for a single particle k_α

$$|\vec{L}| = |\vec{r}\times\vec{p}| = |\vec{r}\times m\vec{v}|$$

$$= r m v_i \underbrace{\sin\beta}_{v_T} \quad \text{or} \quad r v r_i \underbrace{\sin\beta}_{r_\perp}$$

$$= \boxed{m v_\perp r} = \boxed{m v r_\perp}$$

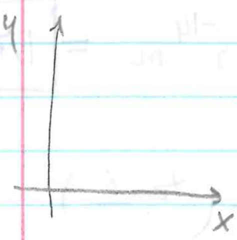
$$|\vec{F}| = \frac{Z_1 Z_2}{4\pi\epsilon_0}\frac{1}{r^2}$$

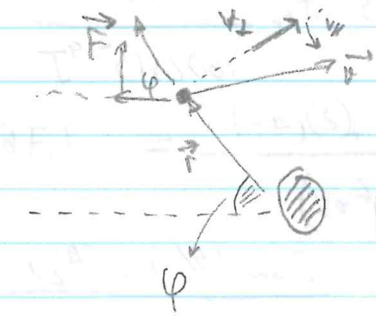
$$\boxed{\vec{\tau} = \vec{r}\times\vec{F} = \vec{0} \Rightarrow \vec{L} \text{ const}}$$

Initially $(t=-\infty)$ $\quad r_\perp = b \Rightarrow \boxed{L = m_\alpha v_\alpha b}$ (const thru motion)

$\vec{F} = m\vec{a}$

Goal
Find $v_y(t\to\infty)$
$\to v_\alpha \sin\theta$



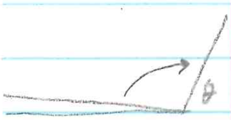


$\boxed{L = m_\alpha \cdot r \cdot v_\perp} \to$ does not depend on how r is changing

$$L = m_\alpha r \cdot \left(r \cdot \frac{d\varphi}{dt}\right) \to r\omega = v$$

$$\Rightarrow \boxed{L = m_\alpha r^2 \cdot \frac{d\varphi}{dt} = m_\alpha v_\alpha \cdot b}$$

$$\Rightarrow \frac{d\varphi}{dt} = \frac{v_\alpha b}{r^2} \Rightarrow \boxed{\frac{1}{r^2} = \frac{1}{v_\alpha b}\cdot\frac{d\varphi}{dt}}$$



$$f_y = \frac{e^2}{4\pi\epsilon_0}\frac{Z_1 Z_2}{r^2}\cdot\sin\varphi = m_\alpha \cdot \frac{dv_y}{dt} \Rightarrow \int_i^f \frac{e^2}{4\pi\epsilon_0}\cdot\frac{Z_1 Z_2}{v_\alpha b}\cdot \overbrace{\frac{d\varphi}{\cancel{dt}}}^{\sin\varphi} = \int_i^f m_\alpha \cdot \frac{dv_y}{\cancel{dt}}$$

$$\frac{1}{r^2} = \frac{1}{v_\alpha b}\cdot\frac{d\varphi}{dt}$$

$$\int_0^{\pi-\theta} \frac{e^2}{4\pi\epsilon_0}\cdot\frac{Z_1 Z_2}{v_\alpha b m_\alpha} \sin\varphi \, d\varphi = \int_{v_y=0}^{v_y = v_\alpha\cdot\sin\theta} dv_y$$

$$v_\alpha \cdot \sin\theta = \frac{e^2}{4\pi\epsilon_0}\cdot\frac{Z_1 Z_2}{\frac{1}{2}m_\alpha v_\alpha}\cdot\frac{1}{2b} \Rightarrow \boxed{\sin\theta = \frac{e^2}{4\pi\epsilon_0}\cdot\frac{Z_1 Z_2}{K_\alpha}\frac{1}{2b}(1+\cos\theta)}$$

$$\cdot\left[-\cos(\pi-\theta)+\cos 0\right]$$

$$k_\alpha = \frac{e^2}{4\pi\varepsilon_0}\frac{Z_1 Z_2}{D}$$

Let $D = \frac{e^2}{4\pi\varepsilon_0} \cdot \frac{Z_1 Z_2}{k_\alpha}$ $\Rightarrow$ $\boxed{\sin\theta = D \cdot \frac{1}{2b}(1+\cos\theta)}$

