PH321 Final Equation Review

VECTOR DERIVATIVES

Cartesian: $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla\cdot \mathrm{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla\times\boldsymbol{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Spherical: $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl: $$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\, v_\phi\right) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}\left(r v_\phi\right)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r v_\theta\right) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Cylindrical: $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}\left(s v_s\right) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla\times\mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}\left(s v_\phi\right) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

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FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_a^b (\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence (Green's) Theorem: $\int (\nabla\cdot\mathbf{A})\, d\tau = \oint \mathbf{A}\cdot d\mathbf{a}$

Curl (Stokes') Theorem: $\int (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint \mathbf{A}\cdot d\mathbf{l}$

Second Derivatives: $\nabla\times(\nabla f) = 0 \qquad \nabla\cdot(\nabla\times\mathbf{A}) = 0$

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) = \nabla^2\mathbf{A}$$

OTHER MATH NOTATION

Separation Vector: $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}' \qquad \hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$

Area Elements:

Cartesian coords: $d\mathbf{a} = dx\, dy\, \hat{\mathbf{z}}$

Cylindrical coords: $d\mathbf{a} = s\, d\phi\, dz\, \hat{\mathbf{s}}$

Spherical coords: $d\mathbf{a} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{r}}$

Volume Elements:

Cartesian coords: $d\tau = dx\, dy\, dz$

Cylindrical coords: $d\tau = s\, ds\, d\phi\, dz$

Spherical coords: $d\tau = r^2 \sin\theta\, d\theta\, d\phi\, dr$

Delta Function: $\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a) \qquad \delta(kx) = \frac{1}{|k|}\delta(x)$

$$\int_{\text{all space}} f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{a})\,d\tau = f(\mathbf{a}) \qquad \nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

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ELECTROSTATICS

Electric force: $\mathbf{F}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}$ $\quad \mathbf{F} = Q\mathbf{E}$ $\quad \varepsilon_0 = 8.85\times10^{-12}\,\mathrm{C/Nm^2}$

Electric field: $\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}^2}\hat{\mathscr{r}}$ $\quad \frac{1}{4\pi\varepsilon_0} = 8.99\times10^{9}\,\mathrm{N\cdot m^2/C^2}$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{1}{\mathscr{r}^2}\hat{\mathscr{r}}\,dq = \frac{1}{4\pi\varepsilon_0}\int_P\frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,dl' = \frac{1}{4\pi\varepsilon_0}\int_S\frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,da' = \frac{1}{4\pi\varepsilon_0}\int_V\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau'$$

Charge Densities: λ = charge/length; σ = charge/area; ρ = charge/volume

Electric flux: $\Phi_E = \int_S \mathbf{E}\cdot d\mathbf{a}$

Gauss's Law: $\oint_S \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\varepsilon_0}Q_{\text{enc}}$ $\quad \nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\rho$

Curl of E: $\oint \mathbf{E}\cdot d\mathbf{l} = 0$ $\quad \nabla\times\mathbf{E} = 0$

Electric Potential: $\mathbf{E} = -\nabla V$ $\quad V(\mathbf{r}) = -\int_{\text{ref}}^{\mathbf{r}}\mathbf{E}\cdot d\mathbf{l}$ $\quad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}}$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{1}{\mathscr{r}}\,dq = \frac{1}{4\pi\varepsilon_0}\int_P\frac{\lambda(\mathbf{r}')}{\mathscr{r}}\,dl' = \frac{1}{4\pi\varepsilon_0}\int_S\frac{\sigma(\mathbf{r}')}{\mathscr{r}}\,da' = \frac{1}{4\pi\varepsilon_0}\int_V\frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

Superposition:

$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots$

$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \ldots$

$V = V_1 + V_2 + V_3 + \ldots$

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ELECTROSTATICS

Work and Energy:

- To move a charge q from ∞ to $\mathbf{r}$: $W = qV(\mathbf{r})$
- For a configuration of point charges, brought in one at a time: $W = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\sum_{i=1}^{n}\sum_{j>1}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$
- For a configuration of point charges, all in place: $W = \dfrac{1}{2}\displaystyle\sum_{i=1}^{n} q_i V(\mathbf{r}_i)$
- For a continuous charge distribution: $W = \dfrac{1}{2}\displaystyle\int_{\substack{\text{all}\\\text{space}}} \rho V \, d\tau$ or $W = \dfrac{\varepsilon_0}{2}\displaystyle\int_{\substack{\text{all}\\\text{space}}} E^2 \, d\tau$

Capacitors:

$C = Q/V$ where $C = A\varepsilon_0/d$ for a parallel-plate capacitor

$W = \dfrac{1}{2}\dfrac{Q^2}{C} = \dfrac{1}{2}CV^2$ (note this V is really ΔV)

Laplace's Equation:

$$\nabla^2 V = 0 \qquad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Boundary Conditions:

$V_{above}\big|_b = V_{below}\big|_b \qquad E^{\parallel}_{above}\big|_b = E^{\parallel}_{below}\big|_b$ (across a boundary b)

$$E^{\perp}_{above}\Big|_b - E^{\perp}_{below}\Big|_b = -\left(\frac{\partial V_{above}}{\partial n}\bigg|_b - \frac{\partial V_{below}}{\partial n}\bigg|_b\right) = \frac{\sigma}{\varepsilon_0}$$

Electric Field Outside a Conductor:

$$\mathbf{E} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}} \quad \text{or} \quad \sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$$

Electrostatic Pressure on a Conductor:

$$P = \frac{1}{2\varepsilon_0}\sigma^2 = \frac{\varepsilon_0}{2}E^2$$

PH321 Final Equation Review

ELECTROSTATICS

Method of Images: For point charge q at $z = d$ above conducting plane,

$$V(x,y,z>0)=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}-\frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

Two alternatives to find energy:

(1) Bring in charges one by one, multiply by fraction that are real
(2) Find energy of entire configuration in place, but count only terms corresponding to real charges

Cartesian Separation of Variables:

$$V(x,y,z)=X(x)Y(y)Z(z)$$

$$\frac{1}{X}\frac{d^2X}{dx^2}=C_1 \quad \frac{1}{Y}\frac{d^2Y}{dy^2}=C_2 \quad \frac{1}{Z}\frac{d^2Z}{dz^2}=C_3 \quad \text{with} \quad C_1+C_2+C_3=0$$

- Positive C $\longrightarrow$ exponential solutions
- Negative C $\longrightarrow$ sinusoidal solutions.
- Fourier's Trick: $\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n'\pi x}{a}\right)dx=\frac{a}{2}$ if $n=n'$, 0 otherwise
- Useful Identities: $\sinh x=\frac{1}{2}\left(e^x-e^{-x}\right)$ $\quad \cosh x=\frac{1}{2}\left(e^x+e^{-x}\right)$

Spherical Separation of Variables:

$$V(r,\theta)=R(r)\Theta(\theta)$$

- General solution:

$$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l\cos\theta$$

- Legendre Polynomials:

$$P_0(x)=1,\ P_1(x)=x,\ P_2(x)=\frac{3x^2-1}{2},\ P_3(x)=\frac{5x^3-3x}{2} \quad \text{with} \quad x=\cos\theta$$

$$\int_{-1}^{1}P_l(x)P_{l'}(x)\,dx=\int_0^{\pi}P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\,d\theta=\frac{2}{2l+1} \text{ if } l=l',\ 0 \text{ otherwise}$$

PH321 Final Equation Review

ELECTROSTATICS

Multipole Expansion:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta')\rho(\mathbf{r}')d\tau'$$

$$\mathbf{p} = \sum_{i=1}^{n} q_i\mathbf{r}_i' = \int \mathbf{r}'\rho(\mathbf{r}')d\tau' \qquad V_{dip}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

$$\mathbf{E}_{dip}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

Electric Fields in Matter:

- Torque and Energy of Dipole in an E field:

$$\mathbf{N} = \mathbf{p}\times\mathbf{E} \qquad U = -\mathbf{p}\cdot\mathbf{E}$$

- Polarization: **P** = dipole moment per unit volume
- Bound Charges:

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}\big|_{\text{surface}} \qquad \rho_b = -\nabla\cdot\mathbf{P}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\oint\frac{\sigma_b}{\mathscr{r}}da' + \frac{1}{4\pi\varepsilon_0}\oint\frac{\rho_b}{\mathscr{r}}d\tau'$$

- Electric Displacement:

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \qquad \mathbf{D} = \varepsilon\mathbf{E} \qquad \oint\mathbf{D}\cdot d\mathbf{a} = Q_{free}^{enc} \qquad \nabla\cdot\mathbf{D} = \rho_f$$

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} \qquad \varepsilon_r = 1 + \chi_e = \varepsilon/\varepsilon_0 \qquad \varepsilon = \varepsilon_0(1+\chi_e)$$

- Boundary Conditions:

$$D_{above}^{\perp} - D_{below}^{\perp} = \sigma_f$$

$$D_{above}^{\parallel} - D_{below}^{\parallel} = P_{above}^{\parallel} - P_{below}^{\parallel}$$

- Energy in Dielectrics:

$$W = \frac{\epsilon_0}{2}\int \epsilon_r E^2\, d\tau \qquad W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\, d\tau$$

- Divergence Summary:

$$\nabla\cdot\mathbf{E} = \frac{\rho_{total}}{\epsilon_0} \qquad \nabla\cdot\mathbf{D} = \rho_{free} \qquad \nabla\cdot\mathbf{P} = -\rho_{bound}$$

PH321 Final Equation Review

MAGNETOSTATICS

Magnetic force:

$$\mathbf{F}_{\text{total}} = Q[E + (\mathbf{v} \times \mathbf{B})] \qquad \mathbf{F}_m = \int I(d\mathbf{l} \times \mathbf{B}) = \int (\mathbf{K} \times \mathbf{B})\, da = \int (\mathbf{J} \times \mathbf{B})\, d\tau$$

Current Densities: $I = \lambda \mathbf{v}$ $\quad \mathbf{K} = \sigma \mathbf{v}$ $\quad \mathbf{J} = \rho \mathbf{v}$ $\quad \nabla \cdot \mathbf{J} = -\dfrac{d\rho}{dt}$

$\mathbf{J}$ = current per unit area; $I = \int \mathbf{J} \cdot d\boldsymbol{a}$

$\mathbf{K}$ = current per unit length; $I = \int \boldsymbol{K} \cdot d\boldsymbol{l}$

Magnetic field (Biot-Savart Law):

$$\mathbf{B} = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\imath}}}{\imath^2} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2} d\mathbf{a}' = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2} d\tau'$$

$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$

Ampère's Law: $\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}$ $\qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

Divergence of B: $\nabla \cdot \mathbf{B} = 0$

Magnetic Vector Potential:

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi_B$$

We can always choose $\nabla \cdot \mathbf{A} = 0$, so that $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$. In that case,

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{I}{\imath} d\mathbf{l}' = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{\imath} da' = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}}{\imath} d\tau'$$

Multipole expansion:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos\theta')\, d\mathbf{l}'$$

$$\mathbf{m} = I \int d\mathbf{a} \qquad \mathbf{A}_{dip}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} \qquad \mathbf{B}_{dip}(\mathbf{r}) = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\, \hat{\mathbf{r}} + \sin\theta\, \hat{\boldsymbol{\theta}}\right)$$

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MAGNETOSTATICS

Magnetic Fields in Matter:

$\mathbf{N} = \mathbf{m} \times \mathbf{B} \qquad \mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$

$\mathbf{M}$ = dipole moment per unit volume

$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}\big|_{\text{surface}} \qquad \mathbf{J}_b = \nabla \times \mathbf{M}$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\,d\tau' + \frac{\mu_0}{4\pi}\oint_S \frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\,da'$$

$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \qquad \mathbf{B} = \mu\mathbf{H} \qquad \oint \mathbf{H} \cdot d\mathbf{l} = I_{free}^{enc} \qquad \nabla \times \mathbf{H} = \mathbf{J}_f$

$\mathbf{M} = \chi_m \mathbf{H} \qquad \mu = \mu_0(1+\chi_m) \qquad \mu = \mu_0(1+\chi_m)$

$\mathbf{H}^{\parallel}_{\text{above}} - \mathbf{H}^{\parallel}_{\text{below}} = \mathbf{K}_f \times \hat{\mathbf{n}} \qquad H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -\left(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}}\right)$

$$W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\,d\tau$$

ELECTRODYNAMICS

Ohm's Law: $\mathbf{J} = \sigma\mathbf{E} \qquad \rho = 1/\sigma \qquad V = IR \qquad P = VI = I^2 R$

Electromotive force: $\varepsilon = \oint \mathbf{f} \cdot d\mathbf{l} \qquad \varepsilon = -\frac{d\Phi_B}{dt} \qquad \Phi_B = \int \mathbf{B} \cdot d\mathbf{a} \qquad \varepsilon = IR$

Faraday's Law: $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt}$

Mutual Inductance: $M = \Phi_1 / I_2 = \Phi_2 / I_1$

Self Inductance: $\Phi = LI \qquad \varepsilon = -L\frac{dI}{dt} \qquad W = \frac{1}{2}LI^2$

$L = \mu_0 n^2 (\pi R^2)$ self-inductance per unit length for a solenoid

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Maxwell's Equations:

Gauss: $\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho$ No monopoles: $\nabla \cdot \mathbf{B} = 0$

Faraday: $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ Ampère-Maxwell: $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

Electrodynamic boundary conditions:

$\varepsilon_0 E^{\perp}_{above} - \varepsilon_0 E^{\perp}_{below} = \sigma_f$ $\mathbf{E}^{\parallel}_{above} - \mathbf{E}^{\parallel}_{below} = 0$

$B^{\perp}_{above} - B^{\perp}_{below} = 0$ $\frac{1}{\mu_0}\mathbf{B}^{\parallel}_{above} - \frac{1}{\mu_0}\mathbf{B}^{\parallel}_{below} = \mathbf{K}_f \times \hat{\mathbf{n}}$

Maxwell's Equations (in matter):

Gauss: $\nabla \cdot \mathbf{D} = \rho_{\text{free}}$ No monopoles: $\nabla \cdot \mathbf{B} = 0$

Faraday: $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ Ampère-Maxwell: $\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$

Electrodynamic boundary conditions (in matter):

$D^{\perp}_1 - D^{\perp}_2 = \sigma_f$ $\mathbf{E}^{\parallel}_1 - \mathbf{E}^{\parallel}_2 = 0$

$B^{\perp}_1 - B^{\perp}_2 = 0$ $\mathbf{H}^{\parallel}_1 - \mathbf{H}^{\parallel}_2 = \mathbf{K}_f \times \hat{\mathbf{n}}$

ELECTROMAGNETIC WAVES

Maxwell's Equations (in vacuum):

Gauss: $\nabla \cdot \mathbf{E} = \mathbf{0}$ No monopoles: $\nabla \cdot \mathbf{B} = \mathbf{0}$

Faraday: $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ Ampère-Maxwell: $\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

The Wave Equation (1D):

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

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Transverse Waves:

$$f(x,t) = A\cos(kx - \omega t + \delta)$$

E and B Wave Relationship:

$$\boldsymbol{B} = \frac{k}{\omega}\left(\hat{\boldsymbol{k}} \times \boldsymbol{E}\right)$$

E and B Amplitudes:

$$B_0 = \frac{k}{\omega}E_0 = \frac{1}{c}E_0$$

Energy Density:

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = \epsilon_0 E^2$$

Average Energy Density:

$$\langle u \rangle = \frac{1}{2}\epsilon_0 E_0^2$$

Power Density (Poynting Vector):

$$\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E} \times \boldsymbol{B})$$

Average Power Density:

$$\langle \boldsymbol{S} \rangle = \frac{1}{2}\epsilon_0 E_0^2 c\,\hat{\boldsymbol{k}} = \langle u \rangle c\,\hat{\boldsymbol{k}}$$

Momentum Density:

$$\mathbf{g} = \frac{1}{c}\epsilon_0 \boldsymbol{E}^2 = \frac{1}{c}u\,\hat{\boldsymbol{k}}$$

Average Momentum Density:

$$\langle \mathbf{g} \rangle = \frac{1}{2c}\epsilon_0 E_0^2\,\hat{\boldsymbol{k}}$$

Intensity:

$$I \equiv \langle S \rangle = \frac{1}{2}\epsilon_0 E_0^2 c$$

Radiation Pressure:

$$P = \frac{I}{c} = \frac{1}{2}\epsilon_0 E_0^2$$

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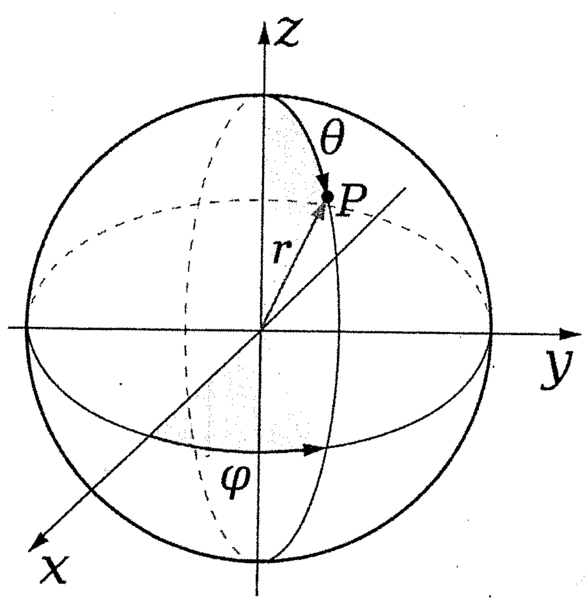
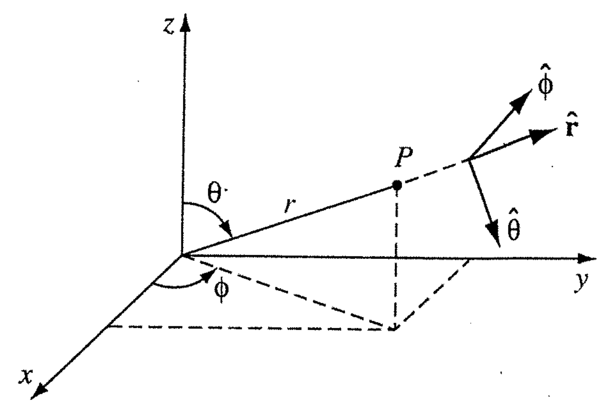
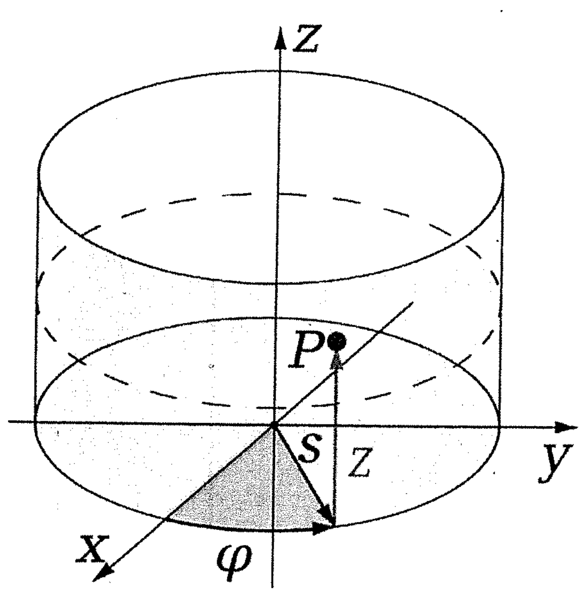
CURVILINEAR COORDINATE SYSTEMS

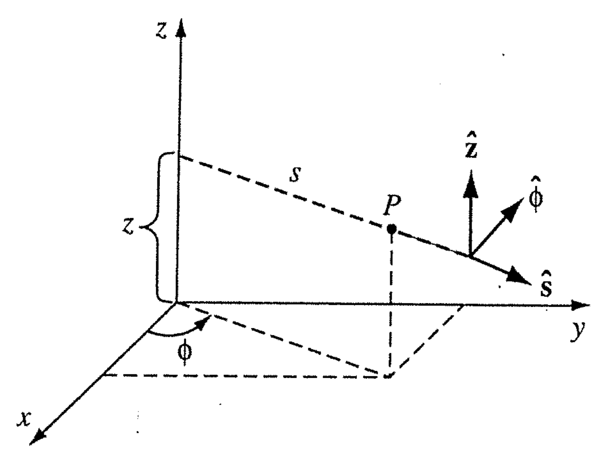
Spherical Coordinates:





Cylindrical Coordinates:





Notes on Laplace's Equations and Electrostatic Potential

1. General Solutions to Laplace's Equation by Separation of Variables in Cartesian Coordinates

We write the Laplace's equation in Cartesian coordinates when the boundary is a rectangular box. The equation is solved to find electrostatic potential in the region $0 \le x \le a$, $0 \le y \le b$, and $0 \le z \le c$. In practice of physical and engineering applications, we often assume that V can be written as the product of three functions, each depending on only one variable, i.e., $V = V_x(x)V_y(y)V_z(z)$. The 3-dimensional Laplace's equation is therefore becoming three second order ordinary differential equations with the same form of (here we take the variable x for example)

$$\frac{\partial^2 V_x}{\partial x^2} = C_x V_x$$

The general solution to this equation can only take the following form:

- If $C_x = k^2 > 0$, V_x are exponential functions given as $V_x = Ae^{-kx} + Be^{kx}$ or $V_x = Asinh(kx) + Bcosh(kx)$, where $sinh(kx) = \frac{1}{2}(e^{kx} - e^{-kx})$, and $cosh(kx) = \frac{1}{2}(e^{kx} + e^{-kx})$ are hyperbolic functions. Note that there is no difference in writing the solution in either expoential functions or hyperbolic functions. In real applications, which form of the functions to take depends on what is the most convenient with given boundary conditions.
- If $C_x = -k^2 < 0$, V_x are sinusoidal functions given as $V_x = Asin(kx) + Bcos(kx)$.
- In the special case of $C_x = 0$, $V_x = ax + b$, which is a linear function of x.

In the above, we only take $V_x(x)$ as an example, yet obviously the conclusion is true for $V_y(y)$ and $V_z(z)$ as well.

The form of the solution and the constants can be determined by boundary conditions, knowing the properties of sinusoidal and exponential (or hyperbolic) functions. The table below lists the form of the solution with certain boundary conditions (again, take $V_x(x)$ as an example):

Table 1. The general solution of $V_x(x)$ at given boundary conditions

boundary condition	solution
$-\infty < x < \infty$	V does not depend on x
$0 \le x \le \infty$	$V_x \sim e^{-kx}$
$x = 0, V(0) = 0$	$V_x \sim sin(kx)$ or $V_x \sim sinh(kx)$
$x = 0, V(0) = 0$ and $x = a, V(a) = 0$	$V_x \sim sin(n\pi x/a)$
$x = -a/2, V(-a/2) = 0$ and $x = a/2, V(a/2) = 0$	$V_x \sim cos(n\pi x/a)$
$x = 0, V(0) = 0$ and $x = a, V(a) = V_0$	$V_x \sim sinh(kx)$
$x = -a/2, V(-a/2) = V_0$ and $x = a/2, V(a/2) = V_0$	$V_x \sim cosh(kx)$

Note that:

- The ks in the solution are not independent; they have to be determined by the Laplace's equation $C_x + C_y + C_z = 0$. The equation indicates that C_x, C_y, and C_z cannot all be positive and cannot all be negative; in other words, $V_x(x)$, $V_y(y)$, and $V_z(z)$ cannot all be sinusoidal functions and cannot all be hyperbolic (exponential) functions. Usually, we first pin down sinusoidal functions by examining boundary conditions: closed boundary at both ends, and the potential is zero at both ends (see the table above).
- For completeness, the general solution is a sum of all possible individual solutions (all ns). The coefficient of each ith term is determined by the (unused) boundary condition.
- As the form of the solution is indeed very similar to the practice of expanding an arbitrary function into Fourier series, these coefficients are determined by the so-called Fourier "tricks" making use of orthogonality of sinusoidal functions.

2. Orthogonality of Sinusoidal Functions and the Fourier Trick (see "Mathemtical Methods in the Physical Sciences" by Baos, pp 308-311)

Orthogonality of sinusoidal functions (n and n' being arbitrary integers):

$$\int_{-\pi}^{\pi} sin(nx)cos(n'x)dx = 0$$

$$\int_{-\pi}^{\pi} sin(nx)sin(n'x)dx = \pi$$

only when $n' = n \neq 0$

$$\int_{-\pi}^{\pi} cos(nx)cos(n'x)dx = \pi$$

only when $n' = n \neq 0$

Or when the argument t is in a region $-L \leq t \leq L$, the orthogonality is reflected by:

$$\int_{-L}^{L} sin(\frac{n\pi x}{L})cos(\frac{n'\pi x}{L})dx = 0$$

$$\int_{-L}^{L} sin(\frac{n\pi x}{L})sin(\frac{n'\pi x}{L})dx = L$$

only when $n' = n \neq 0$

$$\int_{-L}^{L} cos(\frac{n\pi x}{L})cos(\frac{n'\pi x}{L})dx = L$$

only when $n' = n \neq 0$

Furthermore, since the integrand $sin(\frac{n\pi x}{L})sin(\frac{n'\pi x}{L})$ (or $cos(\frac{n\pi x}{L})cos(\frac{n'\pi x}{L})$) is an even function, the orthogonality may also be written as:

$$\int_0^L sin(\frac{n\pi x}{L})sin(\frac{n'\pi x}{L})dx = \int_{-L}^0 sin(\frac{n\pi x}{L})sin(\frac{n'\pi x}{L})dx = \frac{1}{2}\int_{-L}^L sin(\frac{n\pi x}{L})sin(\frac{n'\pi x}{L})dx = \frac{L}{2}$$

only when $n' = n \neq 0$

This is the origin of Equation 3.33 in the book. Strictly speaking, in the book examples, the constants C_n is not derived as coefficients of Fourier series, since the range of x or y is only half the period of a sinusoidal function. These constants are obtained by the Fourier trick using the orthogonality in half the period (as shown above).

Take, for example, the Griffith Problem 3.14, the electrostatic potential is given by

$$V(x,y) = \Sigma_{n=1,2,..}Cnsinh(n\pi x/a)sin(n\pi y/a)$$

At the boundary $x = a$, $V(a,y) = V_0$, or

$$V_0 = \Sigma_{n=1,2,..}Cnsinh(n\pi)sin(n\pi y/a)$$

If we multiplicate the term $sin(n'\pi y/a)$ with both sides of the equation, and then integrate both sides over y in the $0 < y < a$ region, on the right-hand side, only when $n' = n$, the integral is not zero:

$$\int_0^a \Sigma_{n=1,2,..}Cnsinh(n\pi)sin(n\pi y/a)sin(n'\pi y/a)dy = \Sigma_{n=1,2,..}Cnsinh(n\pi)\int_0^a sin(n\pi y/a)sin(n'\pi y/a)dy$$

$$= Cnsinh(n\pi)\int_0^a sin^2(n\pi y/a)dy = Cnsinh(n\pi)\int_0^a \frac{1-cos(2n\pi y/a)}{2}dy = Cnsinh(n\pi)\frac{a}{2}$$

On the left-hand side, we take $n' = n$ to get:

$$\int_0^a V_0 sin(n\pi y/a)dy = \frac{V_0 a}{n\pi}[-cos(n\pi y/a)]_{y=0}^{y=a} = \frac{2V_0 a}{n\pi}$$

when n are odd intergers. Equating the left-hand side and the right-hand side, we arrive at

$$\frac{2V_0 a}{n\pi} = Cnsinh(n\pi)\frac{a}{2}$$

So for this problem

$$C_n = \frac{4V_0 a}{nsinh(n\pi)}$$

where n are odd integers.

Note that in the integral, we use the identity $sin^2(t) = \frac{1-cos(2t)}{2}$.

3. General Solutions in Cylindrical Coordinates (with invariance in the z-dimension)

With the technique of separation of variables, the general solution to the Laplace's equation in cylindrical coordinates (with invariance in the z-dimension, such as electric field near an infinitely long cylinder of radius R) is given by

$$V(s,\phi) = C_0 ln(\frac{s}{R}) + D_0 + \Sigma_{n=1,2,3..}[C_n(\frac{s}{R})^n + D_n(\frac{R}{s})^n][A_n cos(n\phi) + B_n sin(n\phi)]$$

The constants can be determined by boundary conditions, which in general may include:

- inside the cylinder, the $ln(s/R)$ and $(R/s)^n$ terms vanish, because these terms blow up at $s = 0$; the exception case is when the cylinder is a conductor, in which case, the cylinder (including its surface) is equipotential (see the class example).
- outside the cylinder, the $ln(s/R)$ term and $(s/R)^n$ terms vanish, because these terms blow up when $s \sim \infty$; the exception case is when there is a specified external field which does not vanish at infinity (see the class example).
- V is continous at the surface $s = R$, or, $V_{out}(\phi)|_{s=R} = V_{in}(\phi)|_{s=R}$.
- or on the surface $s = R$, $V(\phi)$ is a given function; for a conductor, $V(\phi)|_{s=R} \equiv V_0$, a constant, which is often conveniently set to zero.
- or on the surface $s = R$, the surface charge density $\sigma(\phi)|_{s=R}$ is given;

In the last two situations, in general, noting that the eigen functions in the general solution are sinusoidal functions with the orthogonality property, we can use the same Fourier trick, which is to integrate the product of the solution function and $sin(n'\phi)$ or $cos(n'\phi)$ in the region $0 < \phi < 2\pi$ to find out the coefficients A_n and B_n. Particularly, when the surface charge density is given, we first use the boundary condition

$$\sigma(R) = \epsilon_0(\vec{E}_{out} - \vec{E}_{in}) \cdot \hat{s}|_{s=R} = -\epsilon_0(\frac{\partial V_{out}}{\partial s} - \frac{\partial V_{in}}{\partial s})|_{s=R}$$

And then use the Fourier trick.

However, we often meet situations when the boundary condition $V(\phi)|_{s=R}$ or $\sigma(\phi)|_{s=R}$ is given in such a way that $V(\phi)|_{s=R}$ (or $\sigma(\phi)|_{s=R}$) itself is a sinusoidal function, then we do not have to use the Fourier trick to accomplish an integral of the product of $V(\phi)$ (or $\sigma(\phi)$) and the nth eigen function to find the coefficients; instead we directly compare the terms to match the coefficients (see the class example and Griffiths Problem 3.25).

4. General Solutions in Spherical Coordinates (with azimuthal symmetry)

With the technique of separation of variables, when it concerns a sphere with radius R, the general solution to the Laplace's equation in spherical coordinates (with azimuthal symmetry, i.e., no dependence on the azimuthal angle ϕ), is given by

$$V(r,\theta) = \Sigma_{l=0,1,2,..}[A_l r^l + B_l r^{-(l+1)}]P_l(cos\theta)$$

where $P_l(cos\theta)$ is the lth order Legendre polynomial. The constants can be determined by boundary conditions, which are similar to those in cylindrical coordinates:

- inside the sphere, the $r^{-(l+1)}$ terms vanish, because these terms blow up at $r = 0$; the exception case is when the sphere is a conductor, in which case, the sphere (including its surface) is equipotential.
- outside the sphere, the r^l $(l \geq 1)$ terms vanish, because these terms blow up when $r \sim \infty$; the exception case is when there is a specified external field which does not vanish at infinity (see Griffiths Problem 3.20);
- V is continous at the surface $r = R$, or, $V_{out}(\theta)|_{r=R} = V_{in}(\theta)_{r=R}$.
- or on the surface $r = R$, $V(\theta)$ is a given function; for a conductor, $V(\theta)|_{s=R} \equiv V_0$, a constant, which is often conveniently set to zero.
- or on the surface $r = R$, the surface charge density $\sigma(\theta)|_{r=R}$ is given;

In the last two situations, in general, taking advantage of the orthogonal property of Legendre polynomials, we can use the trick similar to the Fourier trick to find out the coefficients A_l and B_l. Particularly, when the surface charge density is given, we first use the boundary condition

$$\sigma(\theta)|_{r=R} = \epsilon_0(\vec{E}_{out} - \vec{E}_{in}) \cdot \hat{r}|_{r=R} = -\epsilon_0(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r})|_{r=R}$$

and then use the integral trick. See Griffiths Equation 3.68 and 3.69 for the orthogonality of Legendre polynomials and integral with Legendre polynomials.

However, we often meet situations when the boundary condition $V(\theta)|_{r=R}$ or $\sigma(\theta)|_{r=R}$ is given in such a way that $V(\theta)|_{r=R}$ (or $\sigma(\theta)|_{r=R}$) is a function of $cos\theta$ which may be re-written (with a good insight or guess work) into Legendre polynomials, then we do not have to accomplish an integral of the product of $V(\theta)$ (or $\sigma(\theta)$) and the Legendre polynomials to find the coefficients; instead we directly compare the terms to match the coefficients. Note that we shall employ some imagination to be able to re-write the $cos\theta$ terms into the form of $P_l(cos\theta)$ (see the class examples and Griffiths Problem 3.18).

PH321: Electricity and Magnetism

Vector Algebra

General

$$\vec{A}+\vec{B}=\vec{B}+\vec{A} \qquad \vec{A}-\vec{B}=\vec{A}+(-\vec{B})$$

$$(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C}) \qquad a(\vec{A}+\vec{B})=a\vec{A}+a\vec{B}$$

$$\vec{A}\cdot\vec{B}=AB\cos\theta \qquad \vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A} \qquad \vec{A}\cdot(\vec{B}+\vec{C})=\vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$$

$$\vec{A}\times\vec{B}=AB\sin\theta\hat{n} \qquad \vec{A}\times\vec{B}=-\vec{B}\times\vec{A} \qquad \vec{A}\times(\vec{B}+\vec{C})=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

$$\vec{A}\cdot\vec{A}=A^2 \qquad \vec{A}\times\vec{A}=0$$

Cartesian Coordinates

$$\vec{A}=A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k} \qquad \vec{A}+\vec{B}=(A_x+B_x)\hat{\imath}+(A_y+B_y)\hat{\jmath}+(A_z+B_z)\hat{k}$$

$$a\vec{A}=aA_x\hat{\imath}+aA_y\hat{\jmath}+aA_z\hat{k}$$

$$\vec{A}\cdot\vec{B}=A_xB_x+A_yB_y+A_zB_z \qquad A=|\vec{A}|=\sqrt{\vec{A}\cdot\vec{A}}=\sqrt{{A_x}^2+{A_y}^2+{A_z}^2}$$

$$\vec{A}\times\vec{B}=(A_yB_z-A_zB_y)\hat{\imath}+(A_zB_x-A_xB_z)\hat{\jmath}+(A_xB_y-A_yB_x)\hat{k}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

Triple Products

$$\vec{A}\cdot(\vec{B}\times\vec{C})=\vec{B}\cdot(\vec{C}\times\vec{A})=\vec{C}\cdot(\vec{A}\times\vec{B})=-\vec{A}\cdot(\vec{C}\times\vec{B})=-\vec{B}\cdot(\vec{A}\times\vec{C})=-\vec{C}\cdot(\vec{B}\times\vec{A})$$

$$\vec{A}\cdot(\vec{B}\times\vec{C})=(\vec{A}\times\vec{B})\cdot\vec{C}=\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$$

$$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B}) \qquad (\vec{A}\times\vec{B})\times\vec{C}=-\vec{C}\times(\vec{A}\times\vec{B})$$

Quadruple Products

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})=(\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D})-(\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$$

$$\vec{A}\times(\vec{B}\times(\vec{C}\times\vec{D}))=\vec{B}(\vec{A}\cdot(\vec{C}\times\vec{D}))-(\vec{A}\cdot\vec{B})(\vec{C}\times\vec{D})$$

Note: $(\hat{\imath},\ \hat{\jmath},\ \hat{k})=(\hat{x},\ \hat{y},\ \hat{z})$

PH321: ELECTRICITY & MAGNETISM
Prof. Kocershiy

Sep 5, 2019

Introduction

→ 4 fundamental forces, by ↓ strength

(1) Strong force (hold p & n together)
- 137 times stronger than E&M
- effective range → short: 10^{-15} m (femto meter)

$R_{proton} \sim 0.85$ fm

(gluons)

(2) Electromagnetism
- acts on all charged particles
- effective range: large, ∞, $1/r^2$ (photons)

(3) Weak force
- acts on elementary particles (quarks, e^-)
  ↳ fermions ($n/2$ spins, $n = 1, 3, 5, \ldots$) (W/Z Bosons)
- strength: 100,000 weaker than E&M
- effective range → extremely small: 10^{-17} m (1% R_{proton})

(4) Gravity
- acts on all massive particles
- strength: 10^{-42} weaker than E&M
- effective range: ∞, by $1/r^2$ (gravitons)

Why is E&M unique?