Example Silver $Z_2 = 47$ $\quad k_\alpha = 7.71 \text{ MeV}$

$v_\alpha = 1.92 \times 10^7 \text{ m/s}$

$$\frac{e^2}{4\pi\varepsilon_0} = 2.31 \times 10^{-28} \text{ Nm}^2 = 2.31 \times 10^{-28} \text{ Jm}$$

$$= 2.31 \times 10^{-28} \text{ Jm} \frac{1 eV}{1.602 \times 10^{-19} J} = \boxed{1.44 \times 10^{-9} \text{ eVm}}$$

$$D = \frac{(1.44 \times 10^{-9}) \text{ eVm} \cdot (2)(47)}{(7.71 \times 10^6 \text{ eV})} = 1.76 \times 10^{-14} \text{ m} = \boxed{1.76 \times 10^{-4} \text{ Å}}$$

$$\frac{\sin\theta}{1+\cos\theta} = \frac{D}{2b} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\frac{\theta}{2})} = \left(\tan\frac{\theta}{2}\right)$$

$$\boxed{\tan\frac{\theta}{2} = \frac{D}{2b}}$$

c) The Angular distribution function $\Rightarrow$ $\boxed{\cot\frac{\theta}{2} = \frac{2b}{D}}$

Use this relation to relate $N(\theta)d\theta$ to $N(b)db$

corresponding interval...

$$\Rightarrow db = \frac{db}{d\theta} \cdot d\theta = \frac{d}{d\theta}\left(\frac{D}{2} \cdot \cot\frac{\theta}{2}\right) \cdot d\theta$$

$$= \frac{-D}{4} \cdot \frac{1}{\sin^2\frac{\theta}{2}} \cdot d\theta$$

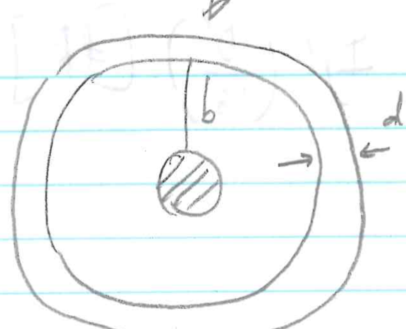
$$N(\theta)d\theta = -N(b)db$$

(further away) $b\uparrow \rightarrow \theta\downarrow$ (less deflection)

$A_{ring} = (2\pi b)\,db$

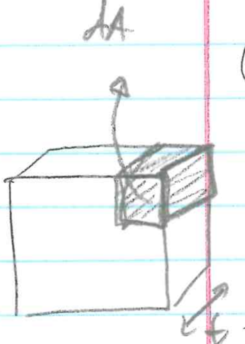
What is $N(b)\,db$?

Head-on view



$N(b)\,db = \dfrac{A_{ring}}{A_{target}} \cdot N_{atom}$ @target

but $\dfrac{N_{atom}}{A_{target}}$ = density!



$$\boxed{N(b)\,db = n \cdot (2\pi b)\,db}$$

[# of atoms/unit area] = [# atom in foil / unit area]

m^2

$n = \dfrac{\#}{m^3} \cdot$ thickness $= \boxed{\rho_\# t = n}$ → thickness

(number density) = $\dfrac{\text{mass density}}{\text{mass of atom}}$

$$\boxed{\#\ \text{density} = \frac{\rho_{mass}}{m_{atom}} = \rho_{mass} \cdot \frac{N_A}{m_{mole}}}$$

incident particles

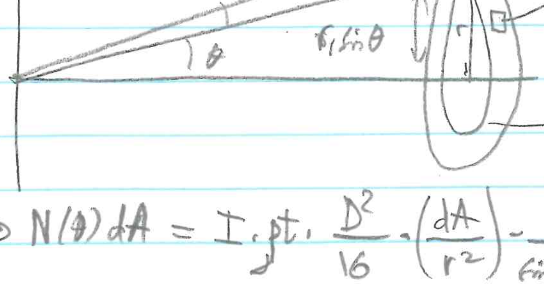
$\Rightarrow \quad N(b)\,db = I \cdot \rho t \cdot (2\pi b)\,db$

$\dfrac{D}{2}\cot\dfrac{\theta}{2}$ $\qquad -\dfrac{D^2}{4}$

$$\boxed{-N(\theta)\,d\theta = -I \cdot n \cdot \left(\frac{D^2}{8}\right)(2\pi)\frac{\cos\theta/2}{\sin^2\theta/2}\,d\theta}$$

$$\boxed{N(\theta)\,d\theta = I \cdot \rho t \cdot \frac{D^2}{16} \cdot \frac{2\pi \sin\theta\, d\theta}{\sin^4\left(\frac{\theta}{2}\right)}}$$

d) The "differential cross section"



$N(\theta)\,d\theta = \dfrac{dA}{A_{ring}}$

$A_{ring} = (2\pi r \sin\theta)(\underbrace{d\theta \cdot r}_{dr}) = [2\pi \sin\theta\, d\theta] \cdot r$

$\hookrightarrow N(\theta)\,dA = I \cdot \rho t \cdot \dfrac{D^2}{16} \cdot \left(\dfrac{dA}{r^2}\right) \cdot \dfrac{1}{\sin^4(\theta/2)}$

$$\frac{dA}{r^2} = \text{solid angle.}$$

$$\rightarrow N(\theta)\,dA = I_0 t \left(\frac{D^2}{16}\right) \cdot \left[dR\right] \frac{1}{\sin^4\left(\frac{\theta}{2}\right)}$$

$\frac{dA}{r^2}$ = solid angle.

$$\rightarrow N(\theta)dA = I_0 t \left(\frac{D^2}{16}\right) \cdot \left[dR\right] \frac{1}{\sin^4\left(\frac{\theta}{2}\right)}$$

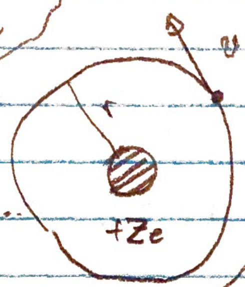
Lec 3_17

D. Bohr's Atomic Model

1) The "Rutherford Memorandum"

hydrogenic atom

with only 1e⁻ orbiting around...



$$F = ma = \frac{mv^2}{r}, \quad v = \omega r = 2\pi\nu r$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{Ze^2}{r^2} = \frac{mv^2}{r} = \frac{m(2\pi\nu r)^2}{r} = m\,4\pi^2\nu^2\cdot r$$

$$E = \frac{1}{2}mv^2 + \left(\frac{-Ze^2}{4\pi\varepsilon_0}\cdot\frac{1}{r}\right)$$

$$\Rightarrow \frac{1}{2}mv^2 = +\frac{1}{2}\,\frac{1}{4\pi\varepsilon_0}\cdot\frac{Ze^2}{r} \rightarrow \boxed{E = \frac{-1}{2}\,\frac{Ze^2}{4\pi\varepsilon_0}\cdot\frac{1}{r}} < 0$$

$$\boxed{r = \frac{-1}{2}\,\frac{Ze^2}{4\pi\varepsilon_0}\cdot\frac{1}{E} > 0}$$

$$\boxed{\nu^2 = \frac{1}{m\,4\pi^2 r}\left(\frac{1}{4\pi\varepsilon_0}\,\frac{Ze^2}{r^2}\right) = \frac{1}{4\pi^2}\left(\frac{Ze^2}{4\pi\varepsilon_0}\right)\cdot\frac{1}{mr^3}}$$ Kepler's law

$$|\vec{L}| = |\vec{r}\times\vec{p}| = |\vec{r}\times m\vec{v}| = rmv = \boxed{2\pi r^2 m\nu = |\vec{L}|}$$

a) Problem with the planetary model

Jose E

1. Electromagnetic instability → accelerating charges radiate (E)
   ↳ e⁻ would "spiral in" to the nucleus...
2. Mechanical instability → planetary system w/ multiple planets are stable ⟺ all forces are attractive.

③ No fundamental atomic length

Observation: $\frac{\hbar^2}{m_e} \cdot \frac{1}{\frac{e^2}{4\pi\varepsilon_0}} = \frac{(\hbar c)^2}{m_e c^2} \cdot \frac{1}{e^2/4\pi\varepsilon_0} = 0.53\,\mathring{A} \ldots$

→ maybe Planck's constant is important in atoms too...

2) "The Trilogy" — 3 papers by Bohr

a) Bohr's understanding of the Balmer / Rydberg + Ritz formula.

$\nu = cR_H\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$

puts h into the eq.