(1) E&M is strong & has large effective range
Magnetism is a relativistic correction. Electric force is fundamental

(2) First unified theory

(3) E&M is the first field theory

//

⊛ Properties of Electric Charge

1. 2 varieties: + & − (Ben Franklin)
2. Charge is always conserved
3. charge is quantized

   proton: $+e$

   electron: $-e$ $e = 1.602 \times 10^{-19}\,C$; $1\,C = 6.24 \times 10^{18}$ electron

   Carbon: $+6e$

   quarks have fractional charges... $\frac{1}{3}e$, $\frac{2}{3}e$ ... but always come in groups of 3

//

Chapter 1: Vector Analysis

(I) Vector Algebra (abstract form)

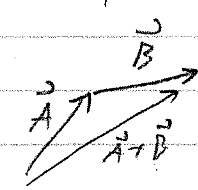
* Basic notation: $\vec{A}$: vector w/ magnitude & direction...

$A \equiv |\vec{A}|$

$\hat{a} \equiv \dfrac{\vec{A}}{|\vec{A}|}$ unit vector, mag = 1, pts twd $\vec{A}$

* vector operations:

(a) Addition & Subtraction... ⟶ tip to tail



↳ Commutative : $\vec{A}+\vec{B} = \vec{B}+\vec{A}$

↳ Associative :

$$(\vec{A}+\vec{B})+\vec{C} = \vec{A}+(\vec{B}+\vec{C})$$

↳ Subtraction via addition...

$$\vec{A}+(-\vec{B}) = \vec{A}-\vec{B}$$

(b) Multiplication by scalar

If scalar > 0, only magnitude changes

If scalar < 0, direction becomes opposite.

Distributive : $\gamma(\vec{a}+\vec{b}) = \gamma\vec{a}+\gamma\vec{b}$

(c) Multiplication by vector

- Multiplying corresponding components (dot product)
- Multiplying non-corresponding components (cross product)

(i) Dot product

(•) $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta = |\vec{A}||\vec{B}|\cos(\vec{A},\vec{B})$

(•) DP is a scalar.

(•) commutative : $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

(•) Distributive ...

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

(•) DP maxes out when vectors parallel. If $\vec{A}\parallel\vec{B}$ then

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|$$

(•) Perp vectors: If $\vec{A}\perp\vec{B}$ then $\vec{A}\cdot\vec{B} = 0$.

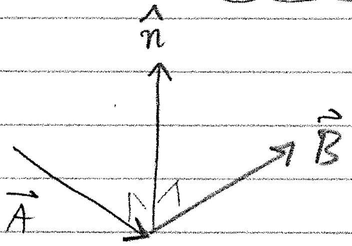
Example

$\vec{C} = \vec{A} - \vec{B}$, then $\vec{C}\cdot\vec{C} = (\vec{A}-\vec{B})(\vec{A}-\vec{B})$

$$= A^2 - 2|\vec{A}||\vec{B}|\cos\theta + B^2$$

So $\boxed{C^2 = A^2 + B^2 - 2AB\cos\theta}$

(ii) Cross product



$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$ where $\hat{n}$ is the normal vector

(right hand rule)

(·) Perp vectors cause $\vec{A}\times\vec{B}$ to max out

(·) Parallel vectors minimize $\vec{A}\times\vec{B}$.

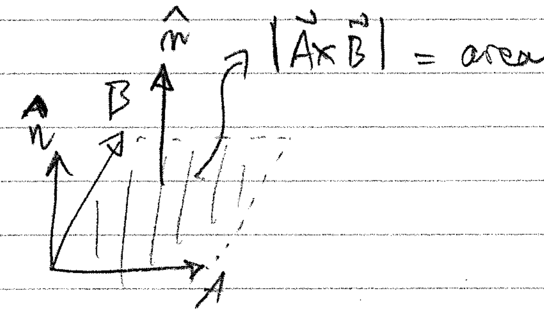
(·) CP is <u>distributive</u>

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$$

(·) <u>Not</u> commutative

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

(·) Geometric interpretation



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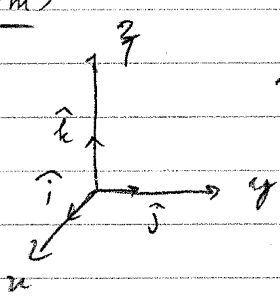
(II) Vector Algebra (component form)

* Cartesian coordinates...



$\hat{i}, \hat{j}, \hat{k} = \hat{x}, \hat{y}, \hat{z}$

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

(1) Adding vectors: component-wise...

$$\vec{A}+\vec{B} = (A_x+B_x)\hat{x} + (A_y+B_y)\hat{y} + (A_z+B_z)\hat{z}$$

(2) Scalar multiplication:

$$a\vec{A} = (aA_x)\hat{x} + (aA_y)\hat{y} + (aA_z)\hat{z}$$

(3) Dot product...

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

(4) Cross - product ...

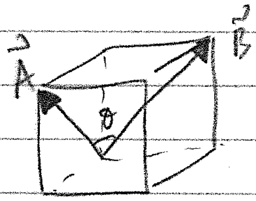
$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - B_y A_z)\hat{i} - (A_x B_z - B_x A_z)\hat{j} + (A_x B_y - B_x A_y)\hat{k}$$

Example 2 Find $\vec{A}\times\vec{B}$ where

$$\vec{A} = 3\hat{x} - \hat{y} + \hat{z} \quad , \quad \vec{B} = \hat{x} + 2\hat{y} - \hat{z}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 3 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = (1-2)\hat{x} + (3+1)\hat{y} + (6+1)\hat{z}$$

Example



$$\vec{A} = 1\hat{x} + 0\hat{y} + 1\hat{z}$$

$$\vec{B} = 0\hat{x} + 1\hat{y} + 1\hat{z}$$

what is θ ?

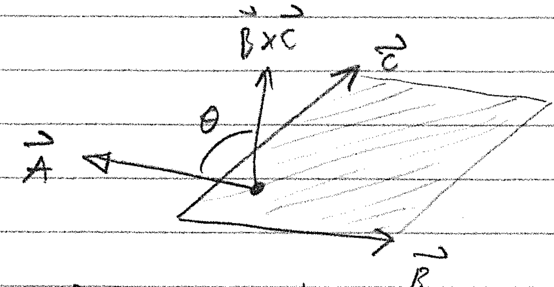
$$\Rightarrow \vec{A}\cdot\vec{B} = AB\cos\theta = 1 = \sqrt{2}\cdot\sqrt{2}\cos\theta \Rightarrow \theta = \pi/3$$

p 9, 2019

~~Differential Vector Calculus~~

(III) Triple products

① Dot triple product $\boxed{\vec{A}\cdot(\vec{B}\times\vec{C})}$



$|\vec{B}\times\vec{C}|$ is area of parallelogram

$|A\cos\theta|$ = height on $\vec{B}\times\vec{C}$

$\Rightarrow |\vec{A}\cdot(\vec{B}\times\vec{C})|$ is volume of prism bounded by $\vec{A},\vec{B},\vec{C}$

Note

$$\boxed{\begin{gathered}\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}) \\ \vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}\end{gathered}}$$ (cyclic)

(2) Cross triple product $\vec{A}\times(\vec{B}\times\vec{C})$

$$\boxed{\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})}$$

result is a vector, that is $\perp$ to all $\vec{A}, \vec{B}, \vec{C}$

(IV) Differential Vector Calculus

* Derivative: $f(x)$

$$\Delta f = \left(\frac{\partial f}{\partial x}\right)\Delta x$$

* In 3D space $T = T(x, y, z)$

$$\Delta T = \left(\frac{\partial T}{\partial x}\right)\Delta x + \left(\frac{\partial T}{\partial y}\right)\Delta y + \left(\frac{\partial T}{\partial z}\right)\Delta z$$

* Alternatively

$$\Delta T = \underbrace{\left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)}_{\vec{\nabla}T}\cdot\underbrace{\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right)}_{d\vec{l}}$$

* Del operator

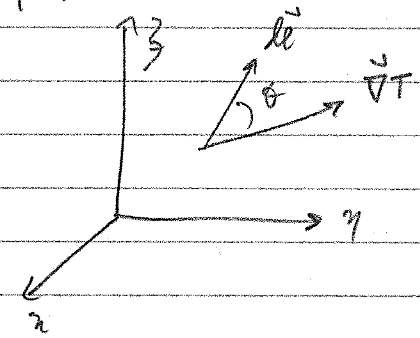
$$\boxed{\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}}$$

* Application

(1) Gradient : $\vec{\nabla} f$
(2) Divergence : $\vec{\nabla} \cdot \vec{F}$
(3) Curl : $\vec{\nabla} \times \vec{F}$

(1) $\boxed{\text{Gradient}}$, $\vec{\nabla} f = (\partial_x f, \partial_y f, \partial_z f)$

▪ properties → gradient is a vector, in what direction?



$\Delta T = (\vec{\nabla} T) \cdot \vec{dl} = (\vec{\nabla} T) |dl| . \cos\theta$

ΔT maxes when $\theta = 0$

⇒ $\boxed{\vec{\nabla} T \text{ points in direction of steepest ascent/descent}}$

↳ Rate of change in direction of maximal change

(2) $\boxed{\text{Divergence}}$ $\vec{\nabla} \cdot \vec{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z$

▪ properties → divergence is a scalar...
→ measures rate of change in the direction that vector is pointing...

→ Measures how much a vector field "spreads out" from a point.

→ Negative divergence → points inwards...

(3) $\boxed{\text{Curl}}$ $\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$

$$= \left(\partial_y A_z - \partial_z A_y \, , \, -\partial_x A_z + \partial_z A_x \, , \, \partial_x A_y - \partial_y A_x \right)$$

- Properties → curl is a vector...
  → measure the rate of change in the perpendicular direction that vector field points...
  → measure how much a vector field "swirls" around a given point.

  → direction is given by the RHR

(IV) Second order derivatives

* Combinations: $\vec{\nabla}\cdot(\vec{\nabla} f)$

$\vec{\nabla}\times(\vec{\nabla} f)$

$\vec{\nabla}(\vec{\nabla}\cdot\vec{A})$

$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})$

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})$

* Notes ① $\vec{\nabla}\times(\vec{\nabla} f) = \vec{0}$ (Laplacian)

② $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

③ $\vec{\nabla}\cdot(\vec{\nabla} f) = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f = \Delta f = \vec{\nabla}^2 f$

Laplacian of a scalar is a scalar.

Laplacian of a vector: $\vec{\nabla}^2 \cdot \vec{A} = \left(\vec{\nabla}^2 A_x, \vec{\nabla}^2 A_y, \vec{\nabla}^2 A_z\right)$

$\boxed{\text{(VI) Sum \& Product Rules}}$

$$\partial_x(f+g) = \partial_x f + \partial_x g$$

$$\partial_x(fg) = (\partial_x f)g + f(\partial_x g)$$

(3) Sum rules are the same for vector fields ...

$$\begin{cases} \vec{\nabla}(\vec{A}+\vec{B}) = \vec{\nabla}\vec{A} + \vec{\nabla}\vec{B} \\ \vec{\nabla}\cdot(\vec{A}+\vec{B}) = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{B} \\ \vec{\nabla}\times(\vec{A}+\vec{B}) = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{B} \end{cases}$$

(4) Product Rules

Grads

$$\vec{\nabla}(fg) = f\vec{\nabla}g + \vec{\nabla}f\,g$$

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}$$

$$(\vec{A}\cdot\vec{\nabla})\vec{B} = \left(A_x\partial_x + A_y\partial_y + A_z\partial_z\right)\left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right)$$

Divs

$$\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f)$$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

Curls

$$\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times(\vec{\nabla}f)$$

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A})$$

Second Derivatives

$$\vec{\nabla}\cdot(\vec{\nabla}T) = \vec{\nabla}^2 T$$

$$\vec{\nabla}^2 \vec{v} = \vec{\nabla}^2 v_x\,\hat{i} + \vec{\nabla}^2 v_y\,\hat{j} + \vec{\nabla}^2 v_z\,\hat{k}$$

$$\vec{\nabla}\times(\vec{\nabla}T) = \vec{0}$$

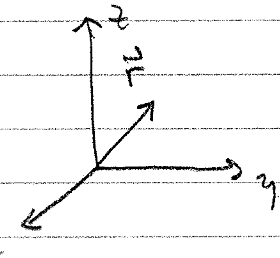
$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = 0$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \vec{\nabla}^2\vec{v}$$

Sep 10, 2019

COORDINATE SYSTEMS

(1) Cartesian Coordinates



Position vector

$$\vec{r} = (x, y, z) = x\hat{x} + y\hat{y} + z\hat{z}$$

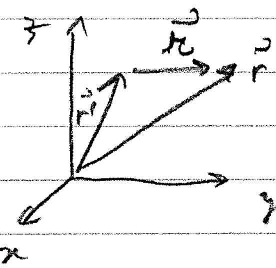
$$|\vec{r}| = (x^2 + y^2 + z^2)^{1/2}$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{(x^2 + y^2 + z^2)^{1/2}}$$

- Infinitesimal Displacement Vector...

$$\boxed{d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}}$$

- Separation vector

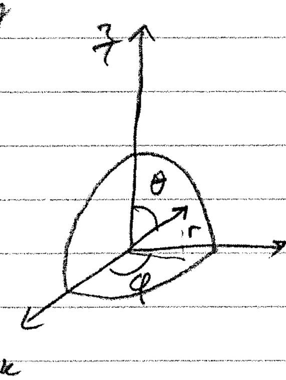


$$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$$

$$|\vec{\mathscr{r}}| = |\vec{r} - \vec{r}'|$$

$$\vec{\mathscr{r}} = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$$

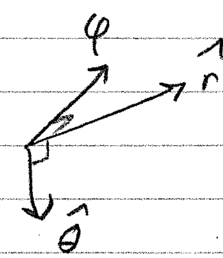
(2) Spherical Coordinates



r : distance to O
θ : distance from z
φ : distance from z

- Relation to Cartesian

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$



- component form : $\boxed{\vec{A} = A_r\hat{r} + A_\theta\vec{\theta} + A_\varphi\vec{\varphi}}$

- Displacement vector: $\boxed{d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\varphi\,\hat{\varphi}}$

- Inf volume element : $\boxed{d\tau = r^2\sin\theta\,dr\,d\theta\,d\varphi}$

- Vector derivatives in spherical coordinates ...

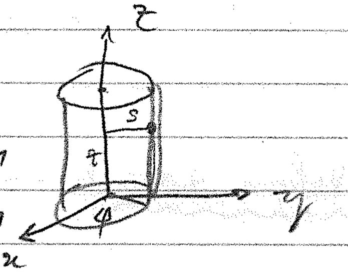
$$\vec{\nabla} T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \varphi}\hat{\varphi}$$

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\partial_r(r^2 v_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\partial_\varphi v_\varphi$$

$$\vec{\nabla}\times\vec{v} = \frac{1}{r\sin\theta}\left[\partial_\theta(\sin\theta\, v_\varphi) - \partial_\varphi v_\theta\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \varphi} - \partial_r(r v_\varphi)\right]\hat{\theta} + \frac{1}{r}\left[\partial_r(r v_\theta) - \partial_\theta v_r\right]\hat{\varphi}$$

$$\vec{\nabla}^2 t = \frac{1}{r^2}\partial_r\left(r^2\partial_r t\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta t\right) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 t$$

(3) Cylindrical Coordinates



s : distance from z

z

φ

To Cartesian: $\hat{s} = \cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}$

$\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$

$\hat{z} = \hat{z}$

- Displacement vector: $\boxed{dl_s = ds\,\hat{s} + s\,d\varphi\,\hat{\varphi} + dz\,\hat{z}}$

- Volume element: $d\tau = s\,ds\,d\varphi\,dz$

$$\vec{\nabla} t = \partial_s t\,\hat{s} + \frac{1}{s}\partial_\varphi t\,\hat{\varphi} + \partial_z t\,\hat{z}$$

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{s}\partial_s(s v_s) + \frac{1}{s}\partial_\varphi v_\varphi + \partial_z v_z$$

$$\vec{\nabla}\times\vec{v} = \left[\frac{1}{s}\partial_\varphi v_z - \partial_z v_\varphi\right]\hat{s} + \left[\partial_z v_s - \partial_s v_z\right]\hat{\varphi} + \frac{1}{s}\left[\partial_s(s v_\varphi) - \partial_\varphi v_s\right]\hat{z}$$

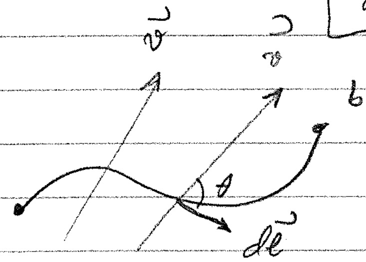
$$\vec{\nabla}^2 t = \frac{1}{s}\partial_s\left(s\,\partial_s t\right) + \frac{1}{s^2}\partial_\varphi^2 t + \partial_z^2 t$$

Sep 12, 2019

INTEGRAL VECTOR CALCULUS

* *Line integrals*

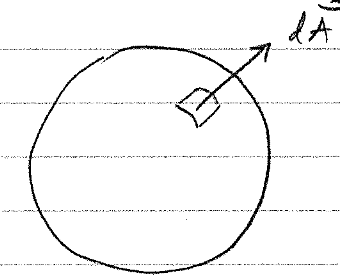
$$\boxed{\int_a^b \vec{v}\cdot d\vec{l} \quad \text{or} \quad \oint_a^b \vec{v}\cdot d\vec{l} \text{ for closed loop.}}$$



Note Path is important, except for conservative fields.

* *Surface integrals*

$$\boxed{\int_S \vec{v}\cdot d\vec{A}}$$



or $\oint_S \vec{v}\cdot d\vec{A}$ for closed surface, called the flux of $\vec{v}$ out of S.

* *Volume integrals* → scalar fn

$$\int_V T\,d\tau = \int_V T\,dx\,dy\,dz$$

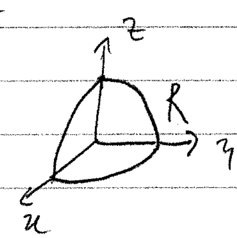
vectors

$$\int_V \vec{v}\,d\tau = \int_V (v_x\hat{x} + v_y\hat{y} + v_z\hat{z})\,dx\,dy\,dz$$

$$= \hat{x}\int_V v_x\,d\tau + \hat{y}\int_V v_y\,d\tau + \hat{z}\int_V v_z\,d\tau$$

In curvilinear coordinates...

- Spherical : $d\tau = r^2 \sin\theta\,dr\,d\theta\,d\varphi$
- Cylindrical $d\tau = s\,ds\,d\varphi\,dz$



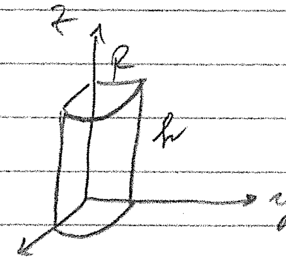
$\boxed{\text{Ex}}$ Volume of 1/8 of a sphere $T = r^2\cos^2\theta$

$$\int_V T\,d\tau = \int_V r^2\cos^2\theta\; r^2\sin\theta\,d\theta\,dr\,d\varphi$$

$$\int_V T\,d\tau = \int_0^{R} r^4\,dr \int_0^{\pi/2} \cos^2\theta \sin\theta\,d\theta \int_0^{\pi/2} d\varphi$$

$$= \frac{R^5}{5}\left(\frac{\pi}{2}\right)\left(-\frac{1}{3}\cos^3\theta\right)\Big|_0^{\pi/2} \qquad R=1$$

$$= \boxed{\frac{\pi}{30}}$$



$R=h=1$

$\boxed{\text{Ex 2}}$ Vol. int in cylindrical coord.:

$$\int_V T\,d\tau = \int_V z^2\, s\,ds\,d\varphi\,dz = \int_0^1 z^2 dz \int_0^1 s\,ds \int_0^{\pi/2} d\varphi = \frac{\pi}{12}$$

FUNDAMENTAL THEOREM OF CALCULUS

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

① Fundamental Theorem for gradients

$$\boxed{\int_a^b \vec{\nabla} T \cdot d\vec{l} = T(b) - T(a)}$$

(a) Line integrals of gradients are path independent.
(b) Closed loop int. of gradients are zero.

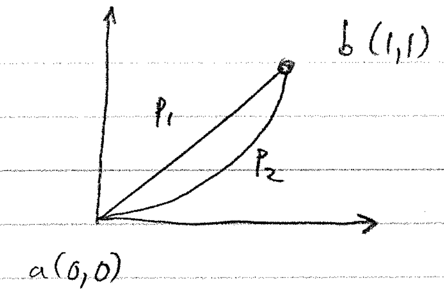
$$\oint \vec{\nabla} T \cdot d\vec{l} = 0$$

true for curless vectors (not really)

beware of "beloved vector field"

(path independence is more fundamental)

Ex



$T(x,y) = xy$

$$\int_a^b \vec{\nabla} T \cdot d\vec{l} = T(b) - T(a) \; ?$$

<u>Path 1</u> : $y = x$
<u>Path 2</u> : $y = x^3$

<u>RHS</u> $T(1,1) - T(0,0) = 1$

<u>Path 1</u> $\vec{\nabla} T = y\hat{x} + x\hat{y}$

$\Rightarrow d\vec{l} = (dx, dy)$

$$\vec{\nabla} T \cdot d\vec{l} = (y, x)\cdot(dx, dy) = y\,dx + x\,dy$$

<u>So</u> $$\int \vec{\nabla} T \cdot d\vec{l} = \int_0^1 y\,dx + \int_0^1 x\,dy = \int_0^1 x\,dx + \int_0^1 y\,dy = \frac{1}{2} + \frac{1}{2} = \boxed{1}$$

<u>Path 2</u> $\cancel{\vec{l} = (x, y) \Rightarrow d\vec{l} = (3x^2, 1)}$

$dy = 3x^2 dx$

$$\cancel{\vec{\nabla} T \cdot d\vec{l} = (y, x)\cdot(3x^2 dx, 1dy) = \int 3x^2 y\,dx + x\,dy}$$

$$\cancel{= \int_0^1 3x^2 x^3 dx + \int_0^1 x\, 3x^2 dx}$$

$$\cancel{= \frac{3}{6} + \frac{3}{}}$$

$$\vec{\nabla} T \cdot d\vec{l} = y\,dx + x\,dy = x^3 dx + x \cdot 3x^2 dx = 4x^3 dx$$

<u>So</u> $$\int \vec{\nabla} T \cdot d\vec{l} = \boxed{1}$$

2 Fundamental Theorem for Divergence (Divergence Theorem) (Gauss' Theorem)

$$\boxed{\int_V (\vec{\nabla}\cdot\vec{v})\, d\tau = \oint_S \vec{v}\cdot d\vec{a}}$$

If $\vec{\nabla}\cdot\vec{v} = 0 \Rightarrow$ no net flux across closed surface

$\boxed{\text{Ex}}$ Heat flow $\vec{H}$ = Heat / cm²/s (area)

Total heat escaping this volume: $\oint_S \vec{H}\cdot d\vec{a}$ = Heat/sec

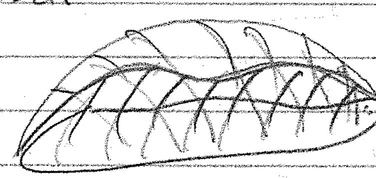
Divergence of $\vec{\nabla}\cdot\vec{H}$ = Heat/cm³/sec, then $\int_V \vec{H}\cdot d\vec{V} =$ Heat/sec

Sep 13, 2019

3 Fundamental Theorem of ~~vector calc~~ Curl (Stokes' theorem)

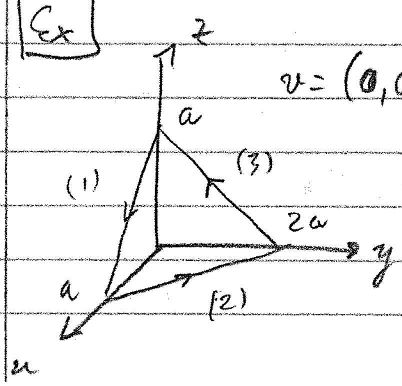
$$\boxed{\int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \oint_P \vec{v}\cdot d\vec{l}}$$

Consequences (a) $\vec{\nabla}\times\vec{v}$ integrated over a surface depend only on the boundary line of that surface, not on the surface itself.



(b) $\oint(\vec{\nabla}\times\vec{v})\cdot d\vec{a} = 0$

$\boxed{\text{Ex}}$



$v = (0,0,y)$ ~~$\vec{\nabla}\times\vec{v} =$~~

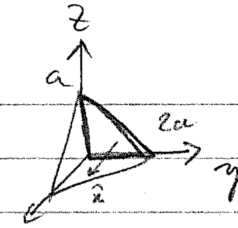
$v = (0,0,y)$ $\quad d\vec{l} = (dx, dy, dz)$

$$\oint_P \vec{v}\cdot d\vec{l} = \int \vec{v}\cdot d\vec{l}_1 + \int \vec{v}\cdot d\vec{l}_2 + \int \vec{v}\cdot d\vec{l}_3$$

$$= \int_1 y\,dz + \int_2 y\,dz + \int_0^a y\,dz = \boxed{a^2}$$

$\underbrace{\int_1 y\,dz}_{0}$, $\underbrace{\int_2 y\,dz}_{0}$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & y \end{vmatrix} = \hat{x}$$



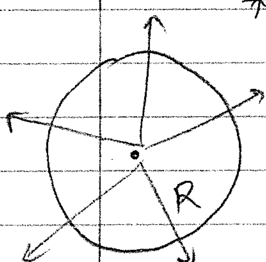
So $\int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \int_S \hat{x}\cdot d\vec{a}$

$$= \int_S (\hat{x})\cdot(dy\,dz)\,\hat{x} = \iint dy\,dz = \frac{1}{2}a(2a) = \boxed{a^2}$$

//

(IX) $\boxed{\text{The Delta Dirac Function}}$

* Divergence of : $\vec{v} = \frac{\hat{r}}{r^2}$ $\quad \vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\partial_r\left(r^2 v_r\right) = \boxed{0}$



With FT:

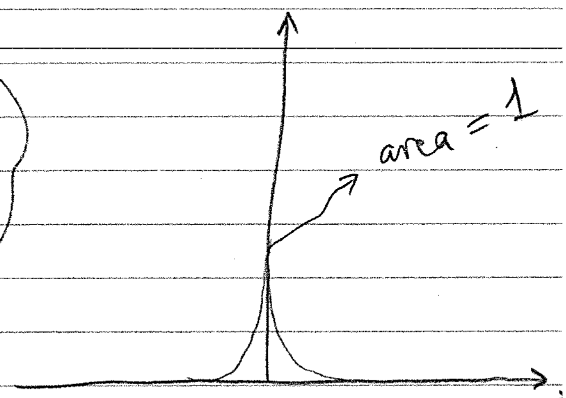
$$\int_V (\vec{\nabla}\cdot\vec{v})\,dv = \oint_S \vec{v}\cdot d\vec{a} = \oint_S \left(\frac{1}{\cancel{R^2}}\hat{r}\right)\overbrace{(\cancel{R^2}\sin\theta)\,\hat{r}\,d\theta\,d\varphi}^{d\vec{a}}$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\theta\,d\theta\,d\varphi = \boxed{4\pi}$$

→ problem $\operatorname{div}(\vec{v}) = 0$ everywhere except $r=0$

$\boxed{\text{Defn}}$

$$\delta(x) = \begin{cases} 0 & \text{if } x\neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$



$$\boxed{\int_{-\infty}^{\infty}\delta(x)\,dx = 1}$$

$$f(x)\,\delta(x) = f(0)\,\delta(x)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)\int_{-\infty}^{\infty}\delta(x)\,dx = f(0)$$

Generalize

$$\delta(x-a) = \begin{cases} 0 & \text{if } x \neq a \\ \infty & \text{if } x = a \end{cases}$$

$$\boxed{\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)}$$

Three-dim delta fn: $\boxed{\delta^3(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)}$

$$\boxed{\int_V \delta^3(\vec{r})\,d\tau = 1}$$

☒ Now, back to "paradox" : $\int (\vec{\nabla}\cdot\vec{v})\,d\tau = 4\pi$

or $\int \vec{\nabla}\cdot\left(\frac{1}{r^2}\hat{r}\right) d\tau = 4\pi$ possible if $\boxed{\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(\vec{r})}$

because $\int 4\pi\,\delta^3(\vec{r})\,d\tau = 4\pi$

☒ Ex $\int_2^6 (3x^2-2x+1)\,\delta(x-3)\,dx$

$= 3\cdot 3^2 - 2\cdot 3 + 1 = 22$

Ex $\int_2^{10} \ln(x+3)\,\delta(x+1)\,dx \neq \ln(-1+3) = \ln(2)$

Bounds $2 \to 10$, but δ centered at -1

$\Rightarrow \int_2^{10} \ln(x+3)\,\delta(x+1)\,dx = 0$

Sep 16, 2019

$\boxed{\text{VECTOR FIELD THEORY}}$

- Force between objects... $\vec{F} = q\left(\vec{E} + \frac{\partial \vec{r}}{\partial t} \times \vec{B}\right)$

↳ Lorentz Force Law.

Maxwell's Eqns

$$\boxed{\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \rho/\varepsilon_0 \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned}}$$

ε_0 : permitivity of free space $8.854 \times 10^{-12}\ C^2/Nm^2$

μ_0 : permeability of a vacuum $4\pi \times 10^{-7}\ N/A^2$

$$\left\{ \frac{1}{\varepsilon_0 \mu_0} = c^2 \right\}$$

ρ = Charge Density $[c/m^3]$ $\quad \vec{J}$ = current density $[A/m^2]$

$\boxed{\text{For static fields}}$

$$\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0 \qquad \vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$\boxed{\text{Helmholtz Thm}}$

- Any field is uniquely determined by its divergence and curl if its boundary conditions are known.
- Boundary conditions: $\vec{E} \to \vec{0}$ as $r \to \infty$
  $\vec{B} \to \vec{0}$ as $r \to \infty$

Theorem 1 Fields with no curl (irrotation fields)

1. $\vec{\nabla} \times \vec{F} = 0$ everywhere
2. $\vec{F} = -\vec{\nabla} V$ where f is a scalar function.
   because $\vec{\nabla} \times (\vec{\nabla} f) = \vec{0} \quad \forall f$.
   V: "scalar potential"
3. $$\oint \vec{F} \cdot d\vec{l} = 0 \quad \text{for any closed loop}$$
   $$\int (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \oint \vec{F} \cdot d\vec{l} = 0$$
4. $$\int_a^b \vec{F} \cdot d\vec{l} = -V(b) + V(a) \rightsquigarrow \text{path independent.}$$
   $$\parallel \qquad \parallel$$
   $$-\int_a^b (\vec{\nabla} \cdot V) \cdot d\vec{l}$$

Theorem 2 Fields with no divergence (solenoidal fields)

1. $\vec{\nabla} \cdot \vec{F} = 0$ everywhere
2. $\vec{F} = \vec{\nabla} \times \vec{A} \rightsquigarrow$ curl of some vector fn.
   $\hookrightarrow \vec{A}$: vector potential ...
3. $$\oint \vec{F} \cdot d\vec{a} = 0 \quad \text{for any closed surface}$$
   $$\oint \vec{F} \cdot d\vec{a} = \int (\vec{\nabla} \cdot \vec{F})\, dV = 0$$
4. $$\int_S \vec{F} \cdot d\vec{a} = \text{independent of surface} = \oint_P \vec{A} \cdot d\vec{l}$$

$\vec{\nabla} \times \vec{A}$

* For all fields : $\boxed{\vec{F} = -\vec{\nabla} V + \vec{\nabla} \times \vec{A}}$ → always true

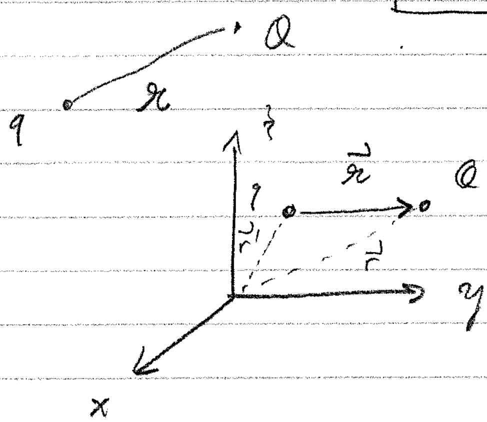
Chapter 2: ELECTROSTATICS

In static fields ...

$$\left\{ \begin{array}{l} \vec{\nabla} \cdot \vec{E}(\vec{r}) = \rho(\vec{r})/\varepsilon_0 \\ \vec{\nabla} \times \vec{E}(\vec{r}) = \vec{0} \end{array} \right. \qquad \left\{ \begin{array}{l} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}(\vec{r}) \end{array} \right.$$

Electrostatics — Magnetostatics

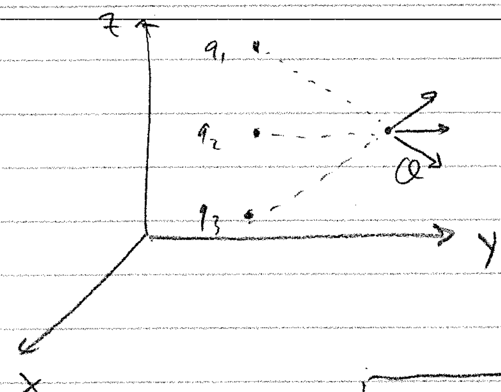
(A) Coulomb's law $\boxed{\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{qQ}{r^2} \hat{r}}$ $\varepsilon_0 \sim 8.854 \times 10^{-12}\ C^2/Nm^2$ → Electric constant.



Note ① $F \propto qQ$, $F \propto r^{-2}$

② Points in direction $q \to Q$, $\hat{r}$

③ F is repulsive when $qQ > 0$
attractive when $qQ < 0$



Force law follows principle of superposition...