$E = h\nu = hcR_H\left(\frac{1}{m^2} - \frac{1}{n^2}\right) = -\Delta E = E_i - E_f$

E_i, E_f → $E_\gamma = E_i - E_f$

This implies: $\boxed{E_n = -hcR_H \cdot \frac{1}{n^2}}$ (*)

works but only by adding an empirical constant R_H (Rydberg...)

different for different Z...

b) The postulates

what he actually wrote

① There are certain radii where an e^- can orbit w/o continuous radiation

② Transitions between 2 "stationary" states occur with the emission of 1 photon ($E_\gamma = h\nu = E_i - E_f$)

$$(*) \Rightarrow \begin{cases} r = \frac{1}{2}\left(\frac{Ze^2}{4\pi\varepsilon_0}\right) \cdot \frac{n^2}{hcR_Z} = \frac{(\hbar c)^2}{(m_e c^2)} \cdot \frac{1}{\left(\frac{e^2}{4\pi\varepsilon_0}\right)} \frac{n^2}{Z^2} \\ \nu^2 = \frac{1}{4\pi^2}\left(\frac{Ze^2}{4\pi\varepsilon_0}\right) \cdot \frac{1}{m} \cdot \left(\frac{hcR_Z}{n^2} \cdot \left(\frac{4\pi\varepsilon_0}{Ze^2}\right) \cdot 2\right)^3 \end{cases}$$

$\hookrightarrow = \ldots$

3 Correspondence principle

In the limit of $n \gg 1$, the behavior of a quantum system must be the same as a classical system...

allows us to find R_Z

c) Using the Correspondence Principle to find R_Z

A classical atom "emits" radiation @ frequency ν — rotating charge

$$\nu^2 = \frac{2}{\pi^2}\left(\frac{1}{\left(\frac{Ze^2}{4\pi\epsilon_0}\right)^2}\right) \cdot \frac{(hcR)^3}{m_e n^6} \qquad (n\gg \rightarrow \nu \ll)$$

A quantum atom emits:

$$\nu = \frac{E_n - E_{n-1}}{h} = \frac{hcR_Z}{h}\left(-\frac{1}{n^2} + \frac{1}{(n-1)^2}\right)$$

$$\nu = cR_Z\left[\frac{2n-1}{n^2(n-1)^2}\right]$$

As $n \gg 1 \Rightarrow \nu = cR_Z \cdot \frac{2}{n^3}$

$$\Rightarrow \nu^2_{n\rightarrow n-1} = \frac{4c^2 R_Z^2}{n^6} \quad \text{whereas} \quad \nu^2 = \text{stuff} \cdot R_Z^3$$

$$\boxed{R_Z = \left(\frac{Ze^2}{4\pi\epsilon_0}\right)^2 \cdot \frac{2\pi^2}{h^3c^2} \cdot m_e c^2}$$

no longer an empirical constant that depends on Z

d. Comparison of $R_{Empirical}$ & $R_{Theoretical}$

1913: $\begin{cases} R_{exp} = 109{,}700 \text{ cm}^{-1} \\ R_{theory} = 103{,}300 \text{ cm}^{-1} \end{cases}$ ... not bad... because e was not known...

e) Quantization of angular momentum

$$\boxed{|\vec{L}_n| = 2\pi m_e \nu r_n^2 = n\hbar = \frac{nh}{2\pi}}$$ ⟹ "quantization condition" ...

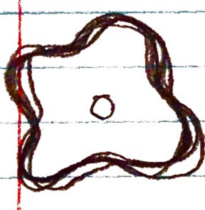
3) deBroglie's understanding of the quantization condition

$$\lambda = \frac{h}{p} \longleftrightarrow p = \frac{h}{\lambda}$$

$$L = rp = r\frac{h}{\lambda} = \frac{nh}{2\pi} \rightarrow 2\pi r = n\lambda$$

circumference of circle = n · wavelength

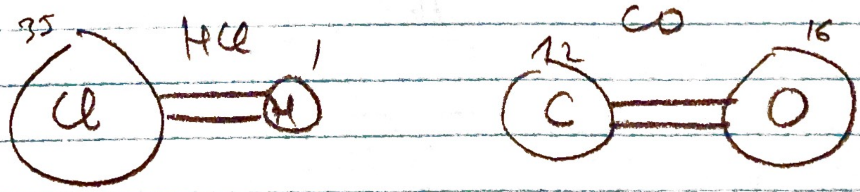
↳ standing wave ...



← orbits are standing waves...

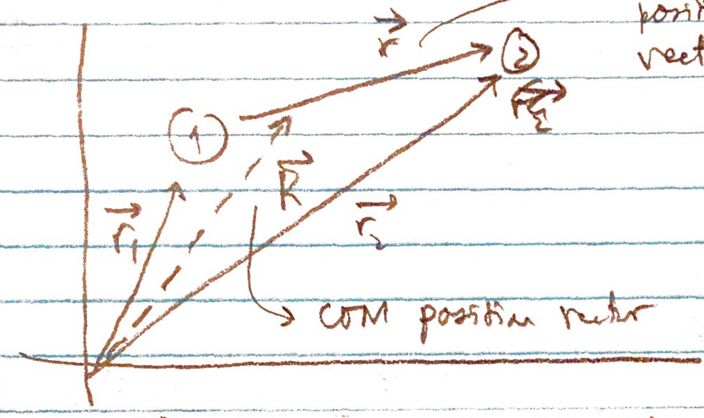
Dec 5, 2017

Molecular Rotation



Do these molecules exhibit the properties $L = J\hbar$?

Consider General Problem



$KE = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$

↳ we want to express the energy in terms of ~~vel~~ $\vec{V}_{com}$ & $\vec{v}$

$\vec{V}_{com} = \frac{d\vec{R}}{dt}$, $\vec{v} = \frac{d\vec{r}}{dt}$

$\vec{r}_1 + \vec{r} = \vec{r}_2 \Rightarrow \vec{r} = \vec{r}_2 - \vec{r}_1$

$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \rightarrow (m_1+m_2)\vec{R} = m_1\vec{r}_1 + m_2(\vec{r}_1 + \vec{r}) = (m_1+m_2)\vec{r}_1 + m_2\vec{r}$

$$\Rightarrow \boxed{\vec{r}_1 = \vec{R} - \frac{m_2}{m_1+m_2}\vec{r}}$$

$$\rightarrow \boxed{\vec{v}_1 = \vec{V}_{com} - \frac{m_2}{m_1+m_2}\vec{v}} \qquad \boxed{\vec{v}_2 = \vec{V}_{com} + \frac{m_1}{m_1+m_2}\vec{v}}$$

$$\rightarrow KE = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1\left(V_{cm}^2 - \cancel{\frac{2m_2}{m_1+m_2}\vec{V}_{cm}\cdot\vec{v}} + \left(\frac{m_2}{(m_1+m_2)}\right)^2 v^2\right)$$

$$+ \frac{1}{2}m_2\left(V_{cm}^2 + \cancel{\frac{2m_1}{m_1+m_2}\vec{V}_{cm}\cdot\vec{v}} + \left(\frac{m_1}{m_1+m_2}\right)^2 v^2\right)$$

$$KE = \frac{1}{2}(m_1+m_2)V_{cm}^2 + \frac{\frac{1}{2}v^2}{(m_1+m_2)^2}\underbrace{\left(m_1 m_2^2 + m_1^2 m_2\right)}_{m_1 m_2 (m_2+m_1)}$$

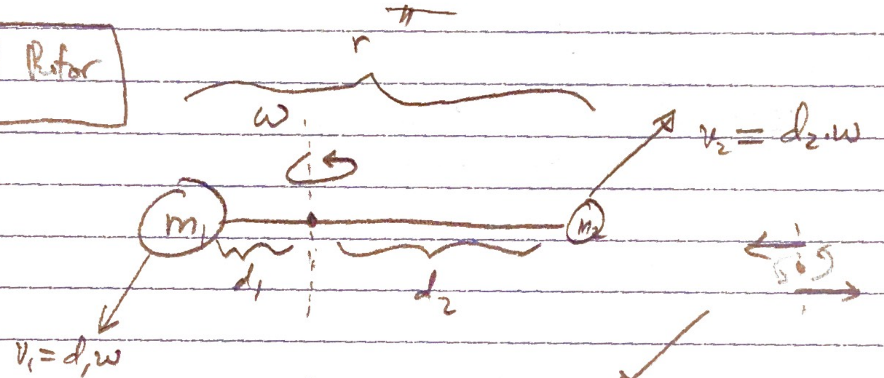
$$\boxed{KE = \frac{1}{2}\underbrace{(m_1+m_2)}_{M}V_{cm}^2 + \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)v_{rel}^2}$$

M (total mass)

μ = "reduced" mass

$HCl \rightarrow \mu_{HCl} = \frac{35 \cdot 1}{35+1} = \frac{35}{36}$, $CO \rightarrow \mu_{CO} = \frac{12 \cdot 16}{12+16} = 6.86$