{ Total force felt is vector sum of individual forces }

$$\vec{F}_Q = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 .$$

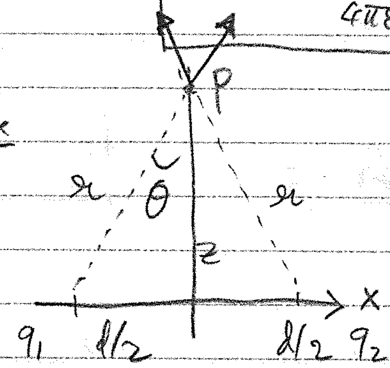
→ Generalized : $\boxed{\vec{F} = \frac{Q}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i^2} \hat{r}_i}$

(B) <u>Electric field</u> $\rightarrow$ $\vec{E}$: force per unit charge (potential force)

$$\vec{F} = q\vec{E}\ , \quad \vec{E} = \vec{F}/Q \rightarrow \text{test charge} \ldots$$

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i^2}\hat{r}}$$

<u>Ex</u>



$$\vec{E}_T = \vec{E}_1 + \vec{E}_2$$

At point P, $\quad E_x = E_1 \sin\theta + E_2 \sin(-\theta)$

$\quad E_y = E_1 \cos\theta + E_2 \cos(-\theta)$

$$\text{So} \quad \vec{E}_p = \left(0\hat{x} + \frac{2q}{r^2}\cos\theta\,\hat{z}\right)\frac{1}{4\pi\varepsilon_0} = \frac{1}{4\pi\varepsilon_0}\,\frac{2q\cos\theta}{r^2}\,\hat{z}$$

Now, $r^2 = x^2 + z^2 = \left(\frac{d}{2}\right)^2 + z^2$

and $\cos\theta = \frac{z}{r} \Rightarrow \quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{2qz}{(z^2 + d^2/4)^{3/2}}\right]\hat{z}$

What happens when $z \gg d/2$

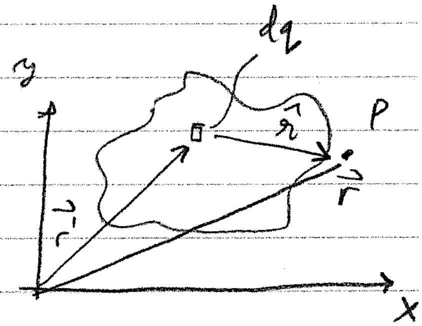
$$\rightarrow \boxed{\vec{E} \sim \frac{1}{4\pi\varepsilon_0}\,\frac{2q}{z^2}\,\hat{z}} \sim \text{like 2 charges} \ldots$$

Sep 17, 2019

CHARGE DISTRIBUTION

(c) Continuous charge distribution

For discrete $\vec{E}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \dfrac{q_i}{r_i^2} \hat{r}_i$

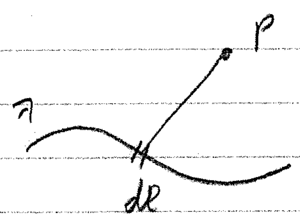


Continuous

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{dq(\vec{r}')}{r^2} \hat{r}$$

Types of charge dist

① Line charge : λ (charge / unit length)

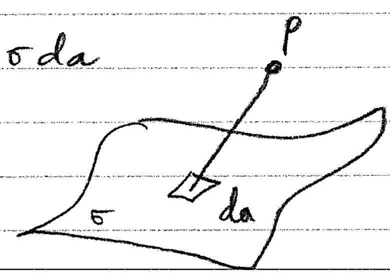


$dq = \lambda dl$

$$\Rightarrow \boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(\vec{r}')}{r^2} \hat{r} \, dl'}$$

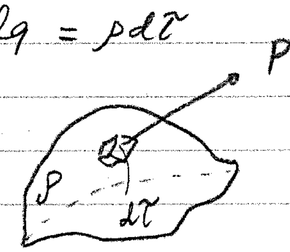
② Surface charge σ (charge / area) $dq = \sigma da$

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma(\vec{r}')}{r^2} \hat{r} \, da'}$$

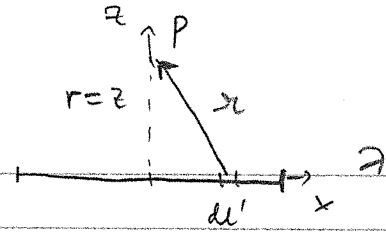


③ Volume charge ρ (charge / volume) $dq = \rho d\tau$

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')}{r^2} \hat{r} \, d\tau'}$$



Example



$$\vec E(\vec r) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\vec r')}{\mathscr{r}^2}\hat{\mathscr{r}}\, dl' \rightarrow \frac{z\hat z - x\hat x}{(z^2+x^2)^{1/2}}$$

$\vec r = z\hat z$

$\vec r' = x\hat x$ $\quad \mathscr{r}^2 = r^2 + x^2$

$dl' = dx$ $\quad \vec{\mathscr{r}} = \vec r - \vec r'$

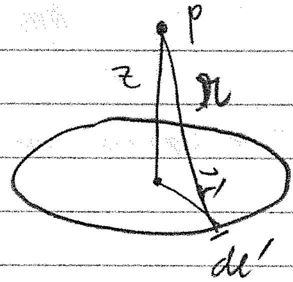
$$\rightarrow \vec E(\vec r) = \frac{1}{4\pi\epsilon_0}\int_{-L}^{L} \frac{\lambda}{z^2+x^2}\,\frac{z\hat z - x\hat x}{(z^2+x^2)^{1/2}}\,dx$$

$$\text{So}\quad \vec E(r) = \frac{\lambda}{4\pi\epsilon_0}\int_{-L}^{L} \frac{z\hat z}{(x^2+z^2)^{3/2}} - \frac{x\hat x}{(x^2+z^2)^{3/2}}\,dx \qquad 0$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[z\hat z\,\frac{x}{z^2(z^2+x^2)^{1/2}} - \cancel{\hat x\left(\frac{-1}{(z^2+x^2)^{1/2}}\right)}\Bigg|_{-L}^{L}\right]$$

$$\boxed{\vec E(\vec r) = \frac{\lambda}{4\pi\epsilon_0}\,\frac{2L}{z(z^2+L^2)^{1/2}}\,\hat z} \rightarrow \text{when } z \to \infty \Rightarrow \vec E(r) = \frac{2\lambda L k}{z^2}$$

Example



$$\vec E(r) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\vec r')}{\mathscr{r}^2}\hat{\mathscr{r}}\, dl'$$

$$\vec E_z(\vec r) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda}{\mathscr{r}^2}\cos\theta\, dl'\,\hat z$$

$$= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} \frac{\lambda}{z^2+R^2}\cdot\frac{z}{(z^2+R^2)^{1/2}}\,\hat z\cdot(R\,d\phi)$$

$$= \frac{1}{4\cancel{\pi}\epsilon_0}\,\frac{\lambda z}{(z^2+R^2)^{3/2}}\,(2\cancel{\pi}R)\,\hat z$$

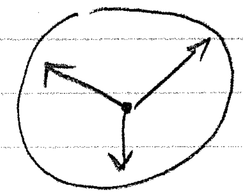
$$= \frac{1}{2\epsilon_0}\,\frac{\lambda z R}{(z^2+R^2)^{3/2}}\,\hat z$$

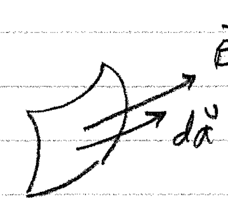
(D) **DIVERGENCE & CURL of $\vec{E}$** → Maxwell's Eqn for Electrostatics.

$$\boxed{\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \quad ; \quad \vec{\nabla}\times\vec{E} = 0}$$

Div Recall $\int_V (\vec{\nabla}\cdot\vec{E})\,dV = \oint \vec{E}\cdot d\vec{A}$

$\oint \vec{E}\cdot d\vec{A}$ for point source ...





$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R^2}\right)\hat{r}. \quad d\vec{a} = R^2\sin\theta\, d\theta\, d\varphi\, \hat{r}$$

$$\text{So} \quad \oint \vec{E}\cdot d\vec{a} = \int \frac{1}{4\pi\varepsilon_0} q \sin\theta\, d\theta\, d\varphi = \frac{q}{4\pi\varepsilon_0}(4\pi) = \frac{q}{\varepsilon_0}$$

$$\text{So} \quad \boxed{\oint \vec{E}\cdot d\vec{a} = q/\varepsilon_0} \rightsquigarrow \text{Gauss' Law.} \quad \boxed{q = \sum Q_i = \int \rho\, dV}$$

Sep 19, 2019 **Notes on Gauss' Law**

↳ Flux of E through closed surface depends on charge enclosed

→ no contribution from charge outside surface.

→ independent of surface shape.

→ independent of surface size.

Holds for multiple charges as well ...

(*) Divergence of $\vec{E}$ $\quad \int_V (\vec{\nabla}\cdot\vec{E})\,d\tau = \oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int \rho\, d\tau$

$$\text{So} \quad \int_V (\vec{\nabla}\cdot\vec{E})\,d\tau = \frac{1}{\varepsilon_0}\int_V \rho\,d\tau \Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \frac{q}{\varepsilon_0}}$$

(*) Alternative way

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}^2}\hat{\mathscr{r}} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{\mathscr{r}^2}\hat{\mathscr{r}}\,\rho(r')\,d\tau'$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \underbrace{\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)}_{=4\pi\delta^3(\mathscr{r})}\rho(r')\,d\tau' = \frac{1}{\varepsilon_0}\int \rho(r')\,\delta^3(\underbrace{\mathscr{r}}_{r-r'})\,d\tau'$$

$$= \frac{1}{\varepsilon_0}\rho(r)$$

$$\text{So} \quad \boxed{\vec{\nabla}\cdot\vec{E}_0 = \rho/\varepsilon_0}$$

(*) $\boxed{\text{Curl of } \vec{E}}$ $\quad \boxed{\vec{\nabla}\times\vec{E} = \vec{0}}$

$dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

Stokes' Thm

$$\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{A} = \oint_P \vec{E}\cdot d\vec{l} = \oint \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}\cdot d\vec{l}$$

$$= \oint \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,dr = 0 \quad (r = r_0,\ r' = r_0)$$

(3) $\boxed{\text{Application of Gauss' law}}$ $\quad \oint \vec{E}\cdot d\vec{a} = Q_{enc}/\varepsilon_0$

(*) 3 symmetries

- Spherical symmetry ~ Spherical surface
- Cylindrical symmetry ~ cylinder surface
- Planar symmetry ~ use pill box surface

Keys to picking Gaussian surface

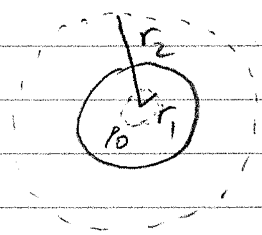
(1) Want to ensure $\vec{E} \parallel d\vec{A}$ and constant over surface

$$\int \vec{E}\cdot d\vec{a} = \int E\,dA = E\int dA = E.A$$

(2) $E \perp d\vec{A}$ over some portion of surface

$$\int \vec{E}\cdot d\vec{a} = 0$$

(3) $E = 0$ inside surface.

Ex 

Find $\vec{E}$ inside & outside of uniformly charged sphere.

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

Inside $\quad E.(4\pi \cancel{r_1^2}) = \frac{1}{\varepsilon_0}\rho_0 \cdot \frac{4}{3}\pi r_1^{\cancel{3}} \Rightarrow \vec{E} = \frac{1}{3}\frac{\rho_0}{\varepsilon_0} r_1 \hat{r}$

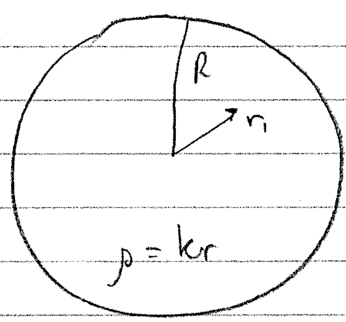
Outside $\quad E(4\pi r_2^2) = \frac{1}{\varepsilon_0}\rho_0 \cdot \frac{4}{3}\pi R^3 \Rightarrow \vec{E} = \frac{1}{3}\frac{\rho_0}{\varepsilon_0} R^3/r_2^2\,\hat{r}$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{r_2^2}\hat{r}$$

What if $\rho_0 = kr$?

$$E(4\pi r_1^2) = \left(\int_0^{r_1} \rho(r) r^2 dr\right) \underbrace{\int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi}\cdot\frac{1}{\varepsilon_0}$$

$Q = \int \rho(r)\, dV$



$$E(\cancel{4\pi} r_1^2) = \int_0^{r_1} k r^3 dr \;\; \cancel{4\pi}$$

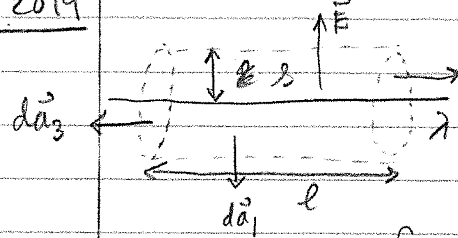
$$\vec{E}(\vec{r}_2) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r_2^2}\hat{r}$$

$$\vec{E} = \frac{1}{4} k r_1^2 \cdot \frac{1}{\varepsilon_0}\hat{r} \Rightarrow \boxed{\vec{E} = \frac{1}{4\varepsilon_0} k r_1^2 \hat{r}}$$

Sep 20, 2019

infinite line of charge

Ex 3 Find E field, distance s from a line charge, λ.



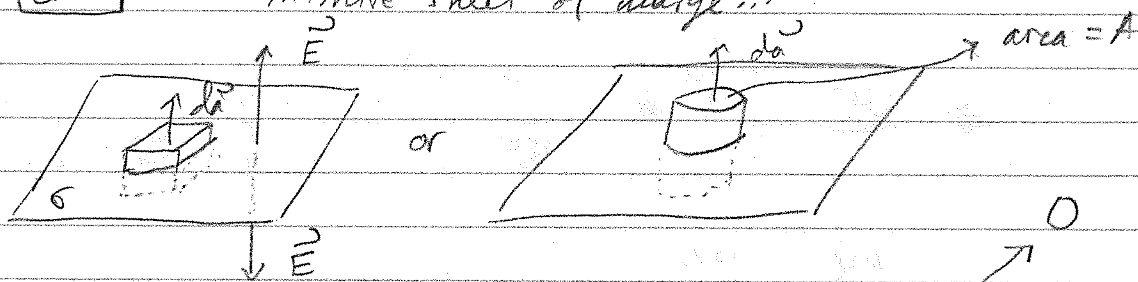
$$\oint \vec{E}\cdot d\vec{a} = Q_{enc}/\varepsilon_0 \qquad \vec{E} \text{ only in } \hat{s}$$

LHS $\rightsquigarrow$ $$\int \vec{E}\cdot d\vec{a} = \int \vec{E}\cdot d\vec{a}_1 + \underbrace{\int \vec{E}\cdot d\vec{a}_2}_{0} + \underbrace{\int \vec{E}\cdot d\vec{a}_3}_{0}$$

$$= E\int da = E\int s\,d\phi\,dz = E\cdot s(2\pi)\,l$$

RHS $$Q_{enc}/\varepsilon_0 = \frac{\lambda l}{\varepsilon_0} = E s 2\pi l \Rightarrow \boxed{\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\cdot\frac{1}{s}\,\hat{s}}$$

Ex 4 $\rightsquigarrow$ infinite sheet of charge...

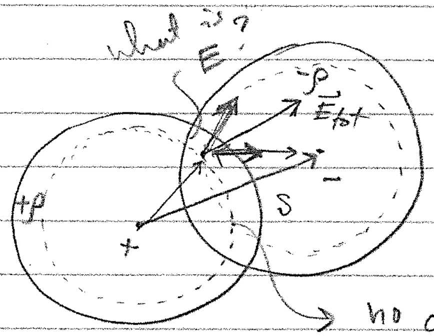


$$\oint \vec{E}\cdot d\vec{a} = Q_{enc}/\varepsilon_0 = \int \vec{E}\cdot d\vec{a}_1 + \underbrace{\int \vec{E}\cdot d\vec{a}_2}_{0} + \int \vec{E}\cdot d\vec{a}_3$$

$$\| \qquad = 2E\int da$$

$$\sigma A/\varepsilon_0 = 2EA \Rightarrow \boxed{\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}}$$

Ex 5



no charge here...

$$\rightsquigarrow \vec{E}_{tot} = \vec{E}_+ + \vec{E}_-$$

$$= \frac{\rho\,\vec{r}_+}{3\varepsilon_0} + \frac{\rho\,\vec{r}_-}{3\varepsilon_0}$$

$$\boxed{\vec{E}_{tot} = \frac{\rho}{3\varepsilon_0}\left(\vec{r}_+ + \vec{r}_-\right) = \frac{\rho}{3\varepsilon_0}\vec{s}}$$

(superposition + Gauss' law)

(F) $\boxed{\text{Electric Scalar Potential}}$

* Curless vectors ... ($\vec{\nabla} \times \vec{E} = 0$) we know:

(1) $\int_a^b \vec{E} \cdot d\vec{l}$ = independent of path.

(2) $\oint \vec{E} \cdot d\vec{l} = 0$

(3) $\vec{E} = -\vec{\nabla} V$ ⤳ V: electric scalar potential ...

* General form of $V(r)$.

$$\int_a^b \vec{E} \cdot d\vec{l} = \int_a^b -\vec{\nabla} v \cdot d\vec{l}$$

$\Rightarrow$ LHS $\int_a^b \vec{E} \cdot d\vec{l} = \frac{q}{4\varepsilon_0\pi} \int_a^b \frac{dr}{r^2} = -\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$

RHS $v(a) - v(b) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$

So $\int_a^b \vec{E} \cdot d\vec{l} = -V(b) + V(a)$

So $\boxed{V(a) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r_a}}$ + Constant

Check $\boxed{E = -\vec{\nabla} v}$

$$E = \frac{-1}{4\pi\varepsilon_0} \partial_r (r^{-1}) \hat{r} \Rightarrow \frac{-q}{4\pi\varepsilon_0}(-r^{-2})\hat{r} = +\frac{q}{4\pi\varepsilon_0} \frac{1}{r^2} \hat{r}$$

* Unit $\boxed{[E] = N/C \quad , \quad [V] = Nm/C = J/C = \text{Volt}}$

* Potential Difference

$$\int_a^b \vec{E}\cdot d\vec{l} = -\left(V(b) - V(a)\right)$$

→ V is not uniquely determined at a & b.

$\vec{E} = -\vec{\nabla} V$ ... then also $-\vec{\nabla}(V + \text{constant})$

$$= -\vec{\nabla} V - \cancel{\vec{\nabla}\,\text{constant}} = -\vec{\nabla} V$$

0

* ⟹ Reference point become important.

(Reference Point) $\quad \int_a^b \vec{E}\cdot d\vec{l} = \int_{ref}^{r} \vec{E}\cdot d\vec{l} = V(ref) - V(r)$

The most useful reference point is $r \to \infty$ because $\vec{E} \to 0$, $V \to 0$ a $r \to 0$.

So by convention... $\boxed{V(r) \equiv -\int_\infty^r \vec{E}\cdot d\vec{r}}$

Sep 23, 2019

Electric potential (cont...)

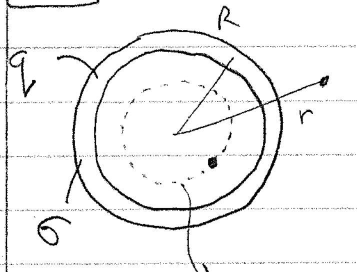
$$\left\{ \begin{array}{l} \vec{E}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \frac{\rho(\vec{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\, d\tau' \\ \displaystyle\oint \vec{E}\cdot d\vec{a} = \int (\vec{\nabla}\cdot\vec{E})\, d\tau \\ \vec{E} = -\vec{\nabla} V \end{array} \right.$$

Recall: V is not unique... $\vec{E} = -\vec{\nabla}(V + \text{constant}) = -\vec{\nabla} V$

$$\int_a^b \vec{E}\cdot d\vec{l} = V(a) - V(b)$$

$$\boxed{V(r) \equiv -\int_{\infty}^{\vec{r}} \vec{E} \cdot d\vec{l}}$$

$\boxed{Ex}$ Find $V(r)$ inside & outside a uniformly charged shell



E inside $= 0$

$$V(r) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l} \quad \text{for outside} \ldots$$

$$= -\int_{\infty}^{r} \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r^2} \hat{r} \cdot d\vec{l} \quad \text{for outside}$$

$$= -\frac{1}{4\pi\varepsilon_0} \int_{\infty}^{r} \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \vec{r} \cdot d\vec{r}$$

$(\sigma \cdot 4\pi R^2)$

$$= \frac{Q}{4\pi\varepsilon_0} \left[\frac{1}{r} - \frac{1}{\infty} \right] \sim \frac{Q}{4\pi\varepsilon_0} \cdot \frac{1}{r}$$

so $\boxed{\cancel{\vec{E}_{out}} = \frac{\sigma R^2}{\varepsilon_0 r}}$ $\boxed{\vec{E}_{inside} = 0}$

Now, inside ...

$$V(r<R) = -\int_{\infty}^{r<R} \vec{E} \cdot d\vec{l}$$

$$= \underbrace{-\int_{\infty}^{R} \vec{E}_{out} \cdot d\vec{l}}_{\cancel{V} = \frac{\sigma R}{\varepsilon_0}} - \underbrace{\int_{R}^{r} \underbrace{\vec{E}_{in}}_{0} \cdot d\vec{l}}_{0}$$

So

$$\boxed{\begin{aligned} V(r)_{out} &= \frac{\sigma R^2}{\varepsilon_0 r} \\ V(r)_{in} &= \frac{\sigma R^2}{\varepsilon_0} \rightsquigarrow \text{constant} \end{aligned}}$$

$$\boxed{\text{Generalize } V \text{ for charge dist.}}$$

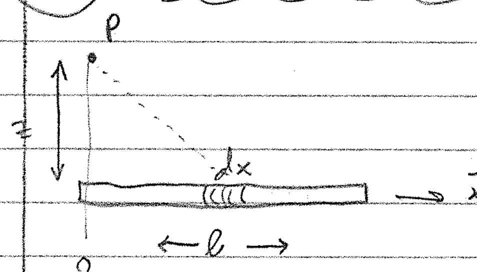
$V(r) = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q}{r}$ ⇝ for point charge ...

$V(r) = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\int \dfrac{\rho(r')}{\mathscr{r}} d\tau'$ ⇝ volume charge ...

$V(r) = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\int \dfrac{\lambda(r')}{\mathscr{r}} dl'$ ⇝ line charge

$V(r) = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\int \dfrac{\sigma(r')}{\mathscr{r}} da'$ ⇝ surface charge ...

Ex Potential for a line charge ...



$$V(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(x)}{\mathscr{r}} dx$$

$$= \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(x)}{\sqrt{z^2 + x^2}} dx$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \int_0^{\ell} \frac{dx}{\sqrt{z^2 + x^2}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \ln \left[x + \sqrt{z^2 + x^2} \right]_0^{\ell}$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \left\{ \ln \left[\ell + \sqrt{z^2 + \ell^2} \right] - \ln(z) \right\}$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \ln \left\{ \frac{\ell + \sqrt{z^2 + \ell^2}}{z} \right\}$$

//

G) Poisson Equation – Laplace's Eqn

Rewrite Maxwell's Eqn in terms of potential ..

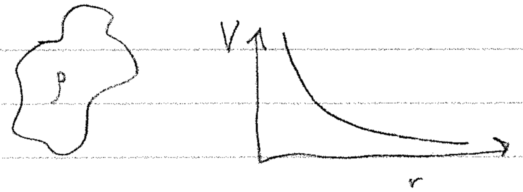
$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 , \quad \vec{\nabla}\times\vec{E} = \vec{0} , \quad \vec{E} = -\vec{\nabla}V$$

So $\vec{\nabla}\cdot(-\vec{\nabla}V) = \rho/\varepsilon_0 \Rightarrow \boxed{\vec{\nabla}^2 V = -\rho/\varepsilon_0}$ → Poisson's Eqn

Away from charge (in free space, $\rho = 0$)

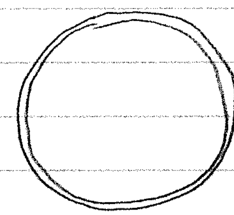
$\boxed{\vec{\nabla}^2 V = 0}$ → Laplace's Eqn.

Solution → are (harmonic functions) → no local max / min

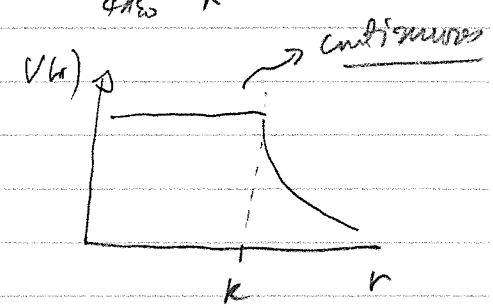


H) Boundary conditions

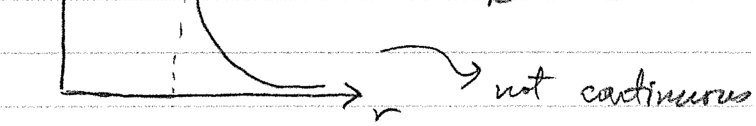
Ex

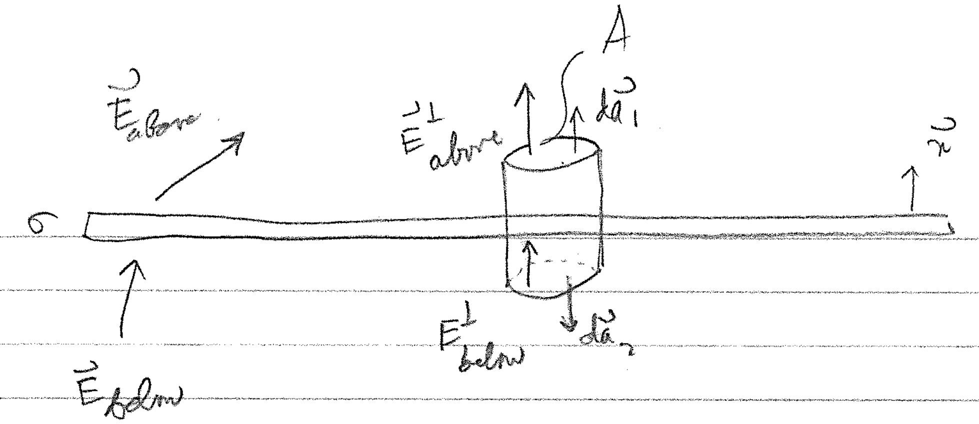


$V_{out} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$, $V_{in} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$



$\vec{E}_{in} = \vec{0}$, $\vec{E}_{out} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$





Perp component ... $\oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$

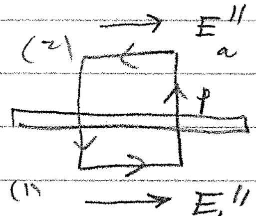
$$= \int_{a_1} \vec{E}^{\perp}_{above} \cdot d\vec{a}_1 + \int_{a_2} \vec{E}^{\perp}_{below} \cdot d\vec{a}_2 = \sigma . A / \epsilon_0$$

$$= E^{\perp}_{above} A - E^{\perp}_{below} . A = \sigma A / \epsilon_0$$

So $\boxed{\vec{E}^{\perp}_{above} - \vec{E}^{\perp}_{below} = \sigma / \epsilon_0 \, \hat{n}}$

→ perpendicular component suffers a discontinuity

Parallel component ... Do a path integral (line integral)



$$\oint_l \vec{E} \cdot d\vec{l} = \int \vec{E}''_b \cdot d\vec{p}_1 + \int \vec{E}''_a \cdot d\vec{p}_2 = \int \underbrace{(\vec{\nabla} \times \vec{E})}_{0} dA$$

$$= E''_b \cdot l - E''_a \cdot l = 0$$

$$\Rightarrow \boxed{E''_a = E''_b}$$

Parallel component always stays ~~[illegible]~~ continuous.

Potential above & below

↳ $\int_a^b \vec{E} \cdot d\vec{l} = V(a) - V(b)$ when $b \to a$, $\Delta V = 0$

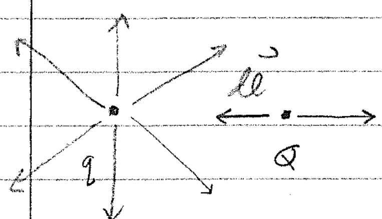
→ V is continuous ... $V_{above} = V_{below}$

$$\boxed{\vec\nabla V_{above} - \vec\nabla V_{below} = -\frac{\sigma}{\epsilon_0}\hat{n}}$$

$$\rightsquigarrow \boxed{\frac{\partial V}{\partial n} = (\vec\nabla V)\cdot\hat{n}}$$

Sep 24, 2019

I) WORK & ELECTROSTATIC ENERGY



$\vec{F} = Q\vec{E}$ $\quad W = \int_a^b \vec{F}\cdot d\vec{l} = -Q\int_a^b \vec{E}\cdot d\vec{l}$

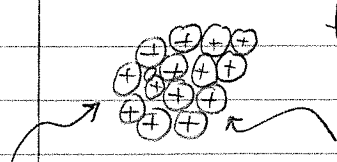
So $$\boxed{W = Q\int_a^b \vec\nabla V\, dl = Q\left[V(b) - V(a)\right]}$$

For reference @ ∞, $\boxed{W = QV(r)}$

So $V(r)$ ~ Energy per unit charge needed to assemble system.

■ Multiple particles...

How much E needed to construct this?



$\Rightarrow$ second charge costs energy $q_2 V_1(r_2)$,

where V_1 is potential from q_1, r_2 = loc of q_2

$q_3 \rightarrow$ feels pot. from 1, 2 ...

$$V_{12} = V_1 + V_2 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}}\right)$$

So $$W_3 = q_3 V_{12} = \frac{q_3}{4\pi\epsilon_0}\left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}}\right)$$

$$W = \frac{\varepsilon_0}{2}\left\{ \int_V \left[\vec{\nabla}(V\vec{E}) - \vec{E}\cdot(\vec{\nabla}V) \right] d\tau \right\}$$

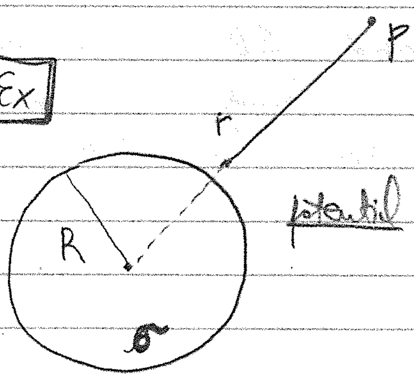
$$= \underbrace{\frac{\varepsilon_0}{2}\int_V \vec{\nabla}\cdot(V\vec{E})\,d\tau}_{} + \frac{\varepsilon_0}{2}\int_V \vec{E}^2\,d\tau$$

Apply FTC ... $= \boxed{\frac{\varepsilon_0}{2}\oint_A V\vec{E}\cdot d\vec{a} + \frac{\varepsilon_0}{2}\int_V |\vec{E}|^2\,d\tau = W}$

Take A to be ∞-sphere, then $V, E \to 0$

so $\boxed{W = \frac{\varepsilon_0}{2}\int_V |\vec{E}|^2\,d\tau}$ all space

$\boxed{Ex}$



Find W for spherical shell ...

potential $W = \frac{1}{2}\int \sigma(r)V(r)\,da$ construct sphere

$$= \frac{1}{2}\int \sigma \cdot \frac{1}{4\pi\varepsilon_0}\frac{1}{R}\,dA$$

so $W = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{R}$ matches!

Electric field

$$W = \frac{\varepsilon_0}{2}\int |\vec{E}|^2 d\tau = \frac{\varepsilon_0}{2}\int \vec{E}_{in}^2\,d\tau_{in} + \frac{\varepsilon_0}{2}\int \vec{E}_{out}^2\,d\tau_{out}$$

$$= \underbrace{\frac{\varepsilon_0}{2}\int (0)\,d\tau_{in}}_{0} + \frac{\varepsilon_0}{2}\int \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)^2 d\tau$$

$$= \frac{\varepsilon_0}{2}\frac{q^2}{(4\pi\varepsilon_0)^2}\int_R^{\infty}\frac{1}{r^4}r^2\,dr\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta\,d\theta = \boxed{\frac{1}{2}\frac{q^2}{4\pi\varepsilon_0}\frac{1}{R}}$$

Four charge ... $W_4 = \frac{q_4}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}}\right)$ ...

So ... $$\boxed{W_{tot} = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{j\neq i}^{N}\frac{q_i q_j}{r_{ij}}}$$

corrects for counting twice

So $$\boxed{W_{tot} = \frac{1}{2}\sum_{i=1}^{N} q_i V(r_i)}$$ → W for constructing charge dist.

Generalizing for dist

$$\boxed{W = \frac{1}{2}\int \rho(r) V(r)\, d\tau}$$ Volume charge dist

Analogue in 2D: $\sigma(r)\,dA$ and 1D: $\lambda(r)\,dl$

Note { Integrals are over space where charge dist is defined }

→ order of assembly doesn't matter
→ Energy is stored = energy need to construct / dispand ... as potential E
→ ~~The~~ Energy is stored in Electric Field ...

Work via Electric field ...

$$W = \frac{1}{2}\int \rho(r) V(r)\, d\tau$$

$$\vec{\nabla}\cdot(V\vec{E}) = (\vec{\nabla}\cdot\vec{E})V + \vec{E}\cdot(\vec{\nabla}V)$$

$$= \frac{1}{2}\int \left[\vec{\nabla}\cdot\vec{E}\right]\varepsilon_0 V(r)\, d\tau$$

→ S...

$\boxed{\text{* Energy needed to construct an electron}}$

$r_e \to 0 \qquad W = \frac{\varepsilon_0}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \int_0^\infty \frac{e^2}{r^2} \sin\theta \, d\theta \, dr \, d\phi$

$= \frac{e^2}{8\pi\varepsilon_0} \int_0^\infty \frac{1}{r^2} dr = \infty \rightsquigarrow$ infinite amount of energy to construct e^-

$\to$ need QM.

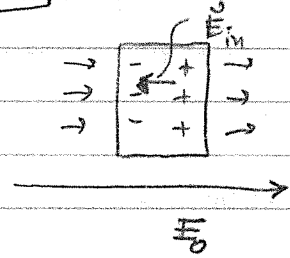
But we can flip this around to find r_e...

Sep 26, 2019

CONDUCTORS

Basic Properties...

① $\boxed{\vec{E} = \vec{0}}$ inside conductors. $\rightsquigarrow$ because of induced charges move to cancel $\vec{E}_{out}$



$\vec{E}_{out} + \vec{E}_{in} = \vec{0}$

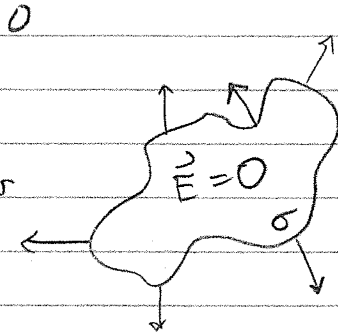
② $\boxed{\rho = 0}$ $\rightsquigarrow$ gauss' law

③ Any charge, Q, resides only on surface (σ)

④ Conductors are equipotentials. (V = constant inside)

$\Delta V = -\int_a^b E \cdot dl = 0 \to \Delta V = 0$

⑤ $\vec{E}$ is perp to surface of conductors



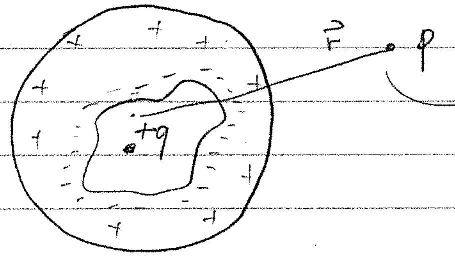
$\boxed{\text{Example 2.9}}$ Energy in charge shell ...

$$W = \frac{\varepsilon_0}{2}\int \vec{E}^2\, d\tau \; ; \quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{C}{r^2}\hat{r}$$

$$\boxed{W_{shell} = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{R}}$$

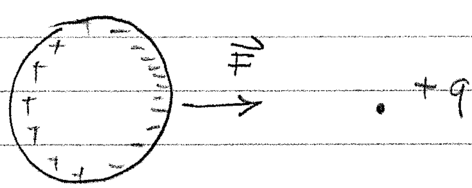
Note that $W_{\substack{filled\\spher}} \cancel{>} W_{shell}$.