Rigid Rotor



$\vec{v}_{rel} = \vec{v}_2 - \vec{v}_1 \Rightarrow |\vec{v}_{rel}| = v_2 + v_1 = w(d_1+d_2) = w \cdot d = \boxed{wr}$

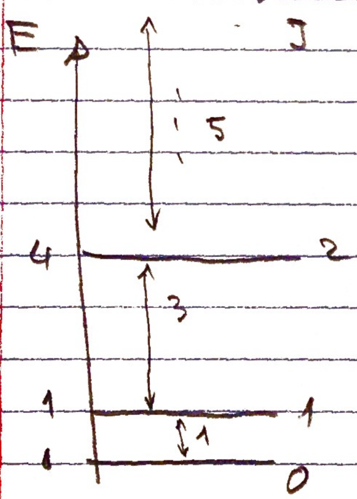
$KE = \frac{1}{2} M V_{cm}^2 + \frac{1}{2}\mu v_{rel}^2 = \boxed{\frac{1}{2} M V_{cm}^2 + \underbrace{\frac{1}{2}\mu \cdot w^2 r^2}_{=\frac{1}{2} I_{cm} w^2}}$

$\mu r^2 = I = m_1 d_1^2 + m_2 d_2^2$

$L = I w = J\hbar$ ← quantization condition

$w = \frac{J\hbar}{\mu r^2} \Rightarrow KE_{rotation} = \frac{1}{2}(\mu r^2) w^2 = \boxed{\frac{1}{2}\frac{\hbar^2}{\mu r^2} J^2 = KE}$

$J = 0, 1, 2, 3, \ldots$



$\Delta E_{J/J+1} = \frac{\hbar^2}{2\mu r^2}\left[(J+1)^2 - J^2\right]$

$$\boxed{\Delta E_{J/J+1} = \frac{\hbar^2}{2\mu r^2}(2J+1)}$$

slope

$$\boxed{\Delta \nu_{J/J+1} = \frac{\Delta E_{J/J+1}}{h} = \frac{\hbar}{4\pi \mu r^2} + \frac{\hbar}{\cancel{4\pi}\mu r^2} J}$$

$\hbar = \frac{h}{2\pi}$

$$\frac{h}{8\pi^2 \mu r^2}$$

twice a

$$\Delta\nu_{HCl} = a + bj = \frac{\hbar}{4\pi\mu r^2} + \frac{h}{(2\pi)^2 \mu r^2} J$$

$$b = \frac{h}{(2\pi)^2 \mu r^2} \Rightarrow \boxed{r^2 = \frac{1}{4\pi^2 \mu b}}$$

$\mu = \frac{35}{36} \cdot (1.66 \times 10^{-27}\,kg)$

$b = 6.52 \times 10^{11}\,1/s$

$h = 6.626 \times 10^{-34}\,Js$

$$\boxed{r_{HCl} \approx 1.3\,\mathring{A}}$$

Same thing for CO $\boxed{r_{CO} \approx 1.13\,\mathring{A}}$ (smaller in r than HCl)

☆ Problem Bohr : $\boxed{|\vec{L}| = J\hbar}$ ← Bohr's h...

$\Rightarrow \Delta\nu = a + bj$, where $\boxed{b = 2a}$

But the correct quantum theory $\Rightarrow$ $|\vec{L}| = \sqrt{J(J+1)}\,\hbar$

$$K = \frac{1}{2} I\omega^2 = \frac{1}{2}(\mu r^2)\omega^2 = \boxed{\frac{1}{2}\frac{\hbar^2}{\mu r^2}(J(J+1)) = K}$$

$$\frac{1}{2}\frac{L^2}{I} \rightarrow \mu r^2$$

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$E = KE + U$

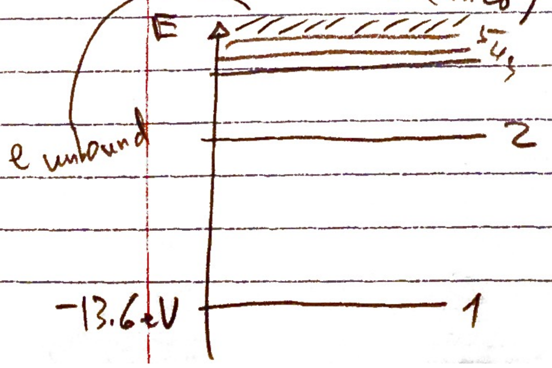
$= -\frac{U}{2}$

$$E_n = \frac{-hcR_Z}{n^2} = -\left(\frac{Ze^2}{4\pi\varepsilon_0}\right)\frac{m_e}{2\hbar}\cdot\frac{1}{n^2} \approx \boxed{-13.6\,eV\,\frac{Z^2}{n^2}}$$