$\boxed{\text{Induced charges}}$ $\rightarrow$ conductors shield charges.



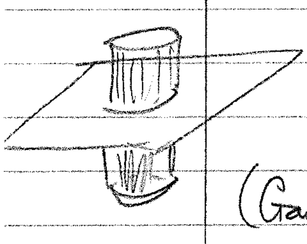
$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\mathscr{r}^2}\hat{\mathscr{r}}$$

(★) Conductors are attracted to external $\vec{E}$ fields



(*) $\boxed{\text{Force on a conductor}}$

$$\vec{F} = Q\vec{E} = \sigma A\vec{E}$$



$$f = F/A = \sigma\vec{E} \quad \text{where } \vec{E} = \vec{E}_{avg}$$

$$= \frac{1}{2}\left(\vec{E}_{above} - \vec{E}_{below}\right)$$

(Gauss)

$\rightarrow \oint \vec{E}\cdot d\vec{a} = Q/\varepsilon_0 = \frac{\sigma A}{\varepsilon_0} \rightarrow \left(\vec{E}_a - \vec{E}_b = \frac{\sigma}{\varepsilon_0}\hat{n}\right)$

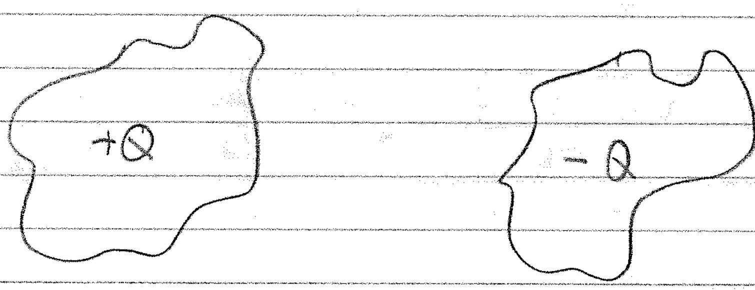
But $\vec{E}_{below} = \vec{0}$ for conductor. $\rightarrow$ $\boxed{\vec{E}_{above} = \frac{\sigma}{\epsilon_0}\hat{n}}$ $\rightsquigarrow$ true for all conductors

So, force experienced $= \vec{f} = \sigma \vec{E}_{avg} = \frac{1}{2}\sigma(\vec{E}_a - \vec{E}_b)$

$= \frac{1}{2}\sigma \vec{E}_{above}$

So $\boxed{f = \frac{\sigma^2}{2\epsilon_0}\hat{n}}$ $\rightsquigarrow$ electrostatic pressure $\boxed{P = \frac{\epsilon_0}{2}E^2}$

CAPACITORS



$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\, d\tau'}{r}$ $\rightsquigarrow$ difficult... $E = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\, d\tau'}{r^2}\hat{r}$ $\rightarrow$ difficult...

But $\Delta V = V_+ - V_- = -\int_-^+ \vec{E}\cdot d\vec{l}$

where $\vec{E} \propto Q_{tot}$ & $V \propto Q_{tot}$

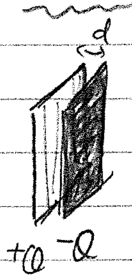
$\rightsquigarrow$ Capacitance...

$$\boxed{C \equiv \frac{Q}{V}}$$

Farad

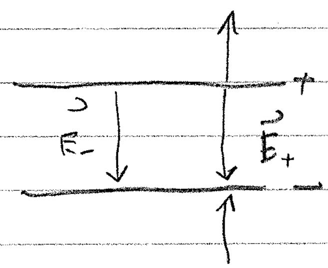
$V = -\int_-^+ \vec{E}\cdot d\vec{l} = \frac{Q}{C}$ $\rightsquigarrow$ (units) $\underbrace{[C] = \frac{\text{Coulombs}}{\text{Volt}}}_{}$

Parallel plate cap:



$$\frac{Q}{C} = V = -\int \vec{E} \cdot d\vec{l} =$$

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0} = 2EA = \frac{\sigma A}{\varepsilon_0} \Rightarrow \vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$$



$$E_{in}\ (\text{net}) = \frac{2\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0} \quad (2 \text{ plates})$$

so $\left|\frac{Q}{C}\right| = |V| = \left|-\frac{\sigma}{\varepsilon_0}\int d\vec{l}\right| = \left|-\frac{\sigma d}{\varepsilon_0}\right| = \left|\frac{Q}{A\varepsilon_0} d\right|$

so $\boxed{\frac{Q}{C} = V = \frac{Q}{A\varepsilon_0} d \rightarrow C = \frac{A\varepsilon_0}{d} = \frac{Q}{V}}$

Energy stored in capacitor

Know that $W = \int_V \frac{\varepsilon_0}{2} \vec{E}^2 \, d\tau = \int_V \left(\frac{\sigma}{\varepsilon_0}\right)^2 \cdot \frac{\varepsilon_0}{2} \, d\tau$

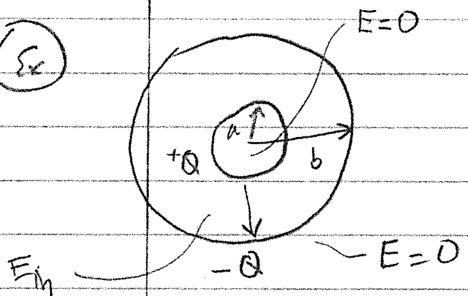
so $W = \frac{\sigma^2}{2\varepsilon_0}(Ad) = \frac{1}{2}\frac{\sigma^2 (Ad)}{\varepsilon_0}$

so $\boxed{W = \frac{1}{2}\frac{Q^2}{C}}$ or $\boxed{W = \frac{1}{2} C V^2}$

Sep 27, 2019

Find cap of 2 nested conducting shells...

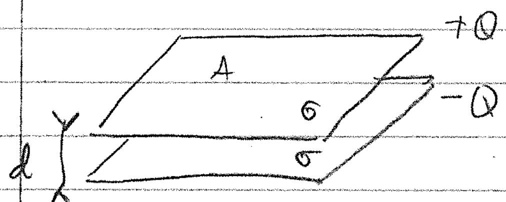
Ex



$$C = \frac{Q}{V} \; ; \; V = -\int \vec{E}\cdot d\vec{l} = -\int_a^b \frac{1}{4\pi\varepsilon_0} Q/r^2 \, \hat{r} \, dr$$

$$\boxed{C = \frac{Q}{V} = 4\pi\varepsilon_0 \left(\frac{ab}{a-b}\right)}$$

Ex



Let go, ⇒ attracts. What's the work done?

(a) work done? by field

$$W = \int \vec{F}\cdot d\vec{x} = \int A \frac{1}{2\varepsilon_0} \sigma^2 dx = \int A \frac{\varepsilon_0}{2} E^2 dx = \frac{\varepsilon_0}{2} E^2 \int A dx$$

(b) Decrease in Energy

$$\Delta W = \frac{(\text{Energy per unit volume}}{\cancel{\text{[illegible]}} \quad 1}) \times \Delta\,\text{volume}$$

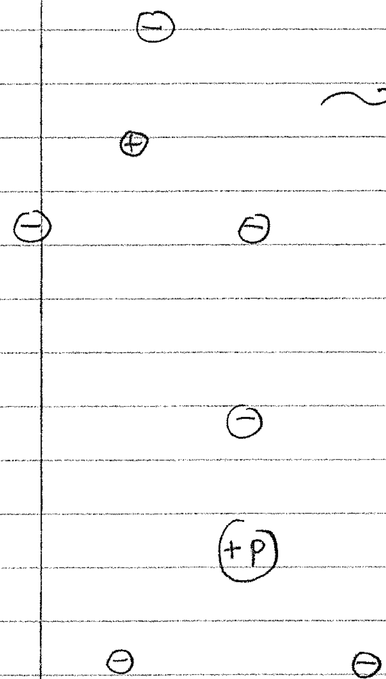
$$W = \frac{1}{2} C V^2 = \frac{1}{2}\left[\frac{A\varepsilon_0}{d}\right]\left[\frac{-\sigma d}{\varepsilon_0}\right]^2 = \frac{1}{2}\sigma^2/\varepsilon_0 \,(Ad) = \frac{1}{2}\frac{\sigma^2}{\varepsilon_0} V$$

$$E = \frac{\sigma}{\varepsilon_0} \rightarrow \boxed{W = \frac{\varepsilon_0}{2} E^2 (Ad)}$$

So $\frac{W}{V} = \frac{\varepsilon_0}{2} E^2$

$$\rightarrow \boxed{\Delta W = \left(\frac{\varepsilon_0}{2} E^2\right)(A dx)}$$

$\boxed{\text{Equilibrium in Electrostatics – Stability of Atoms}}$



$\rightsquigarrow$ restoring force needed for equilibrium to exists.

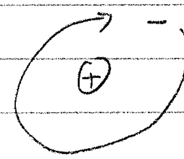
$\boxed{\text{Earnshaw's Thm}}$

there's no stable eq point in Electrostatics
(due to Gauss' law)

Li atom $\to$ counter example ... but there's Plum Pudding model

$\Rightarrow$ But there's Rutherford ...

But there's Bohr ...



But... this ~~is~~ no longer electrostatics ...

But e^- can't crash into p $\rightsquigarrow$ b/c of Heisenberg uncertainty principle...

HU: $\Delta x \Delta p \geq \frac{\hbar}{2}$

Sep 30, 2019

$\boxed{\text{LAPLACE'S EQUATIONS UNIQUENESS THEOREM}}$

(3) Potentials ~ Special Techniques

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r)}{\mathscr{r}^2}\,\hat{\mathscr{r}}\,d\tau$$

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}, \quad V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r)}{\mathscr{r}}\,d\tau, \quad E = -\vec{\nabla} V$$

$\Rightarrow$ $\boxed{\text{Poisson's Equation}}$

$$\boxed{\vec{\nabla}^2 V = -\rho/\varepsilon_0}$$

↳ Special case ⇝ Laplace's Eqn

$$\boxed{\vec{\nabla}^2 V = 0}$$

$$\partial_x^2 V + \partial_y^2 V + \partial_z^2 V = 0$$

Solutions are HARMONIC FUNCTIONS.

→ $\partial f/\partial x$ ~ rate of change... $\partial^2 f/\partial x^2 \rightarrow$ concavity...

. If $\partial^2 f/\partial x^2 < 0$ ⇝ local max ; $\partial^2 f/\partial x^2 > 0$ ⇝ Local min

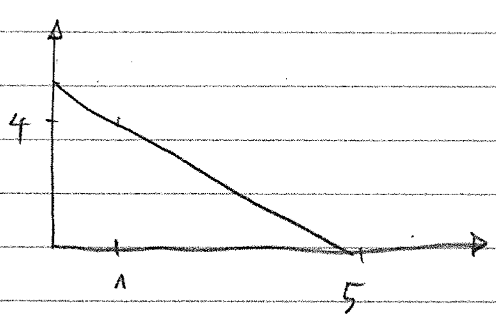
$\partial^2 f/\partial x^2 = 0$: inflection point... no max/min here

So $\boxed{\nabla^2 V = 0 \Rightarrow \text{never peaks anywhere w/ } \rho = 0}$

$\boxed{\text{Laplace in 1D}}$

General soln

$$\nabla^2 V = \partial^2 V/\partial x^2 = 0 \rightsquigarrow \boxed{V(x) = mx + b}$$



BC: $V = 4$ @ $x = 1$
$V = 0$ @ $x = 5$

$$\boxed{\begin{array}{c} b = 5 \\ m = -1 \end{array}}$$

Notes on 1D $\Rightarrow$ Extreme values of $f(x)$ occur only at boundaries.

$\Rightarrow$ $V(x)$ is the average of $V(x+a)$ & $V(x-a)$

$$V(x) = \frac{1}{2}\left[V(x+a) + V(x-a) \right]$$

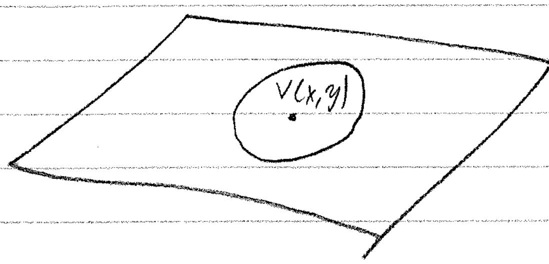
Laplace in 2D

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Notes $\Rightarrow$ $V(x,y)$ is harmonic with no local max/min

$\Rightarrow$ $V(x,y)$ is average of V on path around (x,y)

$$\boxed{V(x,y) = \frac{1}{2\pi r} \oint V \, dl}$$

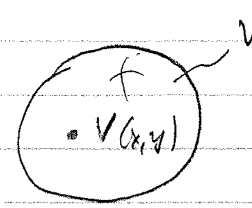


Laplace in 3D $\qquad \partial_x^2 V + \partial_y^2 V + \partial_z^2 V = 0$

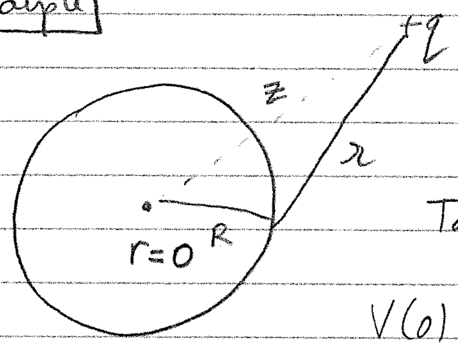
Notes $\Rightarrow$ No local max/min

$\Rightarrow$ $V(r)$ is average of potential on surface surrounding $\tilde{r}$

$$V(r) = \frac{1}{4\pi R^2} \oint V \, dA$$



Example



know that @ $r=0$, $V(r) = \frac{q}{4\pi\varepsilon_0}\frac{1}{z}$

Taking average...

$$V(0) = \frac{1}{4\pi R^2}\oint V(r)\,dA$$

$$= \frac{1}{4\pi R^2}\int \frac{q}{4\pi\varepsilon_0}\frac{1}{\mathscr{r}} R^2 \sin\theta \, d\theta \, d\varphi$$

$\mathscr{r}^2 = z^2 + R^2 - 2zR\cos\theta$

$$= \frac{\cancel{4\pi}}{4\pi}\int \frac{q}{4\pi\varepsilon_0}\frac{1}{r}$$

$$= \frac{1}{4\pi}\int \frac{q}{4\pi\varepsilon_0}\frac{\cancel{R^2}\sin\theta\,d\theta\,d\varphi}{(z^2+R^2-2zR\cos\theta)^{1/2}}$$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} \frac{q}{4\pi\varepsilon_0}\frac{\sin\theta\,d\theta}{(z^2+R^2-2zR\cos\theta)^{1/2}}$$

$$= \frac{q}{8\pi\varepsilon_0}\int_0^{\pi} \frac{\sin\theta\,d\theta}{(z^2+R^2-2zR\cos\theta)^{1/2}}$$

$$= \frac{q}{8\pi\varepsilon_0}\left(\frac{1}{zR}\right)(z^2+R^2-2zR\cos\theta)^{1/2}\Big|_0^{\pi}$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{2zR}\right)\left[(z+R)-(z-R)\right]$$

$$= \boxed{\frac{q}{4\pi\varepsilon_0}\frac{1}{z}}$$

UNIQUENESS THEOREM

V(boundary) = exact solution.

First Uniqueness Thm $\Rightarrow$ $\boxed{\text{uniqueness if } V(r) \text{ is specified on the boundary surface}}$

(if $\rho = 0$)

If $\rho \neq 0$, then uniqueness if $\begin{cases} V(r) \text{ known } \forall \text{ surface} \\ \rho(r) \text{ known everywhere.} \end{cases}$

2^{nd} Uniq. Thm $\Rightarrow$ $\boxed{\text{If volume surrounded by conductors } \& \ \rho \neq 0, \ V(r) \text{ uniq if } Q \text{ known}}$

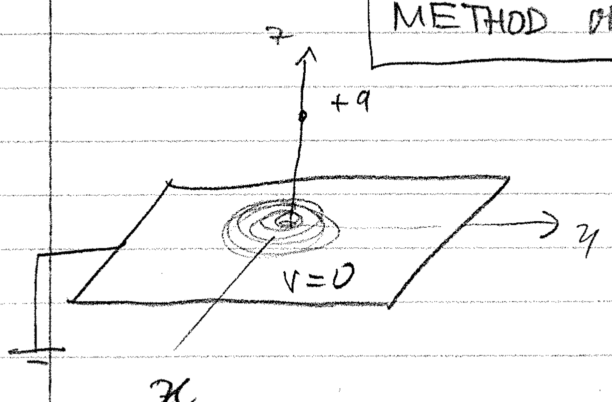
Oct 1, 2019

$\rightsquigarrow$ typical boundary (charge dist, ∞, grounded conductor)

$\rightsquigarrow$ If $V(r)$ is known on bounding surface, then you can uniquely determine $V(r)$ in the bounded volume if $\nabla^2 V = 0$.

$\rightsquigarrow$ If a function satisfies both Laplace's eqn & BC then it is the only solution to the BVP.

METHOD OF IMAGES

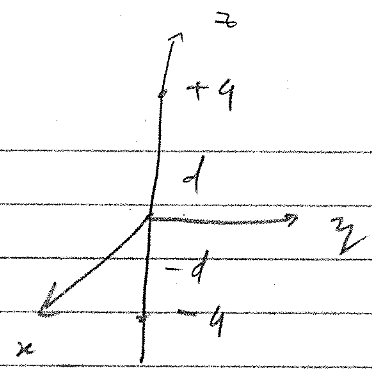


What is $V(x,y,z)$?

Solve the IBVP :

$$\begin{cases} V = 0 & \text{as } r \to \infty \\ V = 0 & \text{at } z = 0 \text{ (grounded)} \end{cases}$$

Similar situation as



$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{(x^2+y^2+(z-d)^2)^{1/2}} - \frac{q}{(x^2+y^2+(z+d)^2)^{1/2}}\right)$$

And so $V(z=0) = 0 \Rightarrow V(x,y,z)$ is a solution by uniqueness thm.

$\Rightarrow$ $\boxed{\text{MIRRORED IMAGE}}$ $\leadsto$ need opposite sign.

$\boxed{\text{PROPERTY OF INDUCED CHARGE DISTRIBUTION}}$

$\hookrightarrow$ surface charge induced on conductor.

$\Rightarrow$ E field above conductor $\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}$, and

$\vec{E} = -\vec{\nabla} V$

So $\boxed{\sigma = -\epsilon_0 \frac{\partial V}{\partial n}}$

Back to example where $V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{(x^2+y^2+(z-d)^2)^{1/2}} - \frac{q}{(x^2+y^2+(z+d)^2)^{1/2}}\right\}$

So $\sigma = -\epsilon_0 \frac{\partial V}{\partial z} = \frac{-1}{4\pi}\left\{\frac{-q(z-d)}{(\ldots)^{3/2}} + \frac{q(z-d)}{(+)^{3/2}}\right\}$

$\boxed{\sigma_{(x,y)} = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}}$ $\leadsto$ where $(z=0)$

Total charge (should be $-q$)

$$Q = \int \sigma \, dA = \iint \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}} \, dx\,dy$$

$$= \iint \frac{1}{2\pi} \frac{-qd}{(r^2+d^2)^{3/2}} \, r\,dr\,d\phi$$

$$= \int_0^\infty \frac{-qd}{(r^2+d^2)^{3/2}} \, r\,dr = \dots = q$$

Force?

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\left(\frac{-q^2}{(2d)^2}\right)(-\hat{z})$$

Energy

$$W = \int_\infty^d \vec{F}\, d\vec{l} = \int_\infty^d \frac{-1}{4\pi\epsilon_0} \frac{q^2}{(2z)^2} \hat{z}\, d\vec{l}$$

$$= \int_\infty^d \vec{F} \cdot d\vec{l} = -\frac{1}{4\pi\epsilon_0} \int_\infty^d \frac{q^2}{(2z)^2} \, dz$$

$$= -\frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{4z}\right)\Bigg|_\infty^d = \boxed{\frac{-1}{4\pi\epsilon_0} \, q^2/4d}$$ first config

What about

$+q$

$\Big\}(2d)$

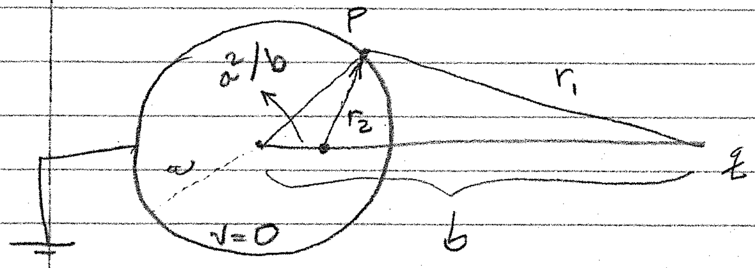
$-q$

$$W_1 = 0, \quad W_2 = \frac{-q^2}{4\pi\epsilon_0} \frac{1}{(2d)} = \frac{-q^2}{4\pi\epsilon_0} \frac{1}{2d} = \boxed{2W_{\text{before}}}$$

So $\boxed{W_{\text{mirror}} = 2W_{\text{original}}}$

→ where image breaks down...

Example Charge near a spherical conductor...

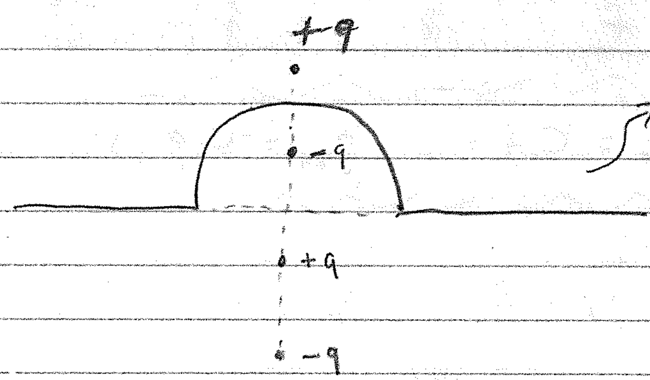


Know $V(P) = \frac{q}{r_1} + \frac{q'}{r_2} = 0$ since ○ is grounded.

$$\Rightarrow \frac{q'}{r_2} = \frac{q}{r_1} \Rightarrow \frac{r_2}{r_1} = \frac{-q'}{q} = \frac{a}{b}$$

Get $\frac{q'}{q} = \frac{-a}{b} \Rightarrow \boxed{q' = \frac{-a}{b} q}$ sitting $\left(\frac{a^2}{b}\right)$ from the origin.

Example

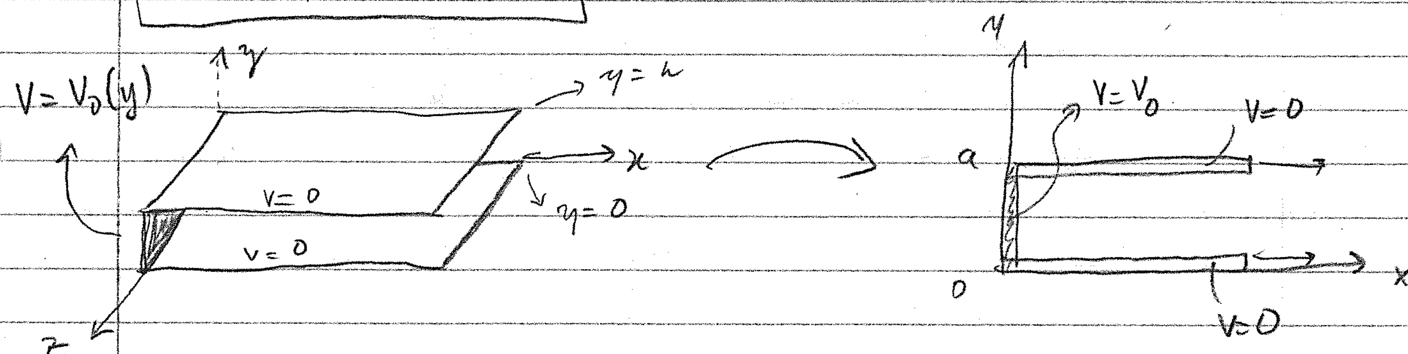


treat this as plane & arc

Sept 3, 2019

BOUNDARY VALUE PROBLEM

Ex: Cartesian Coords



* What is the potential between the plates?

→ Laplace's Eqn ... $\nabla^2 V = \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial x^2} = V_{xx} + V_{yy} = 0$

BC: $\begin{cases} V = 0 \ @ \ y = 0 \\ V = 0 \ @ \ y = a \\ V = V_0(y) \ @ \ x = 0 \\ V = 0 \ @ \ x \Rightarrow \infty \end{cases}$

Assume seperable solution for $V(x,y)$ ..

$$V(x,y) = f(x)\,g(y)$$

Into Laplace ... $g f_{xx} + f g_{yy} = 0$

$$\Leftrightarrow \boxed{\frac{f_{xx}}{f} = \frac{-g_{yy}}{g} = C_1}$$

So $f_{xx} = C_1 f \ , \quad g_{yy} = -C_1 g$

Let $c_1 = k^2 , \quad c_2 = -k^2$

Then $\begin{cases} f(x) = Ae^{ikx} + Be^{-ikx} & \text{(so that } x \to +\infty \to V \to 0) \\ g(y) = C\sin(ky) + D\cos(ky) \end{cases}$

Applying BC ...

@ $x \Rightarrow \infty \to V = 0$ so $A = 0$

@ $x = 0 \to V = V_0$ so $B \in C_0$ good

@ $y = 0 \to V = 0$ so $D = 0$; $y = a, V = 0$ → $\boxed{k = \frac{\pi n}{a}}$

→ $f(x)g(y) = V(x,y) = Be^{-ikx}\sin(ky)$

where $k = \pi n/a$

So $\boxed{V(x,y) = C e^{-\pi n x/a} \sin\left(\frac{n\pi y}{a}\right)}$ ⇝ eigenfunction!

→ general solution...

$$\boxed{V(x,y) = \sum_{n=0}^{\infty} C_n e^{-\pi n x/a} \sin\left(\frac{n\pi y}{a}\right)}$$

- Next, at $x=0$, $V(x,y) = V_0 \Rightarrow \sum_{n=0}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$

⇒ Fourier's trick ... ⇝ $\int \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx = \frac{1}{a}\delta_{mn}$

$$\int_0^a \underbrace{\sum_{n=0}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) \cdot \sin\left(\frac{m\pi y}{a}\right) dy}_{\left(\frac{a}{2}\delta_{mn}\right)} = \int_0^a V_0(y) \sin\left(\frac{m\pi y}{a}\right) dy$$

$$\Rightarrow \frac{a}{2} C_m = \int V_0(y) \sin\left(\frac{m\pi y}{a}\right) dy$$

$$\boxed{C_m = \frac{2}{a}\int_0^a V_0(y) \sin\left(\frac{m\pi y}{a}\right) dy}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = L\,\delta_{mn}$$

→ even when $n=m$

So $\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2}\delta_{mn}$

So $\int_0^a \sin\left(\frac{\pi n x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{a}{2}\delta_{mn}$

So $$\boxed{V(x,y) = \sum_{n=1}^{\infty} \left[\frac{2}{a}\int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy\right] e^{-\pi n x/a} \sin\left(\frac{n\pi y}{a}\right)}$$

Suppose $V_0(y) = V_0$, then

$$V(x,y) = \sum_{n=1}^{\infty} \underbrace{\left[\frac{2V_0}{a} \int_0^a \sin\left(\frac{n\pi y}{a}\right) dy \right]}_{\substack{0 \text{ if } n \text{ even} \\ 2/\pi n \text{ if } n \text{ odd}}} e^{-\pi n x/a} \sin\left(\frac{n\pi y}{a}\right)$$

$$\boxed{V(x,y) = \sum_{n=1,3,5\ldots}^{\infty} \left(\frac{4V_0}{n\pi}\right) e^{-\pi n x/a} \sin\left(\frac{n\pi y}{a}\right)}$$ → by uniqueness thm → this is the solution.

$$V(x,y) = \frac{2V_0}{\pi} \tan^{-1}\left[\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right]$$

Summary of Separation of vars

1. Laplace's Eqn $\nabla^2 V = 0$
2. State BC
3. Assume separation of variables $V(x,y) = f(x)g(y)$,
4. Applying BC

Eigenfn → $V(x,y) = Ce^{-kx}\sin(ky)$ where $k = \frac{\pi n}{a}$, $n = 1, 2, 3, \ldots$

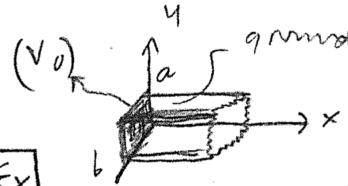
5. General soln
$$V(x,y) = \sum_{n=0}^{\infty} C_n f_n(x) g_n(y)$$

6. Last BC — 1 in this problem

$$V(0,y) = \sum_{n=1}^{\infty} C_n \underset{\sim}{f_n(0)} g_n(y)$$

$$\rightarrow C_m f_m(0) \simeq \int V(0,y)\, g_m(y)$$

In this problem...
$$C_m = \frac{2}{a}\int_0^a V_0 \sin\left(\frac{m\pi y}{a}\right) dy$$



Ex

t 4, 2019

Laplace's Eqn : $V_{xx} + V_{yy} + V_{zz} = 0$

BC $V(y=a) = V(y=0) = V(z=0) = V(z=b) = 0$, $V(x=0) = V_0$, $V(x \to \infty) = 0$

Sep of Var $V(x,y,z) = f(x) g(y) h(z)$

$$\hookrightarrow \frac{1}{f} f_{xx} = C_1 , \quad \frac{1}{g} g_{yy} = C_2 , \quad \frac{1}{h} h_{zz} = C_3 , \quad , \underline{C_1 > 0, \; C_2, C_3 < 0}$$

$(\ell^2 + k^2)$, $-k^2$, $-\ell^2$

$C_1 + C_2 + C_3 = 0$

So we get
$$\left\{ \begin{array}{l} f(x) = A e^{-\sqrt{(k^2+\ell^2)}x} + B e^{\sqrt{(k^2+\ell^2)}x} \\ g(y) = C \sin(kx) + D \cos(kx) \\ h(z) = E \sin(\ell x) + F \cos(\ell x) \end{array} \right\}$$

Apply BC $\rightarrow$ $\boxed{D = F = 0 , \; B = 0}$

$$V(x,y,z) = C e^{-\sqrt{k^2+\ell^2}\,x} \sin(ky) \sin(\ell z)$$

where $k = \frac{\pi n}{a}$, $\ell = \frac{m\pi}{b}$

$$\rightarrow \boxed{V(x,y,z) = C e^{-\sqrt{k^2+\ell^2}\,x} \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{b} z\right)}$$

$\sqrt{(n/a)^2 + (m/b)^2}$

General soln

$$\boxed{V(x,y,z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{m,n} \, e^{-\sqrt{n^2/a^2 + m^2/b^2}\,(\pi x)} \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{b} z\right)}$$

With these... use Fourier' Trick at $x = 0$ on

$$\boxed{V_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{m,n} \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{b} z\right)} \rightarrow \text{to get}$$

$$\int_0^b \int_0^a V_0 \sin\left(\frac{n'\pi}{a} y\right) \sin\left(\frac{m'\pi}{b} z\right) dy\, dz = \sum_{m=1} \sum_{n=1} \int_0^b \int_0^a C_{m,n} \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{n'\pi}{a} y\right)$$

$$\times \sin\left(\frac{m'\pi}{b} z\right) \sin\left(\frac{m\pi}{b} z\right) dy\, dz$$

$$= \sum_m \sum_n C_{m,n} \, \delta_{nn'} \delta_{mm'} \cdot \frac{a}{2} \frac{b}{2}$$

$$\rightarrow \boxed{C_{m,n} = \frac{4}{ab} \int_0^b \int_0^a V_0 \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{b} z\right) dy\, dz}$$

So $\quad C_{m,n} = \dfrac{4V_0}{ab}\displaystyle\int_0^a \sin\left(\frac{n\pi}{a}y\right)dy \int_0^b \sin\left(\frac{m\pi}{b}z\right)dz$ if V_0 constant

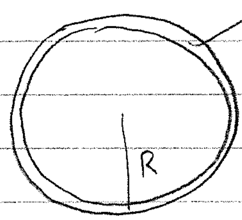
$$\Rightarrow \left\{ C_{m,n} = \frac{16V_0}{\pi^2 mn} \text{ if } n \,\&\, m \text{ are odd}, \ 0 \text{ else} \right.$$

So

$$\boxed{V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{\substack{m\\ \text{odd}}}\sum_{\substack{n\\ \text{odd}}}\left(\frac{1}{nm}\right) e^{-\pi x\sqrt{(n/a)^2+(m/b)^2}} \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}z\right)}$$

Oct 7, 2019

Spherical coords



$V(R,\theta) = k\sin^2(\theta/2) = V_0(\theta)$

$$\boxed{BC} \left\{ \begin{array}{l} V = 0 \text{ as } r \to \infty \\ V = \text{constant } @ \ r = 0 \end{array} \right\}$$

* Laplace's Eqn

$$\nabla^2 V = \frac{1}{r^2}\left(\partial_r\{r^2\partial_r V\}\right) + \frac{1}{r^2\sin\theta}\partial_\theta\{\sin\theta\,\partial_\theta V\} + \underbrace{\frac{1}{r^2\sin^2\theta}\partial_\phi^2 V}_{0} = 0$$

So Azimuthally symmetric! $\rightarrow \partial_\varphi^2 V = 0$

$\rightarrow$ Assume separable solution...

$$V(r,\theta) = f(r)\,g(\theta)$$

$$\Rightarrow \quad \frac{g}{r^2}\partial_r(r^2\partial_r f) + \frac{f}{r^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta g) = 0$$

$$\Rightarrow \quad \frac{1}{f}\partial_r(r^2\partial_r f) + \frac{1}{g\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta g) = 0 \quad \left(\text{divide by } \frac{V}{r^2}\right)$$

<u>So we must have</u>

$$\ell \in \mathbb{N}$$

$$\frac{1}{f}\partial_r\left(r^2\partial_r f\right) = \frac{-1}{g\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta g\right) = C = \ell(\ell+1)$$

<u>So we get</u>

$$\boxed{\partial_r\left(r^2\partial_r f\right) = \ell(\ell+1)f} \longrightarrow \text{Radial}$$

$\rightarrow$ Solution is

$$f(r) = A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}$$

$$\boxed{\partial_\theta\left(\sin\theta\,\partial_\theta g\right) = -g\sin\theta\,(\ell+1)\ell}$$

Solution $g(\theta) = P_\ell(\cos\theta) \longrightarrow$ Legendre polynomials ...

$$\boxed{P_\ell(x) = \frac{1}{2^\ell\,\ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell}$$

where $(x = \cos\theta)$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$\vdots$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{18}(35x^4 - 30x + 3)$

$\vdots$

<u>So</u> general seperable solution ...

$$V(r,\theta) = \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

$$\boxed{V(r,\theta) = \sum_{\ell=0}^{\infty}\left\{A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right\} P_\ell(\cos\theta)}$$

Note Legendre's polynomial are orthogonal + complete!

$\boxed{BC}$ $\rightarrow$ $\boxed{\text{Inside sphere}}$ $(r \leq R)$

Then $V =$ constant @ $r=0 \Rightarrow B_\ell = 0 \ \forall \ell$

So $V(r,\theta) = \sum_{\ell=0}^{\infty} (A_\ell r^\ell) P_\ell(\cos\theta)$

Fourier's Trick... $V = V(R,\theta) = k \sin^2 \theta/2$ @ $r = R$

$$\int_{-1}^{1} V(R,\theta) P_{\ell'}(\cos\theta)\, d\cos\theta = \int_{-1}^{1} \sum_{\ell=0}^{\infty} (A_\ell R^\ell) P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\, d\cos\theta$$

Note
$$\int_{-1}^{1} P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\, d\cos\theta = \int_0^{\pi} P_\ell(\cos x) P_{\ell'}(\cos x) \sin x\, dx$$
$$= \delta_{\ell\ell'} \frac{2}{2\ell+1}$$

So

$$\int_0^{\pi} V(R,\theta) P_{\ell'}(\cos\theta) \sin\theta\, d\theta = \sum_{\ell=0}^{\infty} A_\ell R^\ell \delta_{\ell\ell'} \frac{2}{2\ell+1} =$$

$$\Rightarrow \int_0^{\pi} V(R,\theta) P_\ell(\cos\theta) \sin\theta\, d\theta = A_\ell R^\ell \frac{2}{2\ell+1}$$

$$\Rightarrow \int_0^{\pi} \left(k \sin^2 \frac{\theta}{2}\right) P_\ell(\cos\theta) \sin\theta\, d\theta = A_\ell R^\ell \frac{2}{2\ell+1}$$

$$\Rightarrow \boxed{A_\ell = \frac{2\ell+1}{2R^\ell} \int_0^{\pi} \left(k \sin^2 \frac{\theta}{2}\right) P_\ell(\cos\theta) \sin\theta\, d\theta}$$

Now $V_0(\theta) = k\sin^2\theta/2 = \frac{k}{2}(1-\cos\theta) = \frac{k}{2}\left(P_0(\cos\theta) - P_1(\cos\theta)\right)$

So

$$A_\ell = \frac{2\ell+1}{2R^2}\int_0^\pi \left(k\sin^2\frac{\theta}{2}\right) P_\ell(\cos\theta)\sin\theta\, d\theta$$

$$= \frac{2\ell+1}{2R^\ell}\int_{-1}^{1} \frac{k}{2}\left\{P_0(\cos\theta) - P_1(\cos\theta)\right\} P_\ell(\cos\theta)\, d(\cos\theta)$$

$$A_\ell = \frac{2\ell+1}{2R^\ell}\,\frac{k}{2}\left\{\langle P_0, P_\ell\rangle - \langle P_1, P_\ell\rangle\right\}$$

$$A_\ell = \frac{2\ell+1}{2R^\ell}\,\frac{2}{2\ell+1}\cdot\frac{k}{2}\left\{\delta_{0,\ell} - \delta_{1,\ell}\right\}$$

$$\boxed{A_\ell = \frac{k}{2R^\ell}\left\{\delta_{0,\ell} - \delta_{1,\ell}\right\}}$$

$\Rightarrow$ If $\ell = 0$, $A_0 = \frac{k}{2}$

If $\ell = 1$, $A_1 = \frac{-k}{2R}$

$A_i = 0$ else

So $V(r,\theta) = \left(\frac{k}{2}\right)P_0(\cos\theta) - \frac{k}{2R}P_1(\cos\theta)\cdot r$

$$\boxed{V(r,\theta) = \frac{k}{2} - \frac{k}{2R}r\cos\theta = \frac{k}{2}\left\{1 - \frac{r}{R}\cos\theta\right\}}$$

$\boxed{\text{Outside}}$ $(r \geq R)$

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$ $\rightsquigarrow$ Since want $V \to 0$ as $r \to \infty$

$$V(R,\theta) = k\sin^2\theta/2 = \frac{k}{2}\left(P_0(\cos\theta) - P_1(\cos\theta)\right)$$

So ...

$$\int_{-1}^{1} V(R,\theta)P_{\ell'}(\cos\theta)\,d(\cos\theta) = \sum_{\ell=0}^{\infty}\frac{B_\ell}{R^{\ell+1}}\,\delta_{\ell\ell'}\,\frac{2}{2\ell+1}$$

$$\Rightarrow \int_{-1}^{1}\frac{k}{2}\left\{P_0 - P_1\right\}P_\ell\, d(\cos\theta) = \frac{B_\ell}{R^{\ell+1}}\,\frac{2}{2\ell+1}$$

$$\rightarrow \quad B_\ell = \frac{2\ell+1}{2} R^{\ell+1} \int_{-1}^{1} \left(\frac{k}{2}\right) (P_0 - P_1) P_\ell \, d(\cos\theta)$$

$$= \left(\frac{2\ell+1}{2}\right)\left(\frac{2}{2\ell+1}\right) R^{\ell+1} \left(\frac{k}{2}\right) \left\{ \delta_{0,\ell} - \delta_{1,\ell} \right\}$$

$$= R^{\ell+1} \left(\frac{k}{2}\right) \left\{ \delta_{0,\ell} - \delta_{1,\ell} \right\}$$

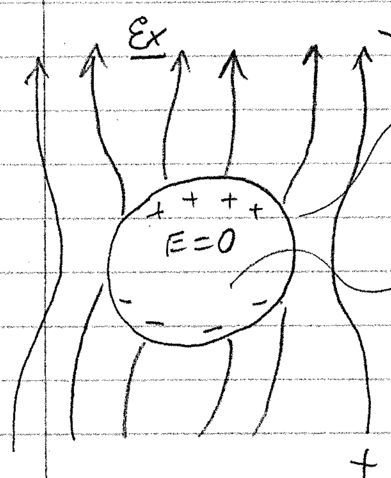
So $\quad B_0 = \frac{Rk}{2}$

$\quad B_1 = -\frac{k}{2} R^2$

So
$$\boxed{V(r,\theta) = \frac{Rk}{2r} - \frac{R^2 k}{2r^2}\cos\theta = \frac{Rk}{2r}\left\{1 - \frac{R}{r}\cos\theta\right\}}$$

oct 8, 2019 $\quad$ More examples

Ex



$\sigma = k\cos\theta$

$\vec{E}_{in} = -E_0 \hat{z}$

$\vec{E} = E_0 \hat{z}$

General solution

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left\{ A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right\} P_\ell(\cos\theta)$$

BC
1. $V = 0$ as $r \to \infty$
2. $V \neq 0$ @ $r = 0$
3. $V_{in} = V_{out}$ @ R

Apply BC

Inside : $V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$

Outside : $V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$

- Continuity @ R : $V(R,\theta)_{in} = V(R,\theta)_{out}$

$$\hookrightarrow \quad A_\ell R^\ell P_\ell(\cos\theta) = \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) \rightarrow \boxed{B_\ell = A_\ell R^{2\ell+1}}$$

- On surface

$$\vec{E}^{\perp}_{out} - \vec{E}^{\perp}_{in}\Big|_R = \frac{\sigma}{\varepsilon_0}\hat{n} \Rightarrow \boxed{\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\Big|_R = \frac{-\sigma}{\varepsilon_0}}$$

So

$$\sum_{\ell=0}^{\infty} -B_\ell(\ell+1)\frac{1}{R^{\ell+2}} P_\ell(\cos\theta) - A_\ell \ell R^{\ell-1} P_\ell(\cos\theta) = \frac{-\sigma}{\varepsilon_0} = \frac{-k\cos\theta}{\varepsilon_0}$$

$$\sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left\{-A_\ell(\ell+1)R^{\ell-1} - A_\ell R^{\ell-1}\ell\right\} = \frac{-k}{\varepsilon_0} P_1(\cos\theta)$$

So

$$\boxed{A_\ell = \frac{+k}{\varepsilon_0}(\delta_{1,\ell})\frac{1}{R^{\ell-1}(2\ell+1)}}$$

So

$$\boxed{B_\ell = \frac{k}{\varepsilon_0}(\delta_{1,\ell})\frac{R^{\ell+2}}{(2\ell+1)}}$$

So

$$\boxed{V_{in}(r,\theta) = \left(\frac{k}{\varepsilon_0}\frac{1}{2\cdot1+1}\right)\left\{r^1\right\}\cos\theta}$$

$$\boxed{V_{out}(r,\theta) = \left(\frac{k}{\varepsilon_0}\frac{1}{2\cdot1+1}\right)\left\{\frac{R^3}{r^2}\right\}\cos\theta}$$

What is $\vec{E}_{in}$?

$$\vec{E}_{in} = -\vec{\nabla} V_{in} = \frac{-\partial}{\partial z}\Big[\frac{k}{3\varepsilon_0}\underbrace{r\cos\theta}_{z}\Big] = \frac{k}{3\varepsilon_0}\cos\theta\,\hat{z} = \frac{-\sigma}{3\varepsilon_0}\hat{z}$$

So $\boxed{\vec{E}_{in} = \frac{-\sigma}{3\varepsilon_0}\hat{z}}$ → cancels the external field

Similarly, $\boxed{\vec{E}_{out} = \frac{+\sigma}{3\varepsilon_0}\hat{z}}$

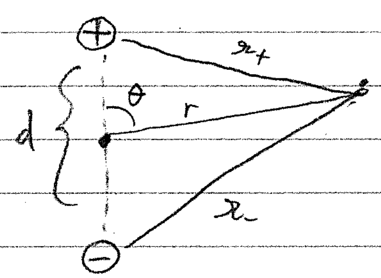
Oct 10, 2019

MULTIPOLE EXPANSION

Potential of Electric Dipole

$Q \bullet \leftarrow V(r) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{r}$

What about



$$V(p) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

Law of cosines ...

$$\left\{\begin{array}{l} r_+^2 = r^2 + (d/2)^2 - rd\cos\theta \\ r_-^2 = r^2 + (d/2)^2 + rd\cos\theta \end{array}\right\}$$

So $r_\pm^2 = r^2 + (d/2)^2 \mp rd\cos\theta$

$$\boxed{r_\pm^2 = r^2\left\{1 + \frac{d^2}{4r^2} \mp \frac{d}{r}\cos\theta\right\}}$$

Almost always $r \gg d$ ... $\to 1/r^2 \ll 1 \Rightarrow \boxed{r_\pm^2 \simeq r^2\left(1 \mp (d/r)\cos\theta\right)}$

So $\boxed{\dfrac{1}{r_\pm} \cong \dfrac{1}{r}\left(1 \mp \dfrac{d}{r}\cos\theta\right)^{-1/2}}$

↳ Expand to estimate more

$$\sqrt{1 \mp (d/r)\cos\theta} \simeq (a+b)^n = a^n + n a^{n-1} b + \frac{n(n-1)}{2} a^{n-2} b^2 + \dots$$

↳ $$\left[1 \mp \frac{d}{r}\cos\theta\right]^{-1/2} \simeq 1 + \left(\frac{-1}{2}\right)\left(\mp\frac{d}{r}\cos\theta\right) + \dots$$

can also get this from Taylor expand...

So $\boxed{\dfrac{1}{r_\pm} \simeq \dfrac{1}{r}\left\{1 \pm \dfrac{1}{2}\dfrac{d}{r}\cos\theta\right\}}$

So

$$\frac{1}{r_+} - \frac{1}{r_-} = \left\{\frac{1}{r} + \frac{d}{2r^2}\cos\theta - \frac{1}{r} + \frac{d}{2r^2}\cos\theta\right\} = \frac{d}{r^2}\cos\theta$$

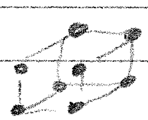
So $\boxed{\dfrac{1}{r_+} - \dfrac{1}{r_-} \simeq \dfrac{d}{r^2}\cos\theta}$

This means... Potential of electric dipole...

$$\boxed{V(r) \simeq \frac{1}{4\pi\epsilon_0}\,\frac{q\, d\cos\theta}{r^2}}$$

Note that $\left\{\begin{array}{l} \text{Monopole} \rightsquigarrow V(r) \sim 1/r \\ \text{Dipole} \longrightarrow V(r) \sim 1/r^2 \end{array}\right\}$

Turns out $\left.\begin{array}{cc} +q & -q \\ -q & +q \end{array}\right\}$ Quadruple $\sim V(r) \cong 1/r^3$

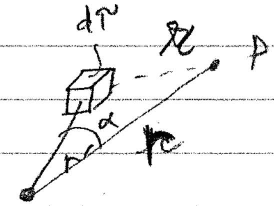
or  $\longrightarrow$ Octapole $\rightarrow V(r) \simeq 1/r^4$

General Multiple Expansion

⊛ For any charges distribution

$$\boxed{V(r) = V_{mon}(r) + V_{dip}(r) + V_{quad}(r) + V_{oct}(r) + \dots}$$

⊛ Let's expand the general potential formula...



Recall that $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{r}')/_{\mathscr{r}}\, d\tau$

Law of cosines? $\mathscr{r}^2 = r^2 + (r')^2 - 2rr'\cos\alpha$

$$= r^2 \left[1 + \left(\frac{r'}{r}\right)^2 - \frac{2r'}{r}\cos\alpha \right]$$

Substitute $\varepsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$

Then

$$\mathscr{r}^2 = r^2 \left[1 + \varepsilon \right] \Rightarrow \mathscr{r} = \sqrt{r^2 + r^2\varepsilon} = \sqrt{r^2(1+\varepsilon)} = r\sqrt{1+\varepsilon}$$

So

$$\mathscr{r} \simeq r(1+\varepsilon)^{1/2} \simeq r\left[1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \dots\right]$$

&

$$\frac{1}{\mathscr{r}} \simeq \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 + \dots\right]$$

$$\simeq \frac{1}{2}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)^2 + \left(\frac{r'}{r}\right)\cos\alpha + \frac{3}{8}\left(\frac{r'}{r}\right)^4 - \frac{3}{2}\left(\frac{r'}{r}\right)^3\cos\alpha + \frac{3}{2}\left(\frac{r'}{r}\right)^2\cos^2\alpha + \dots\right]$$

Group by factor of (r'/r)

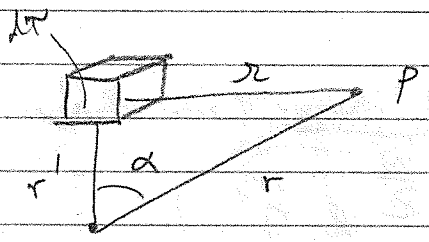
$$\frac{1}{\mathscr{r}} \simeq \frac{1}{r}\left[1 - \frac{r'}{r}\cos\alpha + \left(\frac{r'}{r}\right)^2\left(\frac{3}{2}\cos - 1\right) + \left(\frac{r'}{r}\right)^3 + \dots\right]$$

In terms of Legendre polynomials ...

$$\boxed{\frac{1}{\mathscr{r}} \simeq \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)}$$

⊛ Multipole Expansion $V(\vec{r})$ ...

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\,\rho(r')\,d\tau}$$



$\Longrightarrow$ This is an expansion of $V(r)$ in terms of powers of $1/r$.

$n=0 \longrightarrow$ monopole, $V \propto 1/r$

$n=1 \longrightarrow$ dipole, $V \propto 1/r^2$

$n=2 \longrightarrow$ quadruple, $V \propto 1/r^3$

$\vdots$

Oct 11, 2019

Monopole term: $n=0$

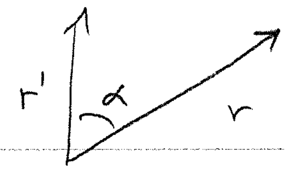
$$V_{mon}(r) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^{0+1}}\int (r')^0 \overbrace{P_0(\cos\alpha)}^{1}\,\rho(r')\,d\tau$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int \rho(r')\,d\tau$$

$$\boxed{V_{mon}(r) = \frac{Q}{4\pi\varepsilon_0 r}}$$

Dipole term: $n = 1$ ... $V_{dip}(r) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{1}{r^2}\displaystyle\int (r')\cos\alpha\,\rho(r')\,d\tau$

$= \ ?$



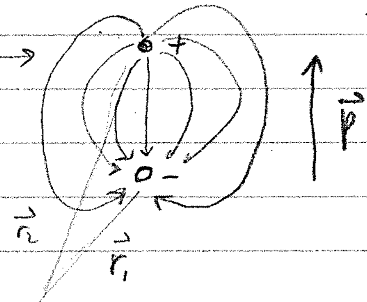
Note $r'\cos\alpha = \hat{r}\cdot\vec{r}\,'$

So $V_{dip} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{1}{r^2}\,\hat{r}\cdot\underbrace{\int \vec{r}\,'\rho(\vec{r}\,')\,d\tau'}_{\text{dipole moment}}$

Dipole moment: $\boxed{\vec{p} \equiv \int \vec{r}\,'\rho(\vec{r}\,')\,d\tau'}$

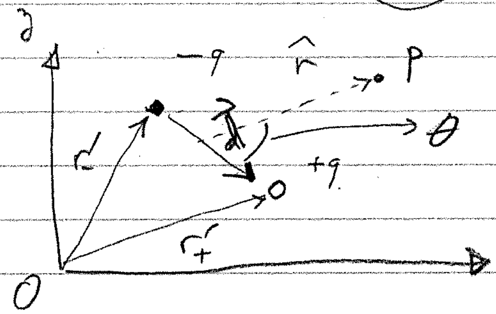
So, for a dipole...

$$\boxed{V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}}$$



- **Dipole moment for collection of charges** (point charges)

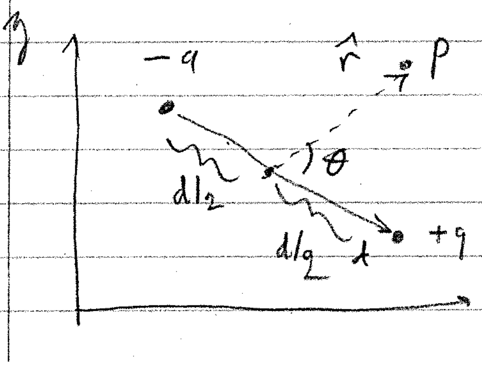
$$\vec{p} = \int \vec{r}\,'\rho(\vec{r}\,')\,d\tau' = \sum_{n=1}^{N} q_i\,\vec{r}_i\,'$$



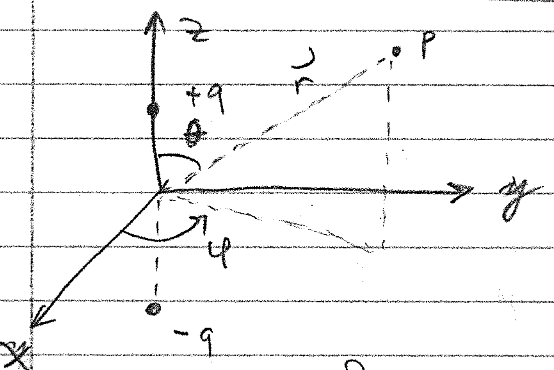
$$\vec{p} = q\vec{r}_+ + (-q)\vec{r}_-$$

$$= q(\vec{r}_+ - \vec{r}_-) = q\vec{d}$$

So $V_{dip}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q\vec{d}\cdot\hat{r}}{r^2} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q\,d\cos\theta}{r^2}$



Electric Field of Pure Dipole



$$V_{dip} = \frac{1}{4\pi\varepsilon_0} \frac{\vec{r}\cdot\vec{p}}{r^2}$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{p\cos\theta}{r^2}$$

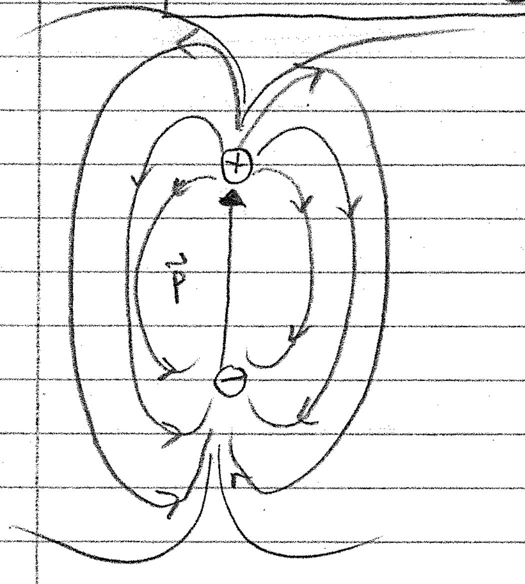
So; $E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$$

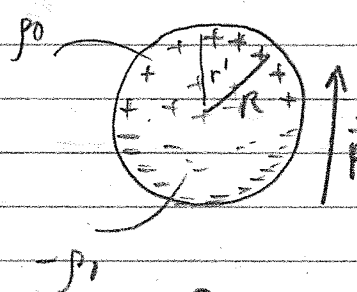
$$E_\varphi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi} = 0$$

So

$$\boxed{\vec{E}_{dip} = \frac{p}{4\pi\varepsilon_0} \frac{1}{r^3} \left\{ 2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta} \right\}}$$



Ex Find $\vec{E}$ field (approx) for polarized sphere...



<u>Note</u> $Q = 0 \Rightarrow$ no monopole

$\vec{p} \longrightarrow$ use <u>dipole</u>.

Dipole moment: $\vec{p} = \int r' \rho(r') d\tau'$

By symmetry ... $r' = z$

So $\vec{p} = \int z \rho(r) d\tau = \int r\cos\theta\, \rho(r) d\tau$

Now, $p = \int r\cos\theta \, \rho(r)\, d\tau = \int r\cos\theta \, \rho(r) \, r^2 \sin\theta \, d\theta \, d\varphi \, dr$

$$= 2\rho_0 \int_0^R r^3 dr \int_0^{2\pi} d\varphi \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta$$

$$= 2\rho_0 \frac{R^4}{4} (2\pi) \cdot \left(\frac{1}{2}\right)$$

$$\boxed{p = \frac{\pi R^4 \rho_0}{2}} \Rightarrow \vec{p} = \frac{\pi R^4 \rho_0}{2} \hat{z}.$$

So E field ... $\vec{E}_{dip}(r,\theta) = \dfrac{p}{4\pi\epsilon_0 r^3} \left\{ 2\cos\theta \, \hat{r} + \sin\theta \, \hat{\theta} \right\}$

↳ same geometry $= \dfrac{\rho_0 \pi R^4}{8\pi\epsilon_0 r^3} \left\{ 2\cos\theta \, \hat{r} + \sin\theta \, \hat{\theta} \right\}$

$\Rightarrow$ only amplitude changes...

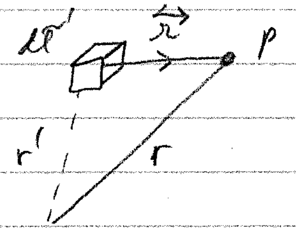
Oct 14, 2019

ELECTROSTATICS REVIEW

$$\vec{F} = Q\vec{E} \quad , \quad \vec{E} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{\mathscr{r}^2} \hat{\mathscr{r}}$$

$$\vec{E}_{tot} = \sum \vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r)}{\mathscr{r}^2} d\tau' \cdot \hat{\mathscr{r}}$$

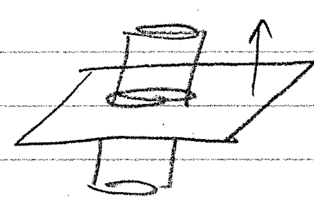
constant



Gauss' $\oint \vec{E} \cdot d\vec{A} = \dfrac{q_{enc}}{\epsilon_0}$, $\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$

$\vec{E} = \dfrac{\sigma}{2\epsilon_0} \hat{n}$

Infinite sheet of charge

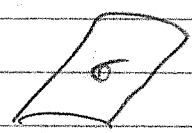


$$V(\vec{r}) \equiv -\int_{ref}^{r} \vec{E} \cdot d\vec{l} \quad , \quad \vec{E} = -\vec{\nabla} V$$

Point charge $\quad V(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$

$$\hookrightarrow \quad V(r) = \sum \frac{1}{4\pi\epsilon_0} \sum \frac{q_i}{r_i} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r')}{r} \, d\tau'$$

$\boxed{BC}$ $\quad \vec{E}^{\perp}_{above} - \vec{E}^{\perp}_{below} = \frac{\sigma}{\epsilon_0} \hat{n} \quad \longrightarrow$ discontinuous...



But $\quad E^{\parallel}_{above} - E^{\parallel}_{below} = 0$

$\boxed{BC}$ $\quad V_{above} = V_{below} \quad \rightsquigarrow$ continuous @ boundary...

$$-\frac{\partial V}{\partial r}_{above} + \frac{\partial V}{\partial r}_{below} = \frac{\sigma}{\epsilon_0} \hat{n}$$

$\boxed{Work}\boxed{PE}$

$$W = QV(r)$$

Energy $\quad W = \frac{1}{2} \sum_{i=1}^{N} q_i V_i(r_i)$

$$W = \frac{1}{2} \int_{\text{over } \rho(r)} \rho V \, d\tau \quad \rightsquigarrow \text{ know } V(\,)$$

$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} |\vec{E}|^2 \, d\tau \quad \rightsquigarrow \text{ know } \vec{E}(\,)$$

$\boxed{\text{CONDUCTORS}}$ (a) $\vec{E} = 0$ inside ...
(b) $\rho = 0$ inside
(c) Net charge sits on surface ... (σ)
(d) $\vec{E} \perp$ surface ...
(e) $V =$ constant throughout conductor ...

Force on conductor

$$P = \frac{F}{A} = \sigma E_{avg} = \sigma \frac{1}{2}\left[E_{above} + \cancel{E_{below}}^{\,0}\right]$$

$$E_{out} - \cancel{E_{in}}^{\,0} = \sigma/\varepsilon_0 \Rightarrow E_{out} = \frac{\sigma}{\varepsilon_0} \dots \rightarrow \boxed{P = \frac{F}{A} = \frac{\sigma^2}{2\varepsilon_0} = \frac{\varepsilon_0 E}{2}}$$

Capacitors

$C \equiv Q/V$ → only dependent on geometric properties ...

Energy stored ... $\boxed{W = \frac{1}{2} C V^2}$

Poisson/Laplace

$$\nabla^2 V = -\rho/\varepsilon_0 \qquad \text{(Poisson)}$$

$$\nabla^2 V = 0 \qquad \text{(Laplace)}$$

Solutions are harmonic functions !!!

Uniqueness theorem

Calculating potentials → Method of Images. (work $= \frac{1}{2} W_{tot}$)
→ BVP.

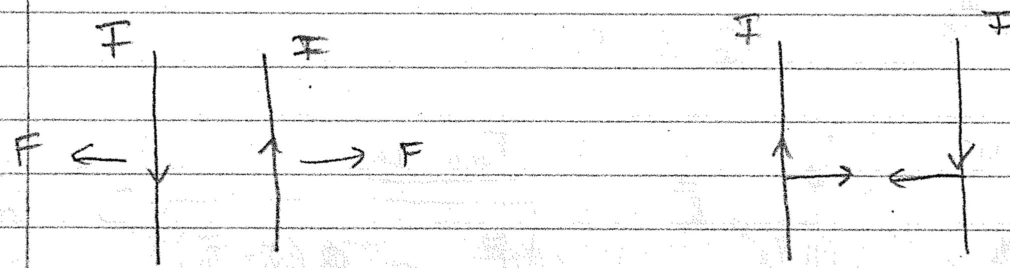
Fourier's trick

$$\int_0^{\pi} \sin(nx)\sin(n'x)\,dx = \left(\delta_{nn'} \cdot \frac{1}{2}\right)$$

$$\int_0^\pi P_\ell(\cos\theta)\,P_{\ell'}(\cos\theta)\,\sin\theta\,d\theta = \int_{-1}^{1} P_\ell(x)\,P_{\ell'}(x)\,dx = \delta_{\ell\ell'}\frac{2}{2\ell+1}$$

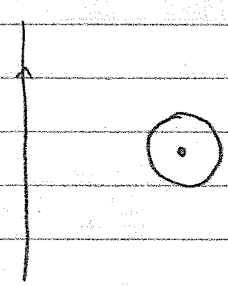
MAGNETOSTATICS

⊛ Nature of $\vec{B}$ fields



opposite currents → repel...

same current → attract



perp currents → no force.

#

$\boxed{\text{Nomenclature}}$ $\vec{B}$: magnetic Field (aka magnetic flux density)

$[B]$: Teslas $(T) = N/A{\cdot}m$

cgs units → 1 Gauss $= 10^{-4}$ T

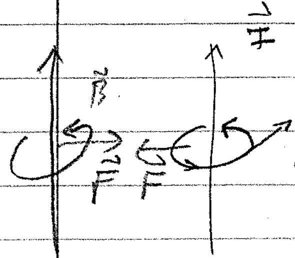
$\vec{I}$: current

$[I]$: Amps = Coulomb / sec

Stationary charges $\Rightarrow$ constant E field (electrostatics

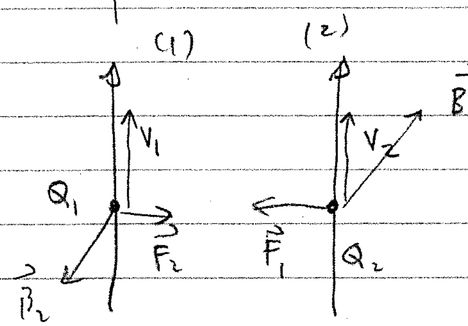
Steady currents $\Rightarrow$ constant B field (magnetostatics)

MAGNETIC FORCE



$\rightsquigarrow$ $\vec{B}$ field follows right hand rule ...

$\vec{F}_B \perp \vec{B}$



Force law

$$\boxed{\vec{F}_{mag} = Q(\vec{v} \times \vec{B})}$$ $\rightarrow$ Lorentz force law

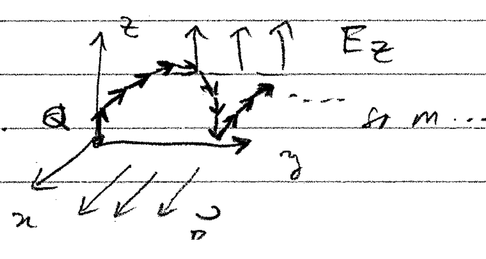
If both $\vec{E}$, $\vec{B}$ exist,

$$\vec{F} = Q\vec{E} + Q(\vec{v} \times \vec{B}) = Q(\vec{E} + \vec{v} \times \vec{B})$$

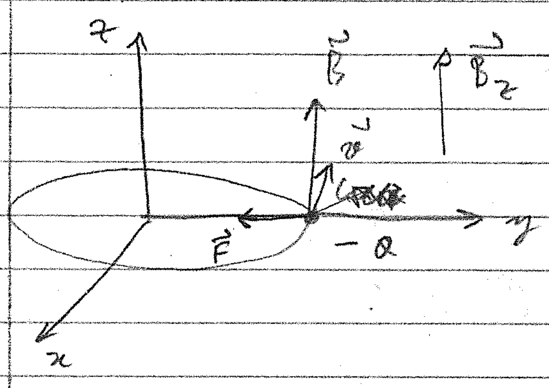
key properties of $\vec{B}$:
1. Current ($\vec{I}$) generates $\vec{B}$
2. Mag force is $\perp$ to $\vec{B}$ and $\vec{I}$
3. $\vec{B}$ fields obey superposition...
4. $\vec{B}$ fields do ~~not~~ no work on charged particles

$$W = \int \vec{F} \cdot d\vec{l} = \int Q \underbrace{(\vec{v} \times \vec{B})}_{\perp v} \cdot (\vec{v}\, dt) = 0$$

$\boxed{\text{Example}}$ Cycloidal motion ... Q



$\boxed{Ex}$ Mass spectrometer ...



$$\vec{F}_{mag} = -Q(\vec{v}\times\vec{B}) = -QvB\,\hat{y}$$

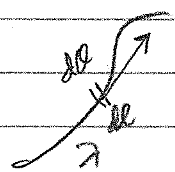
$$|F_{mag}| = QvB = \frac{mv^2}{R}$$

So $\boxed{\frac{Q}{m} = \frac{v}{BR}}$ ⇝ get charge / unit mass

ct 18
2019

CURRENT

$$\boxed{I = \frac{dq}{dt}} \qquad [I] = \text{Amps} = \frac{C}{s}.$$



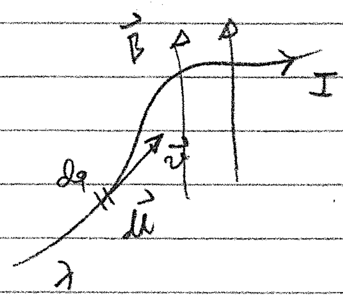
$$\lambda = \text{charge / length} = \frac{dq}{dl} \Rightarrow \Delta Q = \lambda(v\Delta t)$$

So $$\vec{I} = \frac{(\lambda\Delta t)\vec{v}}{\Delta t}$$

So $\boxed{\vec{I} = \lambda\vec{v}}$ (~~constant~~ λ)

$$\frac{dq}{dt} = \frac{dq}{dl}\frac{dl}{dt}$$

Force on a line charge ...



$$d\vec{F}_{mag} = (\vec{v}\times\vec{B})\,dq$$

$$\boxed{\vec{F}_{mag} = \int(\vec{v}\times\vec{B})\,\lambda\,dl}$$

Equivalently ... $\vec{F}_{mag} = \int_{wire} (\underbrace{\lambda \vec{v}}_{\vec{I}} \times \vec{B})\, dl$

So $$\boxed{\vec{F}_{mag} = \int_{wire} (\vec{I} \times \vec{B})\, dl}$$

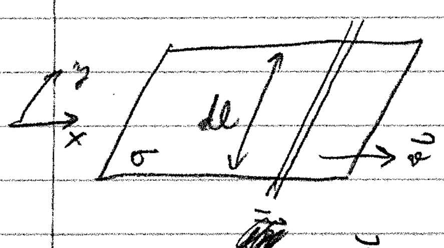
In most cases $\hat{I} = \hat{dl} \Rightarrow \vec{I}\, dl = I\, \vec{dl}$

So ... $$\boxed{\vec{F}_{mag} = \int I\, (\vec{dl} \times \vec{B})}$$

$$\boxed{\vec{F}_{mag} = I \int_{wire} (\vec{dl} \times \vec{B})}$$

CURRENT DENSITIES

(a) Surface current density ... ($\vec{K}$)



$\sigma = \frac{dq}{da}$ $\qquad$ $I = \frac{dq}{dt}$

$$\vec{K} = \frac{\text{current}}{\text{length}} = \frac{d\vec{I}}{dl} = \frac{d\vec{q}}{dt}\, \underbrace{\frac{1}{dy}\, \frac{dx}{dx}} = \frac{dq}{dA}\, \frac{dx}{dt}$$

So $$\boxed{\vec{K} = \sigma \vec{v}}$$

(b) Force on current sheet

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B})\, dq$$

$$= \int (\vec{v} \times \vec{B})\, \sigma\, dA$$

$$= \int (\sigma \vec{v} \times \vec{B})\, dA$$

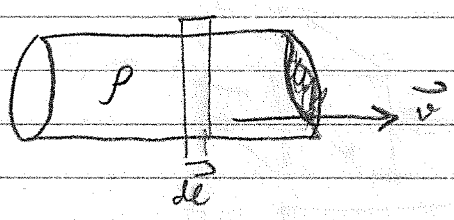
$$\boxed{\vec{F}_{mag} = \int (\vec{K} \times \vec{B})\, dA}$$

$$\cancel{\vec{I} = \int \vec{J}\, dV}$$

$$\cancel{\vec{I} = \int \vec{K}\, dA}$$

$$\cancel{\vec{I} = \int \vec{i}\, dl}$$

(c) Volume current density $(\vec{J})$



$$\vec{J} = \frac{d\vec{q}}{dA} \qquad \vec{I} = \frac{d\vec{q}}{dt}$$

So $\vec{J} = \dfrac{dq}{dt}\dfrac{1}{dA} = \dfrac{dq}{dt}\underbrace{\dfrac{1}{dA}\cdot\dfrac{dl}{dl}} = \dfrac{dq}{dV}\dfrac{dl}{dt} = \rho\vec{v}$

So $\boxed{\vec{J} = \rho\vec{v}}$

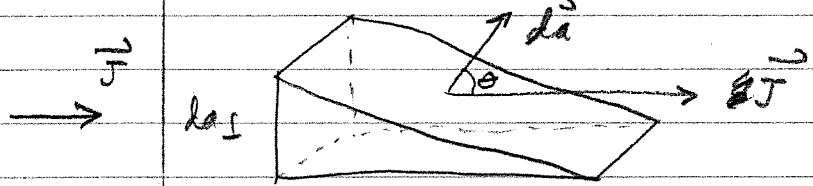
(d) Force on current sheet

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B})\, dq = \int (\vec{v} \times \vec{B})\, \rho\, dV = \int (\rho\vec{v} \times \vec{B})\, dV$$

$$\boxed{\vec{F}_{mag} = \int (\vec{J} \times \vec{B})\, dV}$$

Continuity Equation (conservation of charge eqn)

Consider current through some area...