$$R_1 = \left(\frac{e}{4\pi\varepsilon_0}\right)^2 \frac{m_e c^2}{4\pi(\hbar c)^3}$$

$$r_n = -\frac{1}{2}\left(\frac{Ze^2}{4\pi\varepsilon_0}\right)\frac{1}{E_n} = \frac{(\hbar c)^2}{m_e c^2}\,\frac{1}{\left(\frac{e^2}{4\pi\varepsilon_0}\right)}\cdot\frac{n^2}{Z^2}$$

$$\nu = \frac{\Delta E}{h} = cR\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$



L: ~~Lyman~~: 2→1, 3→1, 4→1

B: 3→2

(?): ...

fine structure constant

Define $\frac{e^2}{4\pi\varepsilon_0} \cdot \frac{1}{\hbar c} = \alpha \simeq \boxed{\frac{1}{137}}$, $a_0 = \frac{(\hbar c)^2}{\left(\frac{e^2}{4\pi\varepsilon_0}\right)} \cdot \frac{[\text{illegible}]}{(m_e c^2)} = 0.529 \times 10^{-10}\,\mathrm{m}$

$= \boxed{0.529\,\text{Å}}$

Bohr radius

$$\Rightarrow \boxed{E = -\frac{1}{2}(m_e c^2)\, Z^2 \alpha^2 \cdot \frac{1}{n^2}}$$

$$\Rightarrow \boxed{r_n = \frac{a_0}{Z} n^2}$$

E. Extension to the Bohr hydrogen model

(1) Isotope Shift

"The Pickering lines" – 1896

↳ looked into stars → see spectrum → He

but didn't agree with Bohr...

Bohr 1913

→ Spectra corresponds to $Z = 2$, $\boxed{R_2 = 4R_1}$

↳ Spectral line from singly ionized He

Bohr's problem

↳ ∞ mass nucleus...

However, the real ratio is $\boxed{4.0016}$ → Bohr's model not really true because e^-, p^+ orbit around their COM

↳ needs small correction

a) Reduced mass & the hydrogen spectrum

→ p^+ & e^- really orbit their mutual COM

Bohr: $E = \frac{-U}{2}$ ... Really: $\boxed{E = \underbrace{\frac{1}{2}(m_e + m_p) V_{COM}^2 + \frac{1}{2}\mu v_{rel}^2}_{KE} - \underbrace{\frac{Ze^2}{4\pi\varepsilon_0} \cdot \frac{1}{r_{rel}}}_{PE}}$

$\mu = \frac{m_e m_p}{m_e + m_p}$

COM motion — internal motion

⇑ quantized...

$\Rightarrow$ have to change $\boxed{m_e \Rightarrow \mu_H}$

→ Rydberg const for H: 1097...

$$\boxed{R_H = \frac{\mu_H}{m_e} R_1 \equiv \frac{\mu_H}{m_e} R_\infty}$$

$\frac{R_\infty}{(1 + m_e/m_p)}$

R_∞ ← infinite mass nucleus...

$$E_n = -\frac{1}{2}(\mu_H c^2) Z^2 \alpha^2 \frac{1}{n^2}$$

$$r_n = \frac{a_0}{Z} \frac{m_e}{\mu_H} n^2$$

L) The Pickering lines

He$^+$

binomial exp...

$$\frac{E_{He^+}}{E_H} = \frac{4R_{He}}{1 \cdot R_H} = \frac{4}{(1+m_e/m_\alpha)} \cdot \frac{(1+m_e/m_p)}{1} \approx 4 \cdot \left(1+\frac{m_e}{m_p}\right)\left(1-\frac{m_e}{m_\alpha}\right)$$

$$\approx 4 \cdot \left(1+\frac{m_e}{m_p} - \frac{m_e}{m_\alpha}\right)$$

$\frac{1}{1836}$ $\frac{1}{7299}$

$$\boxed{\frac{E_{He^+}}{E_H} \approx 4.00163}$$

This is what really was...