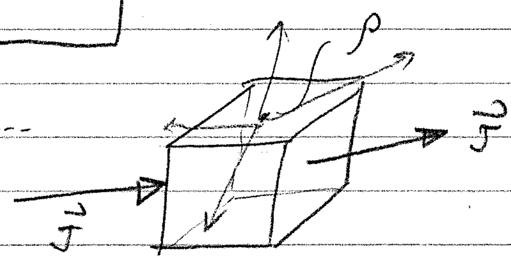


$da_\perp = da\cos\theta$

$$dI = \left|\vec{J}\cdot da_\perp\right| = \vec{J}\cdot d\vec{a} \quad \underline{\text{so}} \quad dI = \vec{J}\cdot d\vec{a}$$

$$\underline{\text{so}} \quad \boxed{I = \int_S \vec{J}\cdot d\vec{a}}$$

Consider closed surface...



$$I = -\frac{dq}{dt}$$

$$\boxed{I = \oint_{\text{surface}} \vec{J}\cdot d\vec{a} = \int_V (\vec{\nabla}\cdot\vec{J})\,dV}$$

But we also know...
$$I = -\frac{dq}{dt} = -\frac{d}{dt}\int \rho\, dV$$

$$\underline{\text{so}} \quad I = -\int_V \left(\frac{\partial\rho}{\partial t}\right) dV$$

$$\underline{\text{so}} \quad \boxed{-\frac{\partial\rho}{\partial t} = \vec{\nabla}\cdot\vec{J}} \quad \text{i.e.} \quad \boxed{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$$

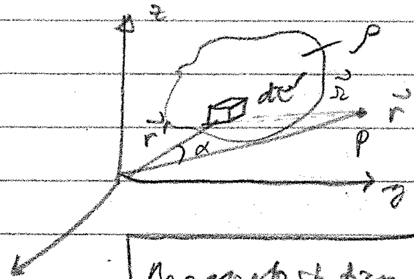
For magnetostatics, $\longrightarrow$ have constant currents $\Rightarrow \boxed{\frac{\partial\rho}{\partial t} = 0}$

$\hookrightarrow$ $\underline{\text{so}}$ for magnetostatics $\rightarrow \boxed{\vec{\nabla}\cdot\vec{J} = 0}$

BIOT – SAVART LAW

Oct 24, 2019

Compare B-S law to Coulomb's law...

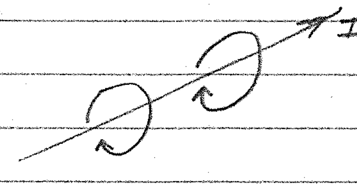


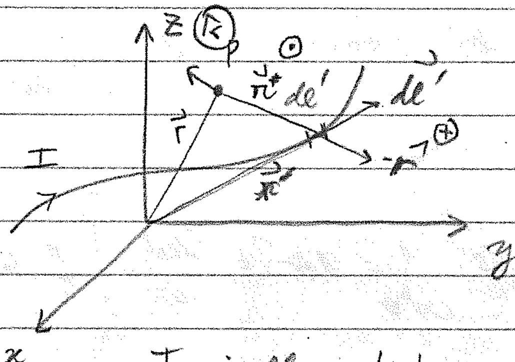
Coulomb's law $\dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \frac{\rho(r')}{\mathscr{r}^2}\,d\tau'$

↳ Electrostatics...

$\boxed{\text{Magnetostatics}}$

Biot – Savart law





$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,dl'}$$

Typically, $|I|$ constant → $\boxed{\vec{B}(\vec{r}) = \dfrac{\mu_0 I}{4\pi}\displaystyle\int \frac{d\vec{l}'\times\hat{\mathscr{r}}}{\mathscr{r}^2}}$

Note

$\boxed{\mu_0 = 4\pi\times 10^{-7}\ N/A^2 \rightsquigarrow \text{permeability of vacuum}}$

Note

$\boxed{\text{Coulomb \& Biot-Savart are inverse square laws}}$

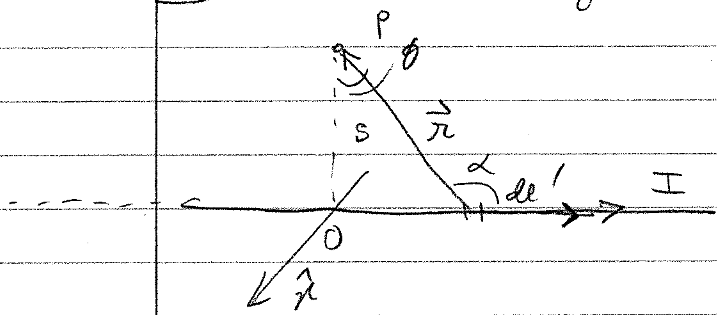
(★) Direction of $\vec{B}$ is given by the RHR.

$\boxed{\text{Biot-Savart for surface current}}$ $\quad I = \int \vec{K}\cdot d\vec{a}$

Surface $\quad \vec{B}(\vec{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\vec{K}(r')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,da'$

$\boxed{\text{B-S for volume current}}$ $\quad$ Volume $\quad \vec{B}(\vec{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\vec{J}(r')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,dV'$

Ex $\vec{B}$ field for long straight wire ... (infinite)



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,dl'$$

$$= \frac{\mu_0 I}{4\pi}\int \frac{\vec{dl}'\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= \frac{\mu_0 I}{4\pi}\int \frac{\sin\alpha}{\mathscr{r}^2}\,dl'\;\hat{x}$$

$\sin\alpha = \sin(90+\theta)$

$\hookrightarrow \sin\alpha = \cos\theta$

$l = r\sin\theta$ or $l = s.\tan\theta \Rightarrow dl' = s\,\frac{1}{\cos^2\theta}\,d\theta$

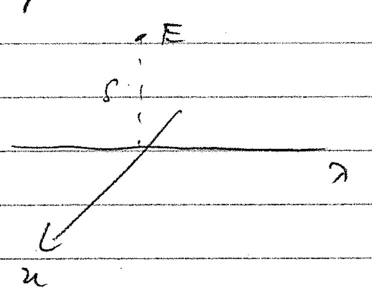
$r\cos\theta = s.$

So $$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{s^2\cancel{\cos^{-2}}\theta}\cdot s\cdot\frac{1}{\cancel{\cos^2\theta}}\,d\theta\,\hat{x} = \frac{\mu_0 I}{4\pi}\int_{-\pi/2}^{\pi/2}\frac{1}{s}\cos\theta\,d\theta\,\hat{x}$$

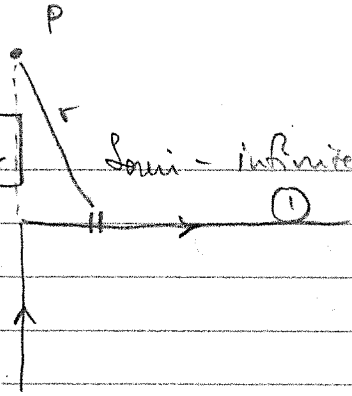
$$= \frac{\mu_0 I}{4\pi}\,\frac{1}{s}\cdot\sin\theta\Big|_{-\pi/2}^{+\pi/2}\,\hat{x}$$

So $$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi s}\,\hat{x}}$$

Compare to electrostatics...



$$\boxed{\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\,\frac{1}{s}\,\hat{z}}$$



Ex: Semi-infinite wire.

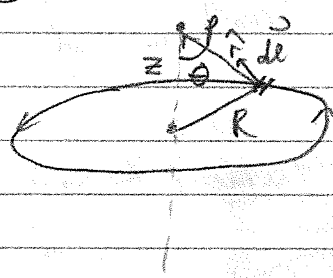
$$\vec{B}_1(\vec{r}) = \frac{\mu_0 I}{4\pi}\left(\sin\left(\frac{\pi}{2}\right) - \sin 0\right)\cdot\frac{1}{s}$$

$$= \frac{\mu_0 I}{4\pi}\frac{1}{s}\hat{\phi}$$

$\vec{B}_2(\vec{r}) = 0$ since $d\vec{l} \parallel \hat{r}$.

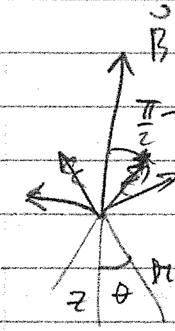
So $\vec{B}_{total}$: $\vec{B}_1 = \frac{\mu_0 I}{4\pi}\frac{1}{s}\hat{\phi}$

Ex: $\vec{B}$ for circular loops



$$\vec{B}(z) = \frac{\mu_0 I}{4\pi}\int \frac{\overbrace{d\vec{l}\times\hat{r}}^{I}}{r^2}$$

$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{1}{r^2}\cdot \underbrace{R\,d\phi}_{\sin\theta}$$



So $\frac{\mu_0 I}{4\pi}(2\pi)\cdot\frac{\sin\theta}{(z^2+R^2)} = \frac{\mu_0 I}{2(z^2+R^2)}\sin\theta = \frac{\mu_0 I}{2(z^2+R^2)}\cdot\frac{R}{\sqrt{z^2+R^2}}$

So $\boxed{\vec{B}(z) = \frac{\mu_0 I}{2}\frac{R^2}{(z^2+R^2)^{3/2}}}$

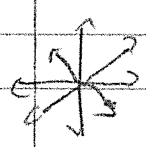
when $z = 0$, $\boxed{\vec{B}(0) = \frac{\mu_0 I}{2R}}$

Oct 25, 2019

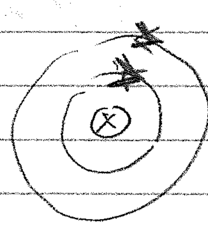
DIV & CURL OF B

Electrostatics

$$\left\{\begin{aligned}\nabla\cdot\vec{E} &= \rho/\varepsilon_0\\ \nabla\times\vec{E} &= 0\end{aligned}\right\}$$



Magneto-statics



$$\boxed{\begin{aligned}\vec{\nabla}\cdot\vec{B} &= 0\\ \vec{\nabla}\times\vec{B} &= \mu_0\vec{J}\end{aligned}}$$

(I) Curl of $\vec{B}$ field

$\vec{B}$ for infinite wire ... $\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$

Stokes' Thm (cylindrical)

$$\int_A (\vec{\nabla}\times\vec{B})\,dA = \oint_{\partial A} \vec{B}\cdot d\vec{l}$$

$$d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$$

So $\oint \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi}\int_0^{2\pi}\frac{1}{s}(s\,d\phi) = \mu_0 I_{enclosed}$

$$\boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed}}$$ → Amp's law (integral form...)

very much like $\oint \vec{E}\cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$

In terms of current density... $I_{enc} = \int \vec{J}\cdot d\vec{a}$

So
$$\boxed{\int_A (\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \oint_{\partial A}\vec{B}\cdot d\vec{l} = \mu_0\int_A \vec{J}\cdot d\vec{a}}$$

So $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ → Amp's law in diff form

Amperian loop.

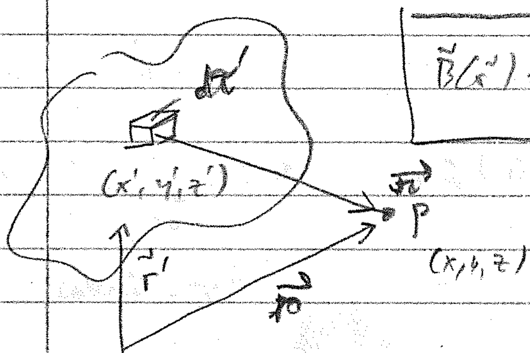
(II) Divergence of $\vec{B}$

$\cancel{\oint_A (\vec{\nabla}\cdot\vec{B})\,da}$ $\quad \oint_A \vec{B}\cdot d\vec{a} = 0 = \int_V (\vec{\nabla}\cdot\vec{B})\,dV$

so $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$ $\rightsquigarrow$ no magnetic monopole...

mathematically... show $\vec{\nabla}\cdot\vec{B} = 0$ w/ Biot-Savart...

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'}$$



$\vec{\nabla}\cdot\vec{B} = \;?$ $\quad$ ($\vec{\nabla}$ in x, y, z, not x', y', z')

$\vec{J}(\vec{r}\,') = f(x', y', z')$

$\mathscr{r} = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$

$d\tau' = dx'dy'dz'$

$$\vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \underbrace{\vec{\nabla}\cdot\left\{\frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right\}}_{\text{product rule...}} d\tau'$$

$$\vec{\nabla}\cdot\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\underbrace{\left(\vec{\nabla}\times\vec{J}\right)}_{\vec{0}} - \vec{J}\cdot\left(\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

since $\left.\begin{array}{l}\vec{\nabla} \sim x, y, z \\ \vec{J} \sim x', y', z'\end{array}\right\} \Rightarrow \vec{\nabla}\times\vec{J} = 0$

Likewise $\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \rightarrow$ radial component $= 0$ because

$$(\vec{\nabla}\times\vec{v})\cdot\hat{r} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \partial_\phi v_\theta\right]\hat{r}$$

So $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$

Summary of Electrostatics vs magnetostatics

Maxwell's Eqn $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ $\quad\quad \vec{\nabla}\cdot\vec{B} = 0$

$\vec{\nabla}\times\vec{E} = 0 \quad\quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

Force laws $\vec{F}_E = \vec{E}q \quad\quad \vec{F}_B = q(\vec{v}\times\vec{B})$

$\boxed{\text{Empirical laws}}$ $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau' \quad\Big|\quad \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$

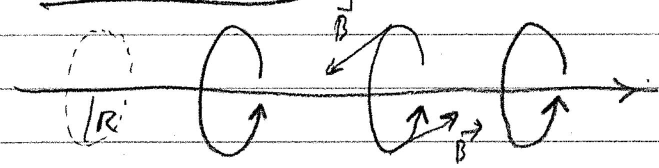
(Coulomb) (Biot-Savart)

$\oint_A \vec{E}\cdot d\vec{a} = \frac{q_{encl}}{\varepsilon_0} \quad\quad \oint_\ell \vec{B}\cdot d\vec{\ell} = \mu_0 I_{encl}$

Note only when $v \to c$ does $|B| \sim |E|$

$\boxed{\text{AMP'S LAW}}$ $\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enclosed}$ → key
- Direction of $\vec{B}$
- Draw Amperian loop $\parallel \vec{B}$.
- Find $I_{enclosed}$

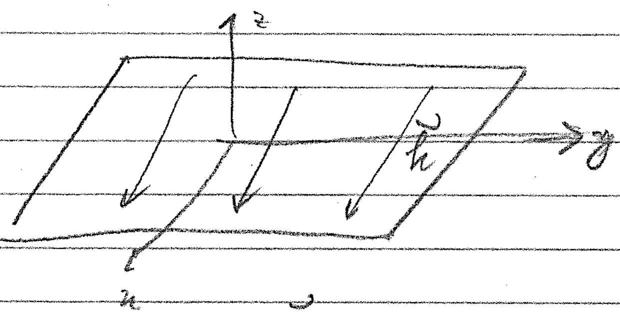
(Ex) Infinite wire



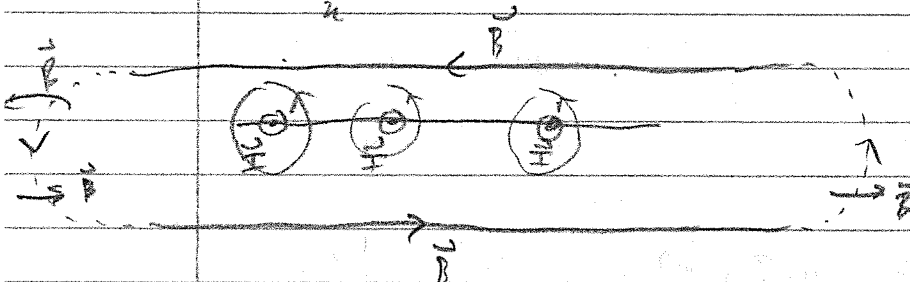
$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi R}\hat{\phi}}$

$\oint \vec{B}\cdot d\vec{\ell} = B(2\pi R) = \mu_0 I \Rightarrow \boxed{B = \frac{\mu_0 I}{2\pi R}}$

Ex Infinite sheet of current.



$\vec{k} = k\hat{x}$



$$\underbrace{\oint \vec{B}\cdot d\vec{l}} = \mu_0 \underbrace{I_{enclosed}}$$

$$2Bl = \mu_0 \int \vec{k}\cdot d\vec{l}_{\perp}$$

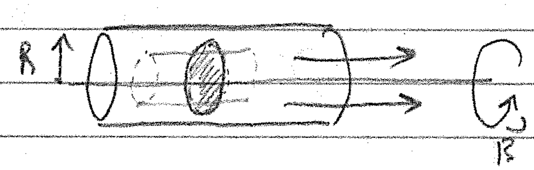
So $\quad 2Bl = \mu_0 k l$

So $\quad \boxed{\vec{B} = \mp \frac{\mu_0 k}{2}\hat{y}}$ $\quad$ (−) above plane, (+) below plane.

Oct 28, 2019

Key steps to Amp's law:
1. Find direction of $\vec{B}$ field.
2. Draw Amp loop $\parallel \vec{B}$, $\perp \vec{J}$.
3. Determine I_{enc}.

Ex Volume current



$$\vec{J}(r) = J_0\left(1-\left(\frac{r}{R}\right)^2\right)\hat{z}.$$

$\vec{B}$ inside

$$\underbrace{\oint \vec{B}\cdot d\vec{l}} = \mu_0 I_{enclosed}$$

$\hat{z}\,da$

$$B(2\pi r) = \mu_0 \int \vec{J}\cdot d\vec{a} = \mu_0 \int_0^{2\pi}\int_0^{r} J_0\left(1-\left(\frac{r'}{R}\right)^2\right) r'dr'd\phi$$

$$= \dots \Rightarrow \quad B(2\pi r_1) = \mu_0 J_0 (2\pi) \int_0^{r} \left(1 - \frac{r'^2}{R^2}\right) r'\,dr'$$

$$= \mu_0 J_0 \frac{(2\pi)}{2}\left(r_1^2 - \frac{1}{2}\frac{r_1^4}{R^2}\right)$$

$$= \mu_0 J_0 \pi \left(r_1^2 - \frac{1}{2}\frac{r_1^4}{R^2}\right)$$

so $\boxed{\vec{B}(\vec{r}) = \frac{\mu_0 J_0}{2}\left(r_1 - \frac{1}{2}\frac{r_1^3}{R^2}\right)\hat{\phi}}$ $\quad (r_1 < R)$

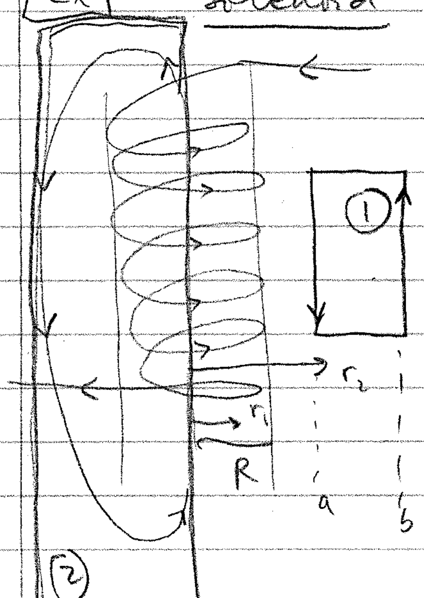
<u>$\vec{B}$ outside</u>

$$B(2\pi r_2) = \mu_0 J_0 (2\pi) \int_0^{R} \left(1 - \frac{r'^2}{R^2}\right) r'\,dr'$$

$$= \mu_0 J_0 \frac{(2\pi)}{2}\left[R^2 - \frac{1}{2}R^2\right] = \frac{\mu_0 J_0 \pi R^2}{2}$$

so $\boxed{\vec{B}(\vec{r}) = \frac{\mu_0 J_0}{4 r_2} R^2 \hat{\phi}}$ $\quad (r_2 > R)$

Ex <u>Solenoid</u>



amp loop

n turns per unit length.

(1) $r_2 > R$ ... Amp's law ...

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed} = 0$$

$B(a)\,l - B(b)\,l = 0 \rightarrow \boxed{B(a) = B(b)}$

$\boxed{B(\infty) = 0 = B(\text{outside})}$

$\rightarrow$ no $\vec{B}$ outside.

② $\vec{B}$ inside?

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \, I_{enclosed}$$

$$B(r_1)\,l - \cancelto{0}{B(r>R)}\,l = \mu_0 I_{enclosed}$$

$$B(r_1)\,l = \mu_0 I_{enclosed} \Rightarrow B(r_1) = \frac{\mu_0}{l} I_{enclosed}$$

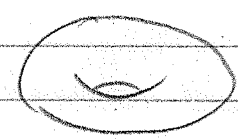
$IN = Iln$

So $$\boxed{\vec{B}_{in} = \frac{\mu_0 I l n}{l}\hat{z} = \mu_0 I n \hat{z}}$$

Ampere law only useful for

① Straight lines (inf)
② Infinite planes
③ Infinite solenoids
④ Toroids

} cylindrical



MAGNETIC VECTOR POTENTIAL

Electrostatics: $\vec{\nabla}\times\vec{E} = 0 \rightarrow \vec{E} = -\vec{\nabla}V \rightsquigarrow$ scalar potential

Magnetostatics: $\vec{\nabla}\cdot\vec{B} = 0 \rightarrow \vec{B} = \vec{\nabla}\times\vec{A} \rightarrow$ vector potential

Helmholtz Thm $\rightarrow$ vector field is uniquely determined by it's curl, divergence.

Electrostatics $\vec{E} = -\vec{\nabla}V_0 = -\vec{\nabla}(V_0 + c)$ constant

↳ gauge fix: Want $V(\infty) = 0 \Rightarrow$ fix V

Same with $\vec{A}$.

$$\vec{B} = \vec{\nabla}\times\vec{A}_0 = \vec{\nabla}\times(\vec{A}_0 + \vec{\nabla}f) \ \ldots$$

How to gauge fix?

for $A = A_0 + \vec{\nabla} f \rightarrow \vec{\nabla} \cdot A = \vec{\nabla} \cdot \vec{A}_0 + \nabla^2 f$.

Gauge fix $\vec{\nabla}^2 f = -\vec{\nabla} \cdot \vec{A}_0 \rightarrow$ Coulomb gauge ...

Require $\boxed{\begin{aligned} \vec{\nabla} \cdot \vec{A} &= \vec{0} \\ \vec{\nabla} \times \vec{A} &= \vec{B} \end{aligned}}$ $\rightarrow$ now $\vec{A}$ is uniquely determined

Side note

There are gauge transformation $\rightarrow$ gauge symmetry.

$$\left\{ \begin{aligned} A &= A_0 + \vec{\nabla} f \\ V &= V_0 + c \end{aligned} \right\}$$

C & $\vec{\nabla} f$ are gauges. $\vec{E}, \vec{B}$ are gauge invariant.

Oct 29, 2019

Functional form of $\vec{A}$ $\left\{ \begin{aligned} &\vec{\nabla} \cdot \vec{A} = \vec{0} \\ &\qquad\qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \\ &\vec{\nabla} \times \vec{A} = \vec{B}. \end{aligned} \right.$

Product Rule $\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} \cancelto{0}{(\vec{\nabla} \cdot \vec{A})} - \nabla^2 \vec{A} = \mu_0 \vec{J}$

So $\boxed{\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J}}$

Looks like Poisson eqn $\nabla^2 V = -\rho/\varepsilon_0 \rightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r')}{\mathscr{r}} d\tau'$

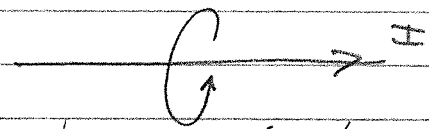
So $\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(r')}{\mathscr{r}} d\tau'}$

Note $\vec{A}$ points in $\vec{J}$. Since $\vec{B}$ doesn't do any work, $\vec{A}$ is not directly related to energy.

Lastly, units.

$[A] \sim T \cdot m$

Ex $\vec{A}$ for infinite wire...



$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$. $\vec{\nabla}\times\vec{A} = \vec{B}$

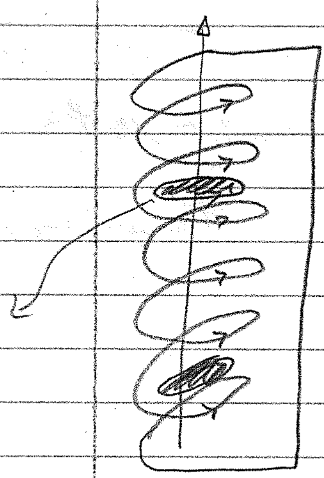
$$\left(\underbrace{\cancel{\frac{\partial A_r}{\partial z}}}_{0} - \frac{\partial A_z}{\partial r}\right)\hat{\phi} = B\hat{\phi}$$

So $-\frac{\partial A_z}{\partial r} = \frac{\mu_0 I}{2\pi r}$

So $A_z = \int -\frac{\mu_0 I}{2\pi}\frac{1}{r}\,dr = -\frac{\mu_0 I}{2\pi}\ln(r)$

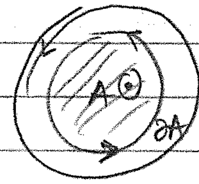
So $$\boxed{\vec{A} = \frac{-\mu_0 I}{2\pi}\ln(r)\,\hat{z}}$$

Ex $\vec{A}$ for infinite solenoid.



Stokes' thm $\oint \vec{A}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{A})\cdot d\vec{A} = \int \vec{B}\cdot d\vec{A}$

So $\oint_{\partial A} \vec{A}\cdot d\vec{l} = \int_A \vec{B}\cdot d\vec{A}$

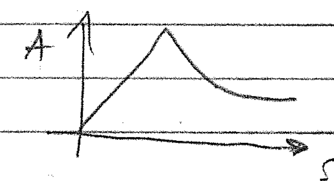


$\rightsquigarrow A(2\pi r) = B(\pi r^2)$

$\rightarrow A = \frac{Br}{2}$

(inside) So $\vec{A} = \left(0, 0, \frac{Br}{2}\right)$ in $(r, s, \hat{\phi})$

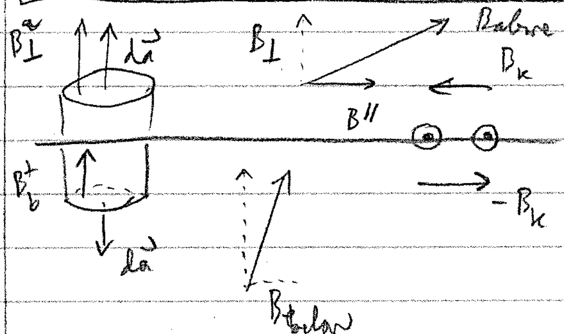
with $B = (\mu_0 I n)$



$$\boxed{\vec{A}_{in} = \frac{\mu_0 I n}{2} s\,\hat{\phi}}$$

(outside) $B_{out} = 0 \Rightarrow$ $$\boxed{\vec{A} = \left(\frac{B_{in}\pi R^2}{2\pi r}\right) = \frac{\mu_0 I n R^2}{2s}\hat{\phi}}$$

Magnetic Boundary Condition



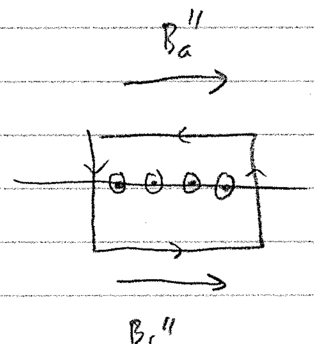
$\perp$ suffers <u>no</u> discontinuity

<u>Perp component</u> $\vec\nabla\cdot\vec B = 0$ and $\oint \vec B\cdot d\vec a = 0$

$$B_a^\perp \cdot A - B_b^\perp \cdot A = 0 \Rightarrow \boxed{B_a^\perp = B_b^\perp}$$

<u>Parallel component.</u> $\vec\nabla\times\vec B = \mu_0 \vec J$

$$\oint \vec B\cdot d\vec l = \mu_0 I_{enclosed}$$



$$B_a^\parallel l - B_b^\parallel l = \mu_0 I_{enclosed} = \mu_0 k l$$

So $\boxed{B^\parallel_{above} - B^\parallel_{below} = \mu_0 k}$ (scalar sense)

So $\boxed{\vec B_{above} - \vec B_{below} = \mu_0 (\vec k \times \vec n)}$

<u>How does $\vec A$ change?</u>

$\vec\nabla\cdot\vec A = 0 \Rightarrow \oint \vec A\cdot d\vec a = 0 \Rightarrow$ $\boxed{A^\perp \text{ above = below continuous}}$

So $\boxed{A^\perp_{above} = A^\perp_{below}}$

<u>Parallel component</u> $\oint \vec A\cdot d\vec l = \int \vec B\cdot d\vec a$ $\boxed{A^\parallel_{above} = A^\parallel_{below}}$

So $\boxed{\vec A \text{ continuous}}$ $\Rightarrow$ $\partial_n \vec A$ discontinuous...

$$\boxed{\left.\frac{\partial \vec{A}}{\partial n}\right|_{above} - \left.\frac{\partial \vec{A}}{\partial n}\right|_{below} = -\mu_0 \vec{k}}$$

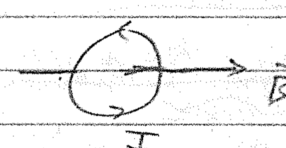
Oct 31, 2014

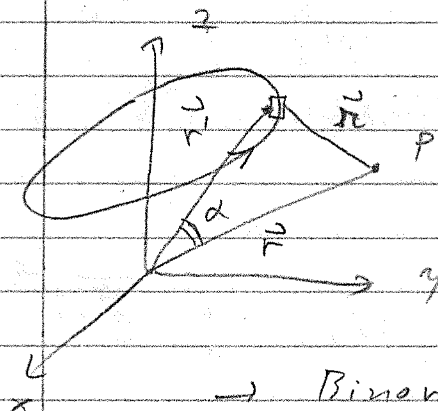
$\boxed{\text{Multipole Expansion of } \vec{A}(\vec{r})}$

↳ Writing $\vec{A}(\vec{r})$ as a power series of $1/r^n$.

$$\vec{A}(\vec{r}) \simeq \underbrace{\vec{A}(1/r)}_{\text{monopole}} + \underbrace{\vec{A}(1/r^2)}_{\text{dipole}} + \underbrace{\ldots}_{\ldots}$$

↳ only valid for closed current loops ~ like magnets

Magnetic dipole ~ 



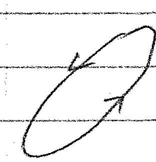
$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{d\vec{l}}{\kappa}}$$

↳ Want to expand $\boxed{\frac{1}{\kappa} = \left(r^2 - (r')^2 - 2rr'\cos\alpha\right)^{-1/2}}$

→ Binomial expansion...

$$\boxed{\frac{1}{\kappa} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)} \leftarrow \text{Eqn. 3.94}$$

So for vector potential...



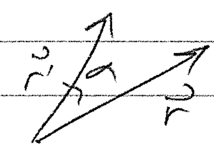
$$\boxed{\vec{A}(\vec{r}) \simeq \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\alpha)\, d\vec{l}'}$$

Monopole term $(n=0) \rightarrow 0$ of course

$$\vec{A}_{mon} = \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint\frac{d\vec{l}}{1} = \boxed{\vec{0}}$$

Dipole $(n = 1)$

$$\vec{A}_{dip}(r) = \left(\frac{\mu_0 I}{4\pi}\right)\frac{1}{r^2}\oint (r')\,d\vec{l}'\,.(\cos\alpha)$$



$$= \frac{\mu_0 I}{4\pi}\,\frac{1}{r^2}\oint (\hat{r}\cdot\vec{r}')\,d\vec{l}'$$

Now, note $\longrightarrow$ $\boxed{(\vec{v}\cdot d\vec{x})\,d\vec{y} = (d\vec{a}\times\vec{v})}$

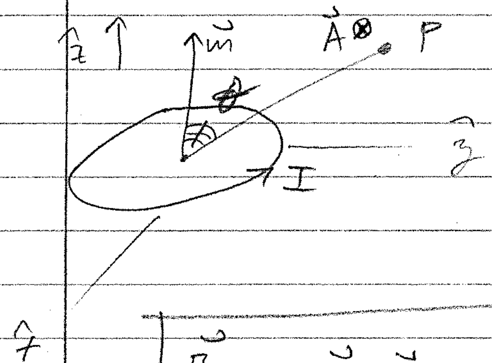
So

$$\boxed{\vec{A}_{dip}(\vec{r}) = \left(\frac{\mu_0 I}{4\pi}\right)\frac{1}{r^2}\oint (\hat{r}\cdot\vec{r}')\,d\vec{l}' = \left(\frac{\mu_0 I}{4\pi}\right)\frac{1}{r^2}\oint d\vec{a}\times\hat{r}}$$

So define magnetic dipole moment

$$\vec{m} = I\int d\vec{a} = I\vec{a}$$

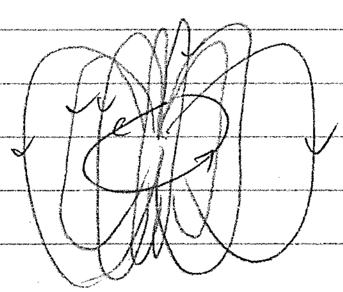
$\boxed{\text{Ex}}$ $\vec{B}$ for pure dipole...



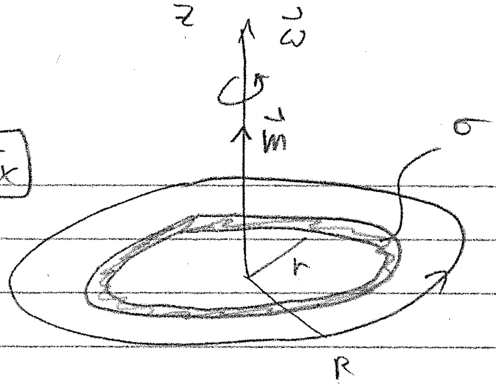
Dipole term $\boxed{\vec{A}_{dip} = \frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\hat{r}}{r^2}}$

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi}\,\frac{m\sin\theta}{r^2}\,\hat{\phi}$$

$$\boxed{\vec{B}_{dip} = \vec{\nabla}\times\vec{A} = \frac{\mu_0 m}{4\pi r^2}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)}$$



Ex



magnetic dipole moment

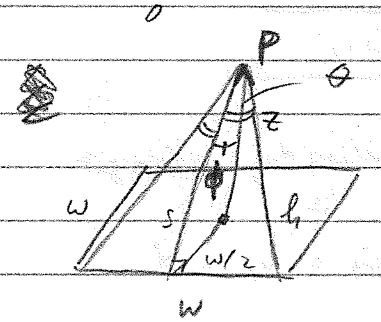
$$\vec{m} = I \int d\vec{a} = I(\vec{a}) = I\pi r^2 \hat{z}$$

$$I = \frac{dq}{dt} = \frac{\sigma\, da}{dt} = \frac{\sigma (2\pi r) dr}{dt} \qquad \omega = \frac{d\phi}{dt} \Rightarrow dt = \frac{2\pi}{\omega}$$

$$I \sim \sigma \omega r\, dr \Rightarrow dm = I(r)\pi r^2 \hat{z} = \sigma \omega r\, dr (\pi r^2)\hat{z}$$

$$\text{So } \vec{m} = \int_0^R \pi \sigma \omega r^3 dr\, \hat{z} = \boxed{\frac{\pi\sigma\omega R^4}{4}\hat{z}}$$

Ex



$\vec{B}$ for (semi-infinite) wire

$$\vec{B} = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)\hat{z}$$

$$\theta_2 = \theta_1 = \theta$$

$$s = (z^2 + (w/2)^2)^{1/2}, \quad \sin\theta = \frac{w/2}{h} = \frac{w}{2h}, \quad h = (s^2 + (w/2)^2)^{1/2}$$

$$h = \left(z^2 + \left(\frac{w}{2}\right)^2 + \left(w/2\right)^2\right)^{1/2} = \left(z^2 + 2\left(\frac{w^2}{2^2}\right)\right)^{1/2} = \left(z^2 + \frac{w^2}{2}\right)^{1/2}$$

$$\Rightarrow \sin\theta = \frac{w}{2}\left(z^2 + \frac{w^2}{2}\right)^{-1/2}$$

$$\text{So } \vec{B} = \frac{\mu_0 I}{4\pi}(z^2 + (w/2)^2)^{-1/2} \cdot \left\{\frac{w}{2}\left(z^2 + \frac{w^2}{2}\right)^{-1/2} + \frac{w}{2}\left(z^2 + \frac{w^2}{2}\right)^{-1/2}\right\}\hat{z}$$

$$= \frac{\mu_0 I}{8\pi}\frac{w}{(z^2 + w^2/2)^{1/2} \cdot (z^2 + (w/2)^2)^{1/2}}\hat{z}$$

To get $B^{\perp} \rightarrow \vec{B}_{tot} = \frac{\mu_0 I}{8\pi}\frac{4w}{(\ldots)(\ldots)}\sin\phi = \frac{\mu_0 I}{8\pi}\frac{4w}{(\ldots)(\ldots)}\frac{w/2}{(z^2 + (w/2)^2)^{1/2}}$

So $\boxed{\vec{B} = \dfrac{\mu_0 I}{4\pi} \dfrac{w^2}{(z^2 + w^2/4)(z^2 + w^2/2)^{1/2}} \hat{z}}$

$z >> w \Rightarrow$

$$\vec{B} \cong \frac{\mu_0 I w^2}{2\pi z^3} \hat{z}$$

In dipole formalism $\Rightarrow$ $\vec{B}_{dip} = \dfrac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta}\right)$

$(z \gg w)$ $= \boxed{\dfrac{\mu_0 I w^2}{2\pi r^3} \hat{z}}$ ✓ (matches)

$r \to z$

$\theta \to 0$

Nov 1, 2019

ELECTRODYNAMICS

Electromotive Force $\rightsquigarrow$ accelerating charges...

what force is needed to drive a current...

Perfect conductor $\Rightarrow$ no force need...
Imperfect conductor $\Rightarrow$ force needed...

Resistivity of substance (ρ) $\rightsquigarrow$ how much current is impeded in an imperfect conductor...

Units: Ohm · Meter Ωm

Values
1. Conductors: (Al, Cu $\sim 10^{-18}\, \Omega m$
2. Semi-conductor: (Si, Ge) $\sim 10^{-1}, 10^{-2}\, \Omega m$
3. Insulators: Rubber... $\sim 10^{14}\, \Omega m$

$\boxed{\text{Conductivity}}$ $\boxed{\sigma = \dfrac{1}{\rho}}$ $\rightsquigarrow$ $\Omega^{-1} m^{-1}$

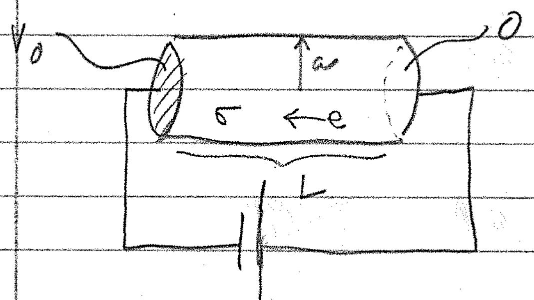
Current density $\boxed{\vec{J} = \sigma \vec{f}}$, $\vec{f}$: force per unit charge.

← Ohm's law.

When $\vec{f}$ is electrical, $\boxed{\vec{J} = \sigma \vec{E}}$

Generalized $\boxed{\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B})}$ $F_E \gg F_B$ unless $v \to c$

$\boxed{\text{Ex}}$ Current in semiconductor...



$$\vec{J} = \sigma \vec{f}$$

$$I = J(\pi a^2) = (\sigma f)\cdot(\pi a^2)$$
$$= (\sigma E)(\pi a^2)$$

$$V = -\int_0^L \vec{E} \cdot d\vec{\ell} \sim EL \Rightarrow E = V/L$$

So $\boxed{V = I\left(\frac{L}{\sigma \pi a^2}\right) = IR}$ ⇝ Ohm's law.

↳ Resistance...

→ Units $[R] = \Omega = V/A$

$$\boxed{R = \frac{L}{\sigma \pi a^2} = \frac{L\rho}{\pi a^2}}$$

How much energy to drive a current?

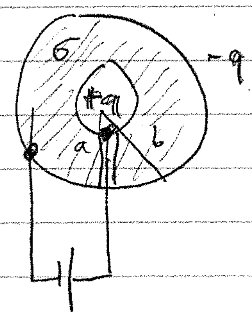
$$dW = \left(\frac{\text{work done}}{\text{charge}}\right) \times \left(\frac{\text{charge per}}{\text{unit time}}\right) \times (\text{time})$$

$$= V \times I \times \Delta t$$

So $dW = VI\,dt \Rightarrow W = \int V \cdot I\,dt \ldots$

Power $\boxed{P = \frac{dW}{dt} = V \cdot I = I^2 R}$

$\boxed{\text{Ex}}$ Resistance of nested shell...



$$V = IR, \quad \vec{J} = \sigma \vec{E} = \sigma \cdot \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}\hat{r}$$

$$I = \oint \vec{J} \cdot d\vec{a} = \oint \sigma \cdot \vec{E}\, d\vec{a} = \sigma \oint \vec{E}\, d\vec{a}$$

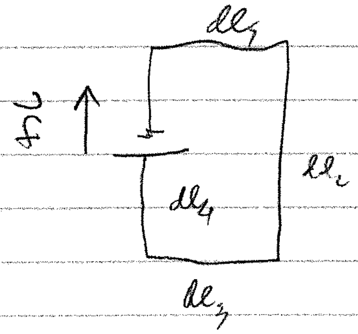
$$\Rightarrow \boxed{I = \sigma \frac{q}{\varepsilon_0}}$$

$$V = -\int_a^b \vec{E} \cdot d\vec{l} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\text{So} \quad I = \frac{\sigma q}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0}\left[\frac{4\pi ab\,\varepsilon_0}{b-a}\right] = \left(\frac{4\pi ab}{b-a}\right)\sigma V$$

$$\text{So} \quad R = \frac{V}{I} = \frac{b-a}{4\pi(ab)\sigma}$$

$\boxed{\text{Electromotive force}} \Longrightarrow \boxed{\mathcal{E} \equiv \oint \vec{f} \cdot d\vec{l}}$



when $\vec{f}$ is purely $\vec{E}$,

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = -V$$

Units $\boxed{\mathcal{E} \rightarrow V}$

Note Electrostatics $\rightarrow \vec{\nabla} \times \vec{E} = 0 \rightarrow$ This is not electrostatics.

$\Rightarrow$ STATIC FIELDS CANNOT DRIVE CURRENTS.

$$\vec{f}_{total} = \vec{f}_{source} + \vec{f}_{ES} \qquad \vec{\nabla}\times\vec{f}_{ES} = 0$$

$$\mathcal{E} = \oint \vec{f}_{total}\cdot d\vec{l} = \oint \vec{E}_{source}\cdot d\vec{l} + \underbrace{\oint \vec{f}_{ES}\cdot d\vec{l}}_{0}$$

$$= \oint \vec{E}_{source}\cdot d\vec{l}$$

&

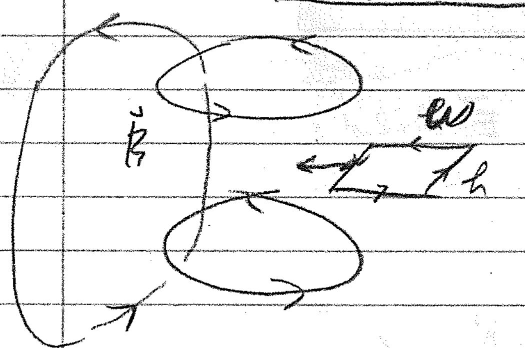
$$\boxed{\mathcal{E} = \oint \vec{E}_{source}\cdot d\vec{l}}$$

Nov 4, 2019

ELECTROMAGNETIC INDUCTION

Faraday: Exp (1831)

(1) Moving loop + constant $\vec{B}$ field...



Motional Emf

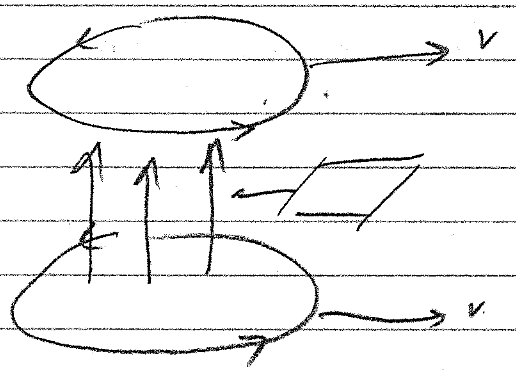
$$\mathcal{E} = \oint \vec{F}_B\cdot d\vec{l}$$

$$= \oint \frac{\vec{F}}{q}\cdot d\vec{l}$$

$$= \oint (\vec{v}\times\vec{B})\cdot d\vec{l} = vB\int dl$$

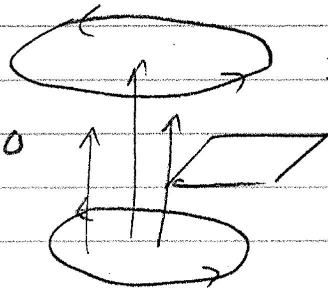
$$\rightarrow \boxed{\mathcal{E} = vBh}$$

(2) Stationary loop + Moving $\vec{B}$ field



(3) Stationary loop + changing $\vec{B}$ field

$\frac{dB}{dt} > 0$



$I = I(t) \quad \frac{dI}{dt} \neq 0 > 0$

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}} \rightarrow \text{Faraday's Law.}$$

Stokes' Thm $\quad \oint \vec{E} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$

so

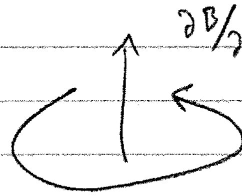
$$\boxed{\begin{aligned} \underbrace{\oint \vec{E} \cdot d\vec{l}}_{} &= -\frac{d}{dt} \int \vec{B} \cdot d\vec{a} \\ \text{Emf} &= -\frac{d\Phi_B}{dt} \end{aligned}}$$

changing $\vec{B}$ fields induce $\vec{E}$ fields

Direction

$\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t \,, \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

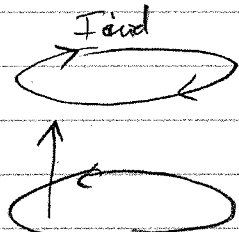
$\partial B/\partial t$



Lenz Law: Nature resists a changes in flux

Ex

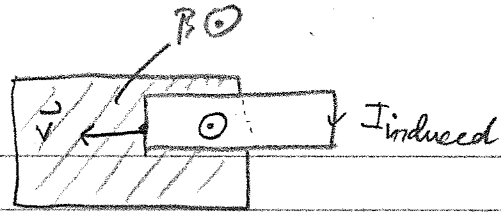
I_{ind}



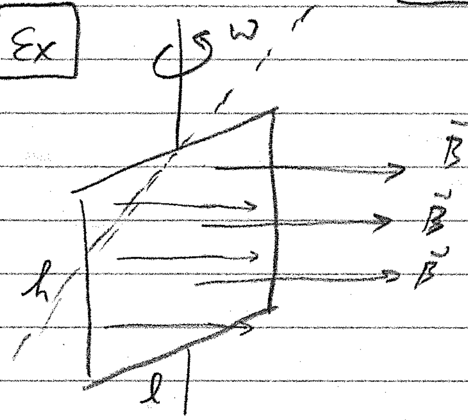
$\partial B/\partial t > 0$

$dI/dt > 0$

Ex Lenz's Law



Ex



$B\,dA\cos\varphi$

$$EMF = -\frac{\partial \Phi_B}{\partial t} = -\frac{\vec{B}\cdot d\vec{A}}{dt}$$

$$= +\omega BA\sin(\omega t)$$

$$\Phi_B = BA\cos\theta \rightarrow \frac{\partial \Phi_B}{dt} = -BA\sin(\theta)\cdot\dot{\theta}$$

$$\rightarrow \boxed{EMF = -\frac{\partial \Phi_B}{\partial t} = BA\omega\sin(\omega t)}$$

Induced Electric Fields

$$\nabla\times\vec{E}_{ind} = -\frac{\partial \vec{B}}{\partial t}, \qquad \vec{\nabla}\cdot\vec{E}_{ind} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}, \qquad \vec{\nabla}\cdot\vec{B} = 0$$

Form for $E_{induced}$ from Biot Savart ...

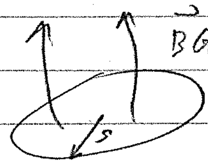
$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\int\frac{I\times\hat{r}}{r^2}\,d\tau' \;;\quad \vec{E}_{ind} = \frac{1}{4\pi}\int\frac{\partial\vec{B}/\partial t\times\hat{r}}{r^2}\,d\tau'}$$

Direction of $\vec{E}_{induced} = \perp\vec{B}$

Amp's Law replaced with Faraday's Law

$$\boxed{\oint\vec{E}\cdot d\vec{l} = \text{Emf} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int\vec{B}\cdot d\vec{a}}$$

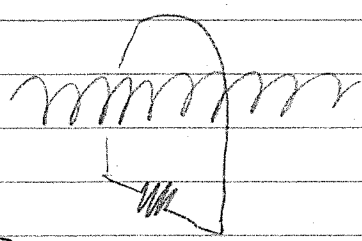
Nov 5/2019 $\boxed{Ex}$ Induced E field



$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\Phi_B = \frac{-d}{dt}\int \vec{B}\cdot d\vec{a}$$

$$E(2\pi s) = \frac{-dB(t)}{dt}(\pi s^2) \Rightarrow \boxed{\vec{E} = \frac{-s}{2}\frac{\partial B}{\partial t}\hat{\phi}}$$

$\boxed{Ex}$ Induced current



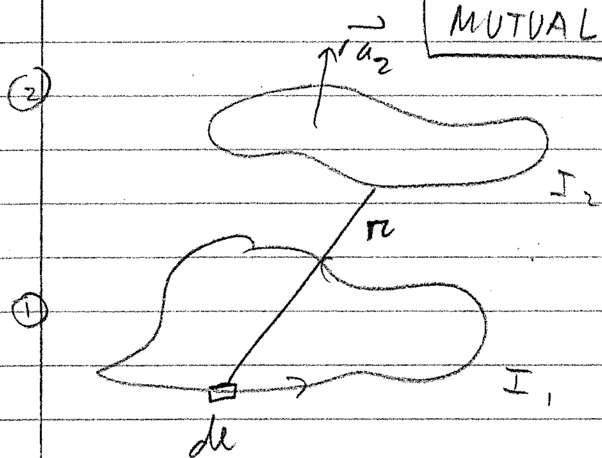
$$\mathcal{E} = IR = \oint \vec{E}\cdot d\vec{l} = -\frac{d\phi}{dt} = \cdots$$

$$B = \mu_0 n I \ldots$$

$\boxed{Ex}$ Lenz Law - Conservation of Energy

$\boxed{Ex}$ Jumping Ring

$\boxed{\text{MUTUAL INDUCTANCE}}$



$$B_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\vec{l}\times\hat{r}}{r^2}$$

$$\Phi_2 = \int \vec{B}_1\cdot d\vec{a}_2 \propto I_1 = M I_1$$

where $\boxed{M \equiv \Phi_2 / I_1}$ ⇝ mutual inductance

↑ a purely geometric factor.

Functional form for M

$$\Phi_2 = \int \vec{B}_1\cdot d\vec{a}_2$$

$$= \int (\vec{\nabla}\times\vec{A}_1)\cdot d\vec{a}_2$$

$$= \oint \vec{A}_1\cdot d\vec{l}_2 \quad \text{where } \vec{A}_1 = \frac{\mu_0 I_1}{4\pi}\oint \frac{d\vec{l}_1}{r}$$

So $\underline{S_1}$ $$\boxed{\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \left(\oint \frac{d\vec{l}_1}{r} \right) \cdot d\vec{l}_2}$$

&

$$\boxed{M = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r}}$$ → mutual inductance for 2 loops.

(Neumann's Formula)

$\underline{N.B}$ $d\vec{l}_1 \cdot d\vec{l}_2 = d\vec{l}_2 \cdot d\vec{l}_1$

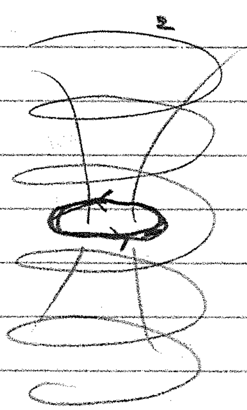
$$\boxed{M = \frac{\Phi_1}{I_1} = \frac{\Phi_2}{I_2}}$$

$$\Phi_1 = \int \vec{B}_2 \cdot d\vec{a}_1 \qquad \Phi_2 = \int \vec{B}_1 \cdot d\vec{a}_2$$

$$\Phi_1 \propto I_2 \qquad \Phi_2 \propto I_1$$

$$\frac{\Phi_1}{I_2} = M_{12}, \quad \frac{\Phi_2}{I_1} = M_{21} \qquad M_{21} = M_{12}$$

$\boxed{Ex}$ Mutual Inductance



$$M = \frac{\Phi_2}{I_1} = \frac{\Phi_1}{I_2}$$

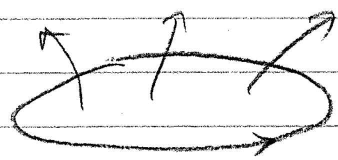
$$\Phi_1 = \int \vec{B}_2 \cdot d\vec{a}_1 = \oint \mu_0 n I_2 \pi a^2$$

$$\rightarrow M = \mu_0 n \pi a^2$$

$$\rightarrow \boxed{\Phi_2 = \mu_0 n \pi a^2 \cdot I_1}$$

Nov 7, 2019

$\boxed{\text{SELF-INDUCTANCE}}$



$\frac{dI}{dt} > 0, \ \frac{d\Phi}{dt} > 0$

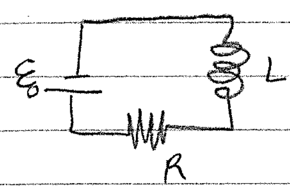
self inductance.

$\mathcal{E} = -\frac{d\Phi}{dt} \qquad \Phi_1 = \int \tilde{B}_1 \cdot d\tilde{a}_1 \propto I_1 \Rightarrow \boxed{\Phi_1 = L I_1}$

$\boxed{\mathcal{E} = -L \frac{dI}{dt}} \leftarrow$ Back Emf.

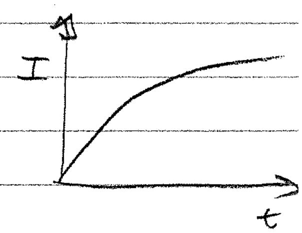
Units
$[L] = \frac{[\mathcal{E}]}{[dI/dt]} = \frac{V}{A/s} = \frac{V}{A} s = \text{Henry} = H$

$\boxed{\text{Ex}}$ Circuits



$\mathcal{E} - L \frac{dI}{dt} = IR$

$\boxed{I(t) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-Rt/L}\right)}$



Energy foreshadow

$V = L \frac{dI}{dt}$	$F = m \frac{dv}{dt}$
I current	v velocity
q charge	x displacement
~~LI~~ "momentum"	mv momentum
$\frac{1}{2} L I^2$ (energy)	$\frac{1}{2} m v^2$

Energy in B fields

$W = \int dw = \iint \tilde{F}_E \cdot d\tilde{l} = \iint \tilde{E} \cdot dq \cdot d\tilde{l} \rightarrow \boxed{\frac{dw}{dq} = \int \tilde{E} \cdot d\tilde{l} = V = -\mathcal{E}_{back}}$

So $\frac{dW}{dt} = -\mathcal{E}_{back} \cdot \frac{dq}{dt} = -\mathcal{E}_{back} \cdot I = -L \frac{dI}{dt} I$

So $\displaystyle\int \frac{dW}{dt} = \int L I \frac{dI}{dt} \rightarrow \boxed{W = \frac{1}{2} L I^2}$

Two alternative energy equations

(1) Vector potential... $\Phi = \int \vec{B} \cdot d\vec{a}$, $\Phi = L \cdot I$.

Stokes' Thm $\rightarrow$ $\Phi = \int \vec{B} \cdot d\vec{a} = \int \vec{\nabla} \times \vec{A} \cdot d\vec{a} = \oint \vec{\nabla} \times \vec{A} \cdot d\vec{l}$

$$\Rightarrow LI = \oint \vec{A} \cdot d\vec{l} \qquad W = \frac{1}{2} L I^2$$

So $W = \frac{1}{2} I \oint \vec{A} \cdot d\vec{l}$

$= \frac{1}{2} \oint \vec{A} \cdot I d\vec{l}$

So $\boxed{W = \frac{1}{2} \int_V (\vec{A} \cdot \vec{J}) \, d\tau}$

(2) Energy related only to B field

$$\vec{\nabla} \times \vec{A} = \vec{B}, \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

So $W = \dfrac{1}{2\mu_0} \displaystyle\int_V (\vec{A} \cdot \mu_0 \vec{J}) \, d\tau = \dfrac{1}{2\mu_0} \int_V \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \, d\tau$

And $\vec{A} \cdot (\vec{\nabla} \times \vec{B}) = \vec{B} \cdot \vec{B} - \vec{\nabla} \cdot (\vec{A} \times \vec{B})$

So $W = \dfrac{1}{2\mu_0} \left[\displaystyle\int_V B^2 d\tau - \int_V \vec{\nabla} \cdot (\vec{A} \times \vec{B}) \, d\tau \right]$

$$\boxed{W = \frac{1}{2\mu_0} \left\{ \int_V B^2 d\tau - \oint_{\partial V} (\vec{A} \times \vec{B}) \cdot d\vec{a} \right\}}$$

Take $S \to \infty$ $\rightarrow$ second term goes to 0

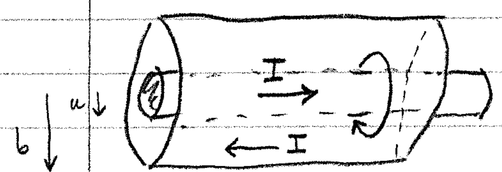
$\rightarrow$ $\boxed{W = \dfrac{1}{2\mu_0} \displaystyle\int_{\text{all space}} B^2 d\tau}$

Recall

$$W_E = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2\,d\tau\,, \quad W_B = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\,d\tau$$

$$W_E = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 \quad W_B = \frac{1}{2}LI^2$$

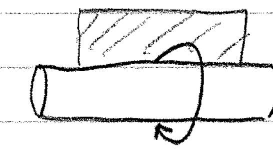
Ex Energy in coaxial cable...



(a) $W = \frac{1}{2}LI^2$

(b) $W = \frac{1}{2\mu_0}\int B^2\,d\tau$

$\vec{B}_{in} = \frac{\mu_0 I}{2\pi s}\hat{\phi}\,, \quad B_{out} = 0.$



(a) $$\Phi = LI = \int \vec{B}\cdot d\vec{a} = \int_a^b \frac{\mu_0 I}{2\pi s}\,\underbrace{d\vec{a}}_{l\,ds} = \frac{\mu_0 I l}{2\pi}\int_a^b \frac{ds}{s} = \frac{\mu_0 I l}{2\pi}\ln\left(\frac{b}{a}\right)$$

so $$\boxed{L = \frac{\mu_0 l}{2\pi}\ln\left(\frac{b}{a}\right)}$$

(b) $$W = \frac{1}{2\mu_0}\int B^2\,d\tau = \frac{1}{2\mu_0}\int\left(\frac{\mu_0 I}{2\pi s}\right)^2 s\,ds\,d\phi\,dz$$

$$= \frac{l}{2\mu_0}\int_a^b\int_0^{2\pi}\left(\frac{\mu_0 I}{2\pi s}\right)^2 s\,ds\,d\phi$$

$$= \dots = \int_a^b\left(\frac{\mu_0 I^2}{8\pi^2 s^2}\right)2\pi l s\,ds$$

$$= \boxed{W = \frac{\mu_0 I^2 l}{4\pi}\ln\left(\frac{b}{a}\right)}$$

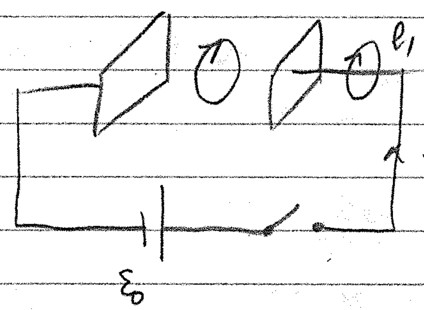
Nov 8, 2019

(E) The Displacement Current

$\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0$ $\quad \vec{\nabla} \cdot \vec{B} = 0$ $\quad \vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$ $\quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}.$

(Gauss' law) (no mag monopoles) Faraday's law Amp's law...

Max Exp



$\oint \vec{B} \cdot d\vec{l}_1 = \mu_0 I_{encl} = \mu_0 I$

$\oint \vec{B} \cdot d\vec{l}_2 = 0$???

$$E = \frac{\sigma}{\varepsilon_0} = \frac{1}{\varepsilon_0}\frac{Q}{A} \rightarrow \boxed{\frac{dE}{dt} = \frac{1}{\varepsilon_0 A}\frac{dQ}{dt} = \frac{1}{\varepsilon_0 A} I} \rightarrow \text{Displacement Current}$$

$$J = I/A \rightsquigarrow \boxed{J_d = \varepsilon_0 \frac{\partial E}{\partial t}}$$

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_{net} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}}$$

$\rightsquigarrow$ changing $\vec{E}$ fields induce $\vec{B}$ field

 changing $\vec{B}$ fields induce $-\vec{E}$ field

(F) Maxwell's Equations Finalized

$$\boxed{\begin{array}{lll}
① & \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 & \oint_{\partial V} \vec{E}\cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0} \\
② & \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} & \oint_{\partial A} \vec{E}\cdot d\vec{l} = \frac{-d}{dt}\int \vec{B}\cdot d\vec{a} = \frac{-d}{dt}\Phi_B \\
③ & \vec{\nabla}\cdot\vec{B} = 0 & \oint_{\partial V} \vec{B}\cdot d\vec{a} = 0 \\
④ & \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t} & \oint_{\partial A} \vec{B}\cdot d\vec{l} = \mu_0\int \vec{J}\cdot d\vec{a} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\underbrace{\int \vec{E}\cdot d\vec{a}}_{\Phi_E}
\end{array}}$$

Extra... magnetic monopoles...

$$\vec{\nabla}\cdot\vec{E} = \rho_e/\varepsilon_0 \qquad \vec{\nabla}\times\vec{E} = -\mu_0 J_m - \partial B/\partial t$$

$$\vec{\nabla}\cdot\vec{B} = \rho_m\mu_0 \qquad \vec{\nabla}\times\vec{B} = \mu_0 J_e + \mu_0\varepsilon_0\,\partial E/\partial t$$

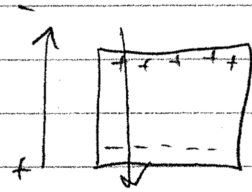
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_e}{r^2}\hat{r}, \quad \vec{B} = \frac{\mu_0}{4\pi}\frac{q_m}{r^2}\hat{r}$$

angular momentum $\rightarrow \frac{d\mathcal{L}}{d\tau} = \varepsilon_0\left[\vec{r}\times(\vec{E}\times\vec{B})\right]$

$$\boxed{\mathcal{L} = \frac{\mu_0}{4\pi}q_e q_m} \rightsquigarrow \text{in QM } \mathcal{L} = \frac{n\hbar}{2}, \quad n = 1, 2, 3, \ldots$$

ELECTRIC FIELDS IN MATTER

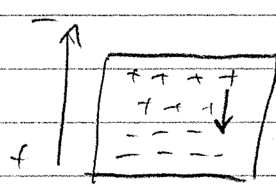
Conductor



$(E_{total} = 0)$

$E_{out} + E_{in} = E_{fin} = 0$

Insulator

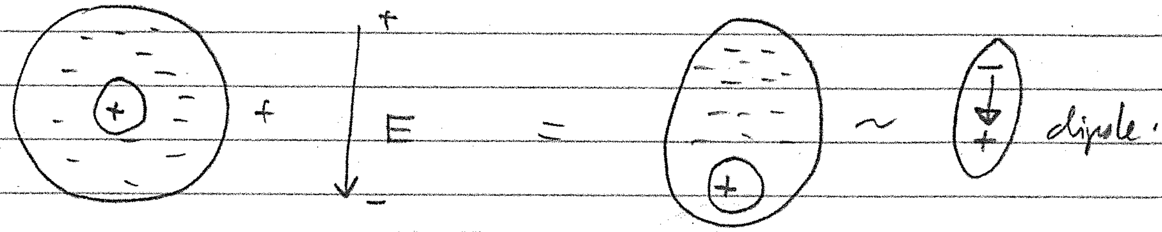


$E_{in} < E_{out}$

$E_{out} + E_{in} = E_{final} < E_{out} \neq 0$

(A) Polarization

(*) External $\vec{E}$ induces → a dipole moment in neutral atoms.



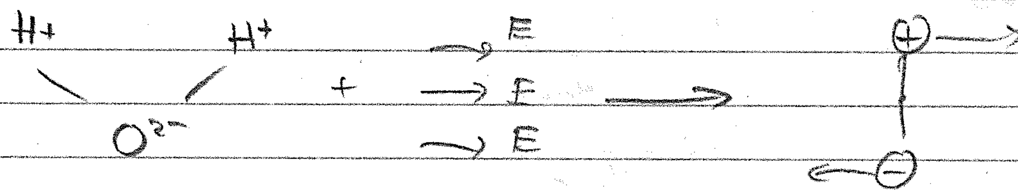
Dipole moment : $\boxed{\vec{p} = q\vec{d}}$

Typical dipole moment $\vec{p} = \alpha \vec{E}$

α → atomic polarizability.

$[\alpha] = C \cdot m^2/\text{volt}$

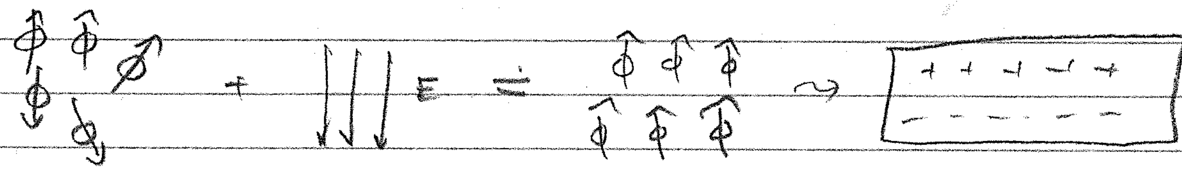
(*) External $\vec{E}$ apply a torque to molecules with existing dipole moment



$$\vec{\tau} = (\vec{r}_+ \times \vec{F}_+) + (\vec{r}_- \times \vec{F}_-)$$

$$|\vec{\tau}| = \frac{d}{2} qE + \frac{d}{2} qE \rightarrow |\vec{\tau}| = qdE$$

$$\boxed{\vec{\tau} = \vec{p} \times \vec{E}}$$



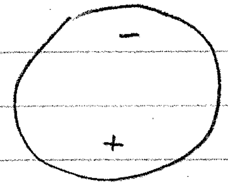
Nov 11, 2019

A) Polarization (con't)

Polarization : net dipole moment induced in insulator.

$\vec{p} \equiv$ dipole moment $\quad \vec{p} = n\vec{p}_{atom}$, n atoms.