c. "Exotic" atoms

1) Deuterium $\rightarrow$ hydrogen with $m = 2u$ nucleus

$$\boxed{\frac{E_D}{E_H} = \frac{1 \cdot R_D}{1 \cdot R_H} = \left(1+\frac{m_e}{m_p} - \frac{m_e}{m_D}\right)}$$

2) Positronium $\rightarrow$ $e^- - e^+$ bound

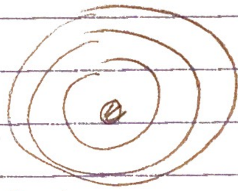
$\mu_{Ps} = \frac{1}{2} m_e \rightarrow \frac{E_P}{E_H} \approx \frac{1}{2}$; $r_{po} = 2 a_0 n^2$

3) Muonium $p^+ + \mu^-$ bound

↳ Experiment shows theory not really correct...?

2. Multi-electron atoms

a) Orbits ~ energy levels



b) X-ray spectra and Moseley's Law

- Bohr's theory: $E_{i-f} \propto Z^2$
- Moseley's Experiment ⟶ x-ray lines $E_x = A(Z-\sigma)^2$

correction... (≈ 1)

$10.2\,eV = \frac{3}{4} hcR_\infty$

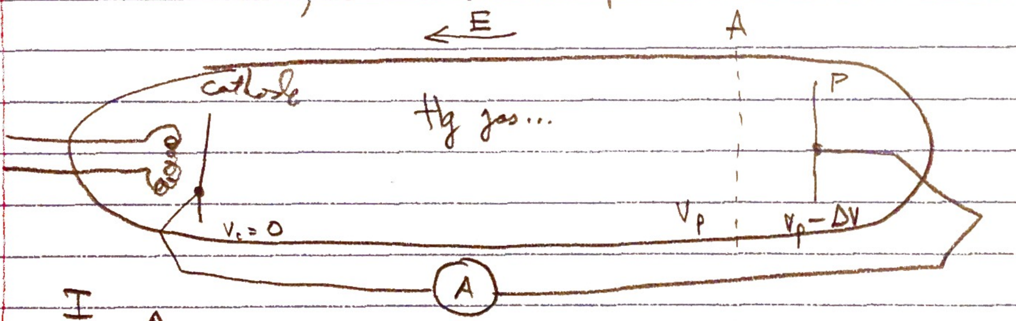
↳ all atoms are Bohr's atom if look at inner electrons...

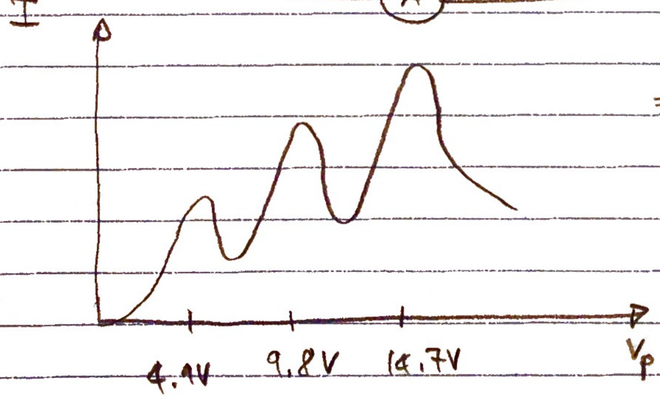
(Heindrich Hertz's nephew...)

Dec 8, 2017

c) The Franck-Hertz experiment (1914)

↳ looking at the behavior of electrons that collide with atoms





⇒ Indicates electrons suffer inelastic collisions and loose energy — but only for k ≥ 4.9V

↳ evidence of quantized energy levels w/o optics...

A glance ahead to PH242

1. Wave-particle duality → superposition of wavelengths λ's $\left(\lambda = \frac{h}{p}\right)$ ⟹ The uncertainty principle
   ↳ particles are described by localized waves

2. Quantized energy & angular momentum → Schrodinger equation...

PH242 ⟹ about putting these two sets of ideas together

$\boxed{\text{EXAM}}$

Saturday 16th, 2017 · 1:30 pm — 4:30 pm (Keyes 105)

Format → Questions (5–10 pts each) 60 pts total
→ Problems (20 pts each) 60 pts total
} 120 total

Overweighting of topics after midterm

{ de Broglie, Uncertainty, Wavepackets

{ Rutherford, Bohr