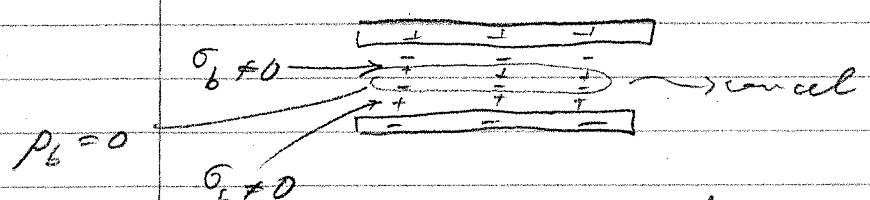
Polarization of uniformly polarized sphere $\vec{P} = N\vec{p} \Big/ \left(\frac{4}{3}\pi R^3\right)$



B) Bound charges

Types: surface charge (σ_b)
volume charge (ρ_b)

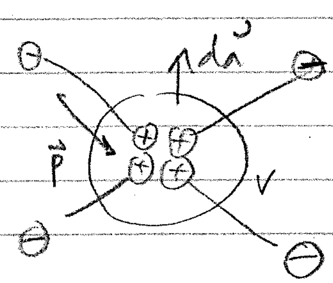
(1) Uniformly polarized (p = constant) $\rightarrow \rho_b = 0$



$$P = \frac{qd}{\text{volume}} = \frac{q}{\text{area}} = \sigma_b .$$

So $\boxed{\sigma_b = \vec{P}\cdot\hat{n} = P\cos\theta}$

(2) Non uniform polarization $\quad \sigma_b \neq 0, \ \rho_b \neq 0$



$Q = \int_V \rho_b \, d\tau \Rightarrow$ charge inside volume

$$\vec{p} = \frac{qd}{\text{vol}} = \frac{q}{\text{area}}$$

$$\oint \vec{P}\cdot d\vec{a} = (-P)\,\text{area} = \boxed{-Q}$$

So $\int_V \rho_b \, d\tau = -\oint \vec{P} \cdot d\vec{a}$

Apply divergence Thm $-\oint \vec{P} \cdot d\vec{a} = -\int_V (\vec{\nabla} \cdot \vec{P}) \, d\tau$

So $\boxed{\rho_b = -\vec{\nabla} \cdot \vec{P}}$ and $\boxed{\sigma_b = \vec{P} \cdot \hat{n}}$

c) E field for polarized object

Potential of single dipole... $V(r) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{\mathscr{r}}}{\mathscr{r}^2}$

$\vec{p} = \vec{P} \, d\tau$

$V(r) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\vec{P}(r) \cdot \hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau$

In terms of bound charges

$\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \rightarrow V = \frac{1}{4\pi\epsilon_0} \int_V \vec{P} \cdot \vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) d\tau$

$\rightarrow$ use product rule...

$\vec{\nabla} \cdot (f\vec{A}) = f(\vec{\nabla} \cdot \vec{A}) + \vec{A} \cdot (\vec{\nabla} f)$

So $V = \frac{1}{4\pi\epsilon_0} \left[\int_V \vec{\nabla} \cdot \left(\frac{\vec{P}}{\mathscr{r}}\right) d\tau - \int_V \frac{1}{\mathscr{r}} (\vec{\nabla} \cdot \vec{P}) \, d\tau \right]$

$= \frac{1}{4\pi\epsilon_0} \left[\oint_A \frac{\vec{P}}{\mathscr{r}} \cdot d\vec{a} - \int_V \frac{\vec{\nabla} \cdot \vec{P}}{\mathscr{r}} \, d\tau \right]$

Remember $\sigma_b = \vec{P} \cdot \hat{n} = \vec{P} \cdot d\vec{a}$

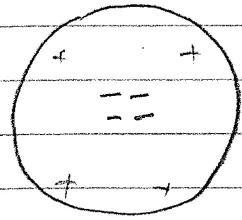
$\rho_b = -\vec{\nabla} \cdot \vec{P}$

$\frac{\sigma\ \text{surface}}{r}$ $\qquad$ $\frac{\rho\ \text{vol}}{r}$

So $\boxed{V(r) = \frac{1}{4\pi\epsilon_0}\oint \frac{\sigma_b}{r}\,da + \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho_b}{r}\,d\tau}$

② $E_{induced} = -\vec{\nabla}.V = \dots$

$\boxed{Ex}$



$\vec{P}(r) = k\vec{r}$

ⓐ Find bound charges $\sigma_b = \vec{P}\cdot\hat{n} = \vec{P}\cdot\hat{r}\Big|_{r=R}$

$= (k\vec{r}\cdot\hat{r})\Big|_{r=R} = kR$

So $\boxed{\sigma_b = kR}$

ⓑ Volume charge $\boxed{\rho_b = -\vec{\nabla}\cdot\vec{P} = \frac{-1}{r^2}\partial_r\left(r^2(kr)\right) = \dots = -3k}$

ⓒ Electric field ... $\oint \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\epsilon_0} = E.\,4\pi r^2$

when $r < R \rightarrow Q_{enc} = \int_V \rho_b\,d\tau = (-3k)\left(\frac{4}{3}\pi r^3\right)$

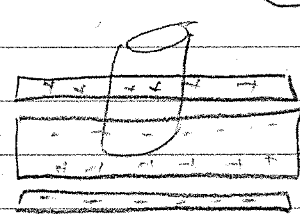
So $\boxed{\vec{E} = \frac{-k}{\epsilon_0}\,r\,\hat{r}}$ $\qquad (r < R)$

$r > R \dots$ $\boxed{E = 0}$

Nov 12, 2019 $\boxed{\text{D) Linear dielectric}}$

$\uparrow E_0 + \downarrow E_p = \vec{E}_{tot} < \vec{E}_0$

* Capacitor



$\leftarrow \sigma_f$

σ_b

$\leftarrow \sigma_p$

$\cancel{E_{out}}^{\,0}.A + E_{in}.A = \frac{Q_{encl}}{\epsilon_0}$

$$E_{in} A = \left(\frac{\sigma_f - \sigma_b}{\varepsilon_0}\right) A \Rightarrow \boxed{E_{in} = \frac{\sigma_f - \sigma_b}{\varepsilon_0}}$$ $\sigma_b = \vec{P}\cdot\hat{n} = p.$

$$\boxed{E_{in} = \frac{\sigma_f - p}{\varepsilon_0}}$$

→ $\boxed{\vec{P} = \varepsilon_0 \chi_e \vec{E}}$ → electric susceptibility (unitless)

→ linear dielectric...

So $E_{in} = \dfrac{\sigma_f - \varepsilon_0 \chi_e E_{in}}{\varepsilon_0} \Rightarrow E_{in}(1+\chi_e) = \dfrac{\sigma_f}{\varepsilon_0}$

$$\boxed{E_{in} = \frac{\sigma_f}{\varepsilon_0}\left(\frac{1}{1+\chi_e}\right)}$$

Permittivity: $\boxed{\varepsilon = \varepsilon_0(1+\chi_e)}$

$\boxed{\varepsilon_0 = \text{permittivity of vacuum}}$

Dielectric constant (relative permittivity)

$$\boxed{\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = 1 + \chi_e}$$ → 1 for vacuum, > 1 for insulators...

ELECTROSTATICS in insulators

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_{tot}}{\varepsilon_0} \xrightarrow{\text{insulator}} \vec{\nabla}\cdot\vec{E} = \frac{\rho_f + \underbrace{\rho_b}_{-\vec{\nabla}\cdot\vec{P}}}{\varepsilon_0} = \frac{\rho_f - \vec{\nabla}\cdot\vec{P}}{\varepsilon_0}$$

So $\vec{\nabla}\cdot\left(\vec{E} + \dfrac{\vec{P}}{\varepsilon_0}\right) = \dfrac{\rho_f}{\varepsilon_0}$

$\boxed{\vec{\nabla}\times\vec{E} = \vec{0}}$

For linear dielectric, $\vec{P} = \varepsilon_0 \chi_e \vec{E}$

So $\boxed{\vec{\nabla}\cdot\left((1+\chi_e)\vec{E}\right) = \rho_f/\varepsilon_0}$ → new Gauss' Law

E) $\boxed{\text{ELECTRIC DISPLACEMENT}}$

Gauss' Law: $\vec{\nabla}\cdot(\varepsilon_0\vec{E}+\vec{P}) = \rho_f$

Define $\boxed{\vec{D} = \varepsilon_0\vec{E}+\vec{P}}$ ⇝ new field...

$$\vec{D} = \varepsilon_0\vec{E} + \chi_e\varepsilon_0\vec{E} = \varepsilon_0(\vec{E})(1+\chi_e) = \varepsilon\vec{E} \ldots$$

$$\text{So}\quad \boxed{\vec{D} = \varepsilon\vec{E}} \rightarrow \boxed{\vec{\nabla}\cdot\vec{D} = \rho_f}$$

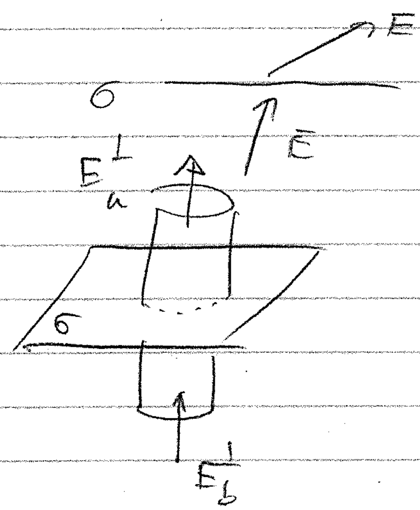
$$\vec{\nabla}\times\vec{D} = \varepsilon_0\underbrace{\cancel{\vec{\nabla}\times\vec{E}}}_{0} + \vec{\nabla}\times\vec{P}$$

$$\text{So}\quad \boxed{\vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P}}$$

We now have Gauss's laws:

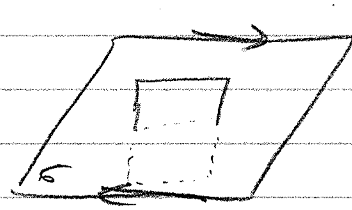
$$\boxed{\vec{\nabla}\cdot\vec{E} = \rho_{total}/\varepsilon_0} \Rightarrow \boxed{\begin{array}{l}\vec{\nabla}\cdot\vec{P} = -\rho_{bound}\\ \vec{\nabla}\cdot\vec{D} = \rho_{free}\end{array}}$$

F) $\boxed{\text{BOUNDARY CONDITIONS}}$



- perpendicular... $E^\perp_{above} - E^\perp_{below} = \frac{\sigma}{\varepsilon_0}$

- parallel component...



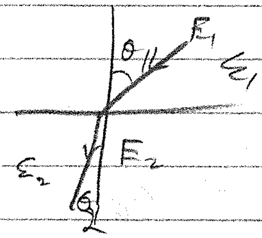
$$E^{\parallel}_{above} - E^{\parallel}_{below} = 0.$$

. <u>For displacement field</u> $\quad \vec{\nabla}\cdot\vec{D} = \rho_{free}$

$$\vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P}.$$

$$\boxed{D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f \,, \quad D^{\parallel}_{above} - D^{\parallel}_{below} = P^{\parallel}_{above} - P^{\parallel}_{below}}$$

Ex show



then $\dfrac{\tan\theta_1}{\tan\theta_2} = \dfrac{\varepsilon_1}{\varepsilon_2}$

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f = 0 \Rightarrow D^{\perp}_{above} = D^{\perp}_{below}$$

$$\vec{D} = \epsilon\vec{E} \Rightarrow \varepsilon_1 \vec{E}_1^{\perp} - \varepsilon_2 \vec{E}_2^{\perp} = \sigma_f = 0$$

so $\varepsilon_1 \vec{E}_1^{\perp} = \varepsilon_2 \vec{E}_2^{\perp}$.

With $\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$, we have
$E^{\perp}_{above} = E_1 \cdot \cos\theta_1$
$E^{\parallel}_{above} = E_1 \cdot \sin\theta_1$

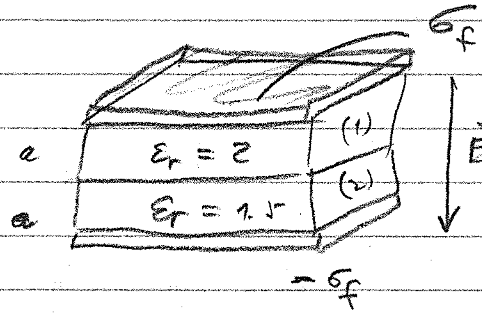
so $\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2$

$E_1 \sin\theta_1 = E_2 \sin\theta_2$

so $\boxed{\dfrac{\tan\theta_1}{\tan\theta_2} = \dfrac{\varepsilon_1}{\varepsilon_2}}$

~~ENERGY IN DIELECTRICS~~

Nov 14, 2019 $\boxed{\text{Ex}}$ 4.18 Griffiths.



$\varepsilon_r = \varepsilon/\varepsilon_0 = 1 + \chi_e$.

(a) $\vec{D} = ?$ $\quad \cancel{D_{out}}^{\,0} A - D_{in} A = \sigma_f A$

$\rightarrow \boxed{D_1 = -\sigma_f \hat{z}}$

$\boxed{\vec{D}_2 = -\sigma_f \hat{z}}$

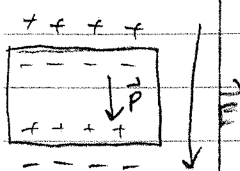
(b) $E_1 = D_1/\varepsilon_1$, $E_2 = D_2/\varepsilon_2$.

$= D_1/2\varepsilon_0$, $= D_2/1.5\varepsilon_0$ $\qquad \varepsilon_r = 1 + \chi_e$

$= \dfrac{\sigma_f}{2\varepsilon_0}(-\hat{z})$, $= \dfrac{2\sigma}{3\varepsilon_0}(\hat{z})$

(c) Find $\vec{P}$. $|\vec{P}| = \chi_e \varepsilon_0 \vec{E}$ $= \chi_e \varepsilon_0 \dfrac{\sigma}{\varepsilon} = \chi_e \cancel{\varepsilon_0} \cdot \dfrac{\sigma}{\cancel{\varepsilon_0}\varepsilon_r}$

$= \dfrac{\sigma \chi_e}{\varepsilon_r} = \dfrac{\sigma \chi_e}{\varepsilon_r} = \dfrac{\sigma(\varepsilon_r - 1)}{\varepsilon_r}$



So $\boxed{\vec{P} = \dfrac{\sigma_f(\varepsilon_r - 1)}{\varepsilon_r}(-\hat{z})} \rightarrow \boxed{\begin{array}{l} \vec{P}_1 = \sigma_f\left(\frac{1}{2}\right)(-\hat{z}) \\ \vec{P}_2 = \sigma_f \frac{1}{3}(-\hat{z}) \end{array}}$

$\vec{P}$ created by σ_b.

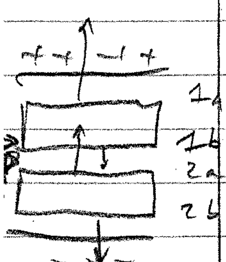
(d) Find V. $V = \int \vec{E} \cdot d\vec{l} = E_1 \cdot d_1 + E_2 \cdot d_2$

$= E_1 a + E_2 a = a(E_1 + E_2)$

$= a\left(\dfrac{\sigma}{2\varepsilon_0} + \dfrac{2\sigma}{3\varepsilon_0}\right) = \boxed{\dfrac{7a\sigma}{6\varepsilon_0}}$

$C = \dfrac{Q}{V} = \dfrac{\sigma A}{(7\sigma A/6\varepsilon_0)} = \boxed{\dfrac{6\varepsilon_0 A}{7a}}$ $\rightarrow$ goes up when there's insulator present vs vacuum

(e) Find bound charge $\boxed{\rho_b = -\vec{\nabla} \cdot \vec{P} = 0}$

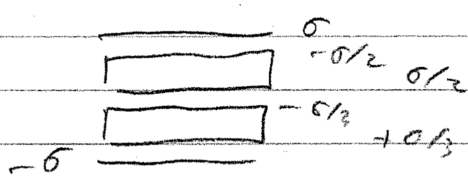


$\sigma_b = \vec{P} \cdot \hat{n}$

$\sigma_{1a} = \dfrac{-\sigma_f(\varepsilon_r - 1)}{\varepsilon_r} = -\sigma_{1b}$, $\quad \frac{1}{2}$

$\sigma_{2a} = \dfrac{-\sigma_f(\varepsilon_r - 1)}{\varepsilon_r} = -\sigma_{2b}$. $\quad \frac{1}{3}$

(f) Refind $\vec{E}$.



$$\begin{cases} E_1 A = \dfrac{A\sigma}{\varepsilon_0} - \dfrac{A\sigma}{2\varepsilon_0} = \dfrac{A\sigma}{2\varepsilon_0} \\ E_2 A = \dfrac{-A\sigma}{\varepsilon_0} + \dfrac{A\sigma}{3\varepsilon_0} = \dfrac{-2A\sigma}{3\varepsilon_0} \end{cases}$$

(G) $\boxed{\text{ENERGY IN DIELECTRICS}}$

Energy in cap $W = \frac{1}{2}CV^2$.

Insulator: $C = \varepsilon_r C_{vac} \ldots \rightarrow \boxed{W = \frac{1}{2}\varepsilon_r C_{vac} V^2}$

Energy in E field...

$$W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E_{vac}^2\, d\tau \rightarrow \boxed{\frac{\varepsilon_0}{2}\int_{\text{all space}} \varepsilon_r E^2\, d\tau}$$

Or $\varepsilon_r = \varepsilon/\varepsilon_0 \rightarrow W = \frac{\varepsilon_0}{2}\int (\varepsilon E)\cdot E\, d\tau$

$$= \frac{1}{2}\int (\varepsilon E)\cdot E\, d\tau$$

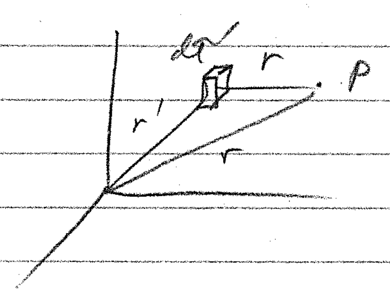
$$\rightarrow \boxed{W = \frac{1}{2}\int_{\text{all space}} \vec{D}\cdot\vec{E}\, d\tau}$$

Nov 15, 2019

EXAM 2 review...

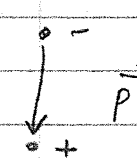
Electric Multipole Expansion

$$V(r) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta)\rho(r')\, d\tau'$$

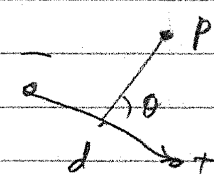


* Dipole moment

$$\vec{p} = \int \vec{r}'\rho(r')\, d\tau'$$



Point charges ⇒ $\vec{p} = \sum_{n=1}^{\infty} q_n \vec{r}_n$.

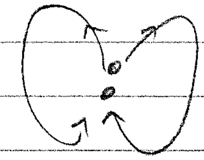


~~Two point charges..~~

$$\boxed{V_{dip}(r) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}} \rightarrow \boxed{V_{dip}(r) = \frac{1}{4\pi\varepsilon_0}\frac{qd\cos\theta}{r^2}}$$

E field.. for dipole...

$$\boxed{E_{dip} = \frac{p}{4\pi\epsilon_0}\frac{1}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)}$$



Magnetostatics Lorentz force law: $F_{mag} = Q(\vec{v}\times\vec{B})$

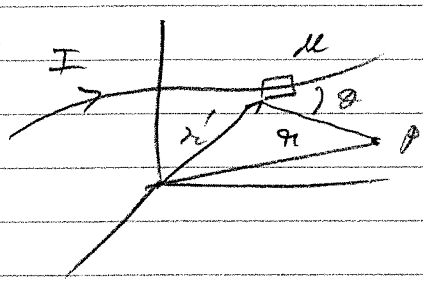
Current densities
→ line current $\vec{I} = \frac{dq}{dt}$
→ surface current: $\vec{k} = \frac{dI}{dl} = \sigma\vec{v} \Rightarrow I = \int \vec{k}\cdot d\vec{l}$
→ volume current $\vec{j} = \frac{dI}{dA} = \rho\vec{v} \Rightarrow \vec{I} = \int \vec{j}\cdot d\vec{A}$

Continuity Eqn

$$\boxed{\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}}$$

For magnetostatics → $\vec{\nabla}\cdot\vec{J} = 0$.

Biot-Savart law



$$\boxed{\vec{B}(r) = \frac{\mu_0 I}{4\pi}\int\frac{d\vec{l}\times\hat{r}}{\mathscr{r}^2}}$$

$$= \frac{\mu_0 I}{4\pi}\int\frac{\sin\theta\, dl}{\mathscr{r}^2}\hat{\phi}\ldots$$

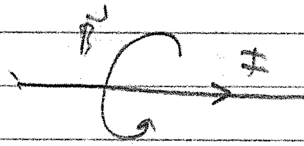
Surface currents...
$$\vec{B}(r) = \frac{\mu_0}{4\pi}\int_A\frac{\vec{k}(r')\times\hat{\mathscr{r}}}{\mathscr{r}^2}da'$$

Volume currents...
$$\vec{B}(r) = \frac{\mu_0}{4\pi}\int_V\frac{\vec{J}(r')\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

Maxwell's Eqn for magnetostatics

$$\boxed{\vec{\nabla}\cdot\vec{B} = 0 \quad , \quad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}.}$$

Ampère's law: $\boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{\text{enc}}}$

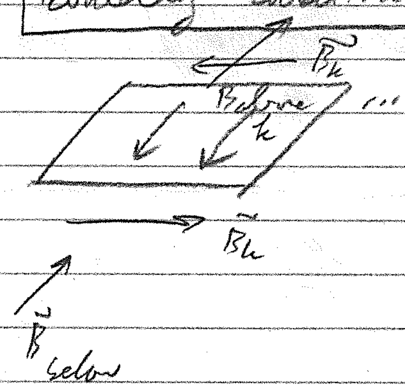
Infinite wire ... 

$$\boxed{B(s) = \frac{\mu_0 I}{2\pi s}\hat{\phi}}$$

Magnetic Vector potential

$$\begin{cases} \vec{\nabla}\times\vec{A} = \vec{B}. \\ \vec{A}(r) = \dfrac{\mu_0}{4\pi}\displaystyle\int \dfrac{J(r')}{\mathscr{r}}\, d\tau' \end{cases}$$

Boundary conditions (magnetics)



$\vec{B}_k$ only affects $\parallel B$.

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}$$

$$B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 k$$

$$\boxed{\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0(\vec{k}\times\hat{n})}$$

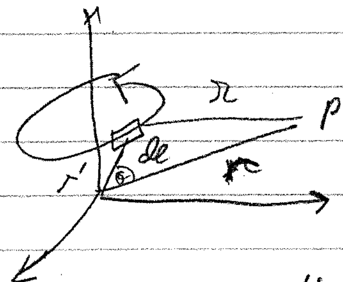
$$\boxed{\vec{A}_{\text{above}} - \vec{A}_{\text{below}} = 0} \rightarrow \vec{A} \text{ continuous}$$

$$\frac{\partial \vec{A}_{\text{above}}}{\partial n} - \frac{\partial \vec{A}_{\text{below}}}{\partial n} = -\mu_0 \vec{k}$$

Many multipole expansions

$$\vec{A}(r) = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos\theta)\, dl'$$

$$\boxed{A_{mm}(n=0) = 0}$$



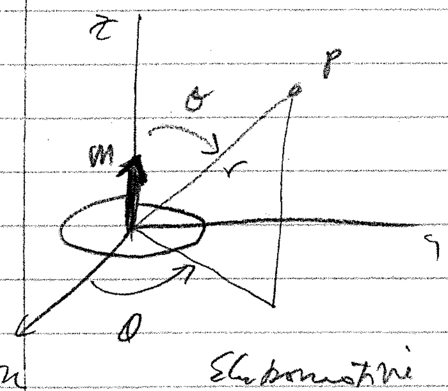
$\theta = (\vec{r}', \vec{r})$

magnetic dipole moment:

$$\boxed{\vec{m} = I\int d\vec{a} = I\vec{a}}$$

Potential for magnetic dipole moment...

$$\boxed{\vec{A}_{dip}(r) = \frac{\mu_0}{4\pi} \frac{\vec{m}\times\hat{r}}{r^2}} = \frac{\mu_0 m \sin\theta}{4\pi r^2}\hat{\phi}$$



$$\boxed{\vec{B}_{dip}(r) = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta}\right)}$$

like Electric dipole ...

Electromotive force $\quad \mathcal{E} = \oint F/q \cdot dl = \oint E\cdot dl = V$

current driven: $\boxed{\mathcal{E}mf = IR}$

Power needed to drive a current. $\quad P = I^2 R$

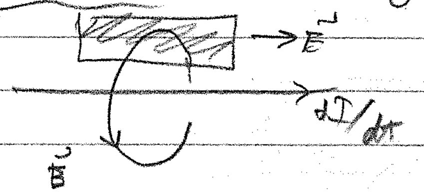
Faraday's law ... $\quad \boxed{\vec{\nabla}\times\vec{E} = \frac{-dB}{dt}}$

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\Phi_B = \mathcal{E}mf$$

$$\boxed{\mathcal{E} = -\frac{d\Phi_B}{dt}}$$

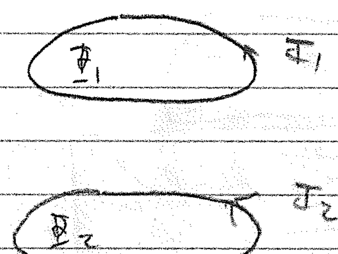
Lenz' law: ~~app~~ → nature wants to maintain mag flux

Apply Faraday's law: $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\Phi_B = \frac{-d}{dt}\int \vec{B}\cdot d\vec{a}$



realize loop $\parallel \vec{E} \Rightarrow \vec{a} \perp \vec{B}$.

Inductance & Mutual Inductance



$M = \Phi_2/I_1 = \Phi_1/I_2 \ldots$

$$\boxed{M = \frac{\mu_0}{4\pi}\oint\oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{\mathscr{r}}}$$

Self-inductance: $\boxed{L = \Phi_1/I_1}$ ⇝ fights change in Φ → back emf

$$\boxed{\mathcal{E} = -L\frac{dI}{dt}}$$ ⇝ back emf.

Energy & Magnetic fields $\boxed{W = \frac{1}{2}LI^2}$

or $$\boxed{W = \frac{1}{2}\int_V (\vec{A}\cdot\vec{J})\,d\tau = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\,d\tau}$$

Displacement current $\boxed{\vec{J}_d = \varepsilon_0 \frac{\partial E}{\partial t}}$

Modified Amp's Law ⇒ $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$

Polarization $\boxed{\vec{P} = \varepsilon_0 \chi_e \vec{E} = \text{dipole moment/unit volume}}$

for linear dielectric

Bound charges $\left\{ \begin{array}{l} \sigma_b = \vec{P}\cdot\hat{n} \\ \rho_b = -\vec{\nabla}\cdot\vec{P} \end{array} \right\}$ → sum to zero for a neutral insulator.

Permittivity $\boxed{\varepsilon = \varepsilon_0 (\underbrace{1+\chi_e}_{\varepsilon_r})}$

ε_r → relative permittivity...

For insulators, $\boxed{\varepsilon_r > 1}$

Displacement field $\boxed{\vec{D} = \varepsilon_0 \vec{E} + \vec{P}}$

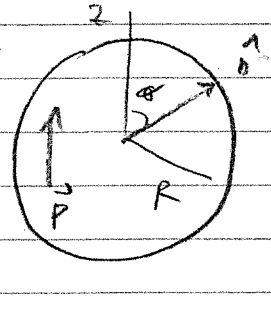
Divergence summary

$$\boxed{\begin{array}{l} \vec{\nabla}\cdot\vec{E} = \rho_{total}/\varepsilon_0 \\ \vec{\nabla}\cdot\vec{P} = -\rho_b \\ \vec{\nabla}\cdot\vec{D} = \rho_{free} \end{array}}$$

Energy in dielectric

$$W = \frac{1}{2}\varepsilon_r C V^2 = \frac{\varepsilon_0}{2}\int \varepsilon_r E^2 \, d\tau$$

Nov 18, 2019 $\boxed{Ex}$ **Uniformly polarized sphere.**

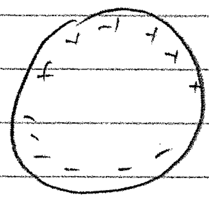


What is E?

① **Bound charges** $\sigma_b = \vec{P}\cdot\hat{n} = P\hat{z}\cdot\hat{n}$
$= P\cos\theta$

Volume bound charge $\rho_b = -\vec{\nabla}\cdot\vec{P} = 0$

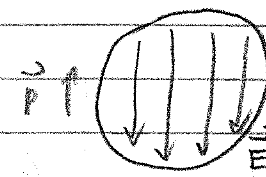
(b) <u>E field</u> $\quad \sigma_b = P\cos\theta \rightarrow$ Ex 3.9



$$V(r,\theta) = \begin{cases} \dfrac{P}{3\varepsilon_0} r\cos\theta & \text{(inside)} \\ \dfrac{P}{3\varepsilon_0} R^3/r^2 \cos\theta & \text{(outside)} \end{cases}$$

$\Rightarrow$ <u>Inside</u>, $V(r,\theta) = \dfrac{P}{3\varepsilon_0} r\cos\theta = \dfrac{P}{3\varepsilon_0} z$

$$\Rightarrow \boxed{\vec{E}_{in} = -\vec{\nabla}\cdot V = \frac{-P}{3\varepsilon_0}\hat{z}}$$

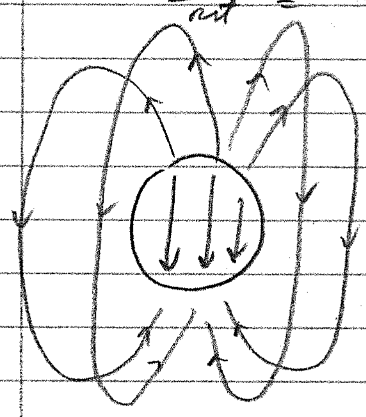


$\Rightarrow$ <u>Outside</u>, $V(r,\theta) = \dfrac{P}{3\varepsilon_0}\dfrac{R^3}{r^2}\cos\theta \ldots$ $\quad \boldsymbol{p} = (\tfrac{4}{3}\pi R^3)\vec{P}$? dipole moment

$$\Rightarrow \vec{E}_{out} = -\vec{\nabla}V = -\vec{\nabla}\left\{\frac{1}{4\pi\varepsilon_0}\frac{p}{r^2}\cos\theta\right\}$$

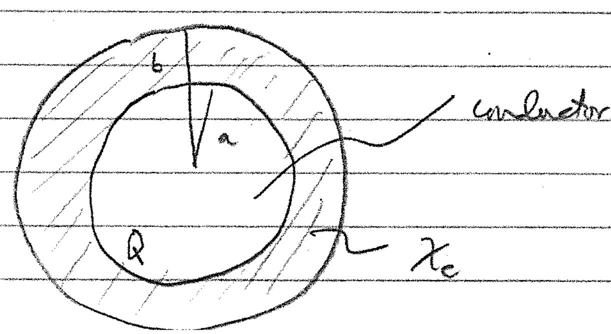
$$= -\vec{\nabla}\left\{\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}\right\}$$ perfect ... dipole ...

$E_{out} = -\vec{\nabla}V_{out} = \ldots$ dipole ...



$\boxed{\text{Ex}}$ <u>Problem 4.26</u>

$$W = \frac{1}{2}\int_{\text{all space}} \vec{D}\cdot\vec{E}\, d\tau$$



$$E = \begin{cases} ? & r < a \\ ? & a < r < b \\ ? & r > b \end{cases}$$

$$E = \begin{cases} 0 & r<a \\ ? \longrightarrow Q/4\pi\varepsilon r^2 \longrightarrow \text{scaled differently} \\ \frac{Q}{4\pi\varepsilon_0 r^2} & r>b \end{cases}$$

$$D = \begin{cases} 0 & r<a \\ Q/4\pi r^2 \\ Q/4\pi r^2 & r>b \end{cases}$$

So, ... $W = \frac{1}{2}\int_{(1)} \vec{0}\cdot\vec{0}\, d\tau + \frac{1}{2}\int_{(2)} \vec{D_2}\cdot\vec{E_2}\, d\tau + \frac{1}{2}\int_{(3)} \vec{D_3}\cdot\vec{E_3}\, d\tau$

$$= \frac{4\pi}{2}\int_a^b \frac{Q^2}{(4\pi)^2} \cancel{\frac{1}{\varepsilon_0}} \frac{1}{\varepsilon} \frac{r^2}{r^4} dr + \frac{4\pi}{2}\int_b^\infty \frac{1}{\varepsilon_0}\frac{r^2}{r^4}\left(\frac{Q}{4\pi}\right)^2 dr$$

$$= \frac{Q^2}{8\pi\varepsilon}\left\{\frac{1}{a} - \frac{1}{b}\right\} + \frac{Q^2}{8\pi\varepsilon_0}\left\{\frac{1}{b} - 0\right\}$$

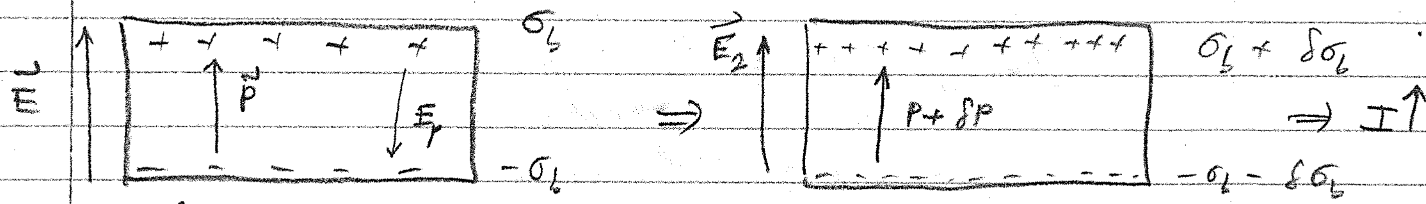
$$= \frac{Q^2}{8\pi}\left\{\frac{1}{\varepsilon a} - \frac{1}{b}\left\{\frac{1}{\varepsilon} - \frac{1}{\varepsilon_0}\right\}\right\}$$

$\varepsilon = \varepsilon_r \varepsilon_0$
$= (1+\chi_e)\varepsilon_0$

$$= \frac{Q^2}{8\pi\varepsilon_0}\left\{\frac{1}{(1+\chi_e)}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{1}{b}\right\}$$

$$\boxed{W = \frac{Q^2}{8\pi\varepsilon_0(1+\chi_e)}\left\{\frac{1}{a} + \frac{\chi_e}{b}\right\}}$$

ELECTRODYNAMICS IN MATTER



current inside $\frac{dq}{dt} = \frac{d\sigma_b da}{dt} = \frac{dP}{dt} da$

<u>Define</u> "Polarization current density"

$$\boxed{J_p = \frac{dP}{dt}} \Rightarrow dI = J_p da \equiv \frac{dP}{dt} da \qquad \varepsilon_0 \chi_e \vec{E}$$

<u>Amp's Law in matter</u> $\boxed{\vec{\nabla} \times \vec{B} = \mu_0 J_{free} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} + \mu_0 \frac{\partial P}{\partial t}}$

But, in linear dielectric...

$$\boxed{\varepsilon_0 \vec{E} + \vec{P} = \varepsilon E = D.}$$

So, $\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon \frac{\partial E}{\partial t} + \mu_0 J_{free}}$

But also know $D = \varepsilon E \Rightarrow \vec{\nabla} \times \vec{B} = \mu_0 \underbrace{\frac{\partial D}{\partial t}}_{\text{displacement current}} + \mu_0 J_{free}$

where $\boxed{\mu_0 J_{free}}$ free charge current density, $\boxed{\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}}$ Vacuum displacement current

and $\boxed{\mu_0 \frac{\partial P}{\partial t}} \rightarrow$ Bound charge current density.

Nov 19, 2019

MAGNETISM IN MATTER

1. Ferromagnetism ⇒ (permanent magnet)
   - → strongly attracted
   - → matter with baked in B field...
   - (Fe, Ni, Co)
   - → no temp dependence (until very high temps)

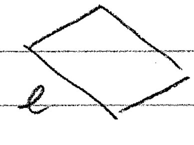
2. Paramagnetism → weakly attracted
   - → 10^3 weaker than ferromag
   - → all substance with an odd number of electrons.
   - → temp dependence...

3. Diamagnetism ⇒ weakly repelled
   - → 10^6 weaker than ferromagnetism
   - → true for all matter...
   - → only noticeable with even # electrons...
   - ⇒ No temp dependence.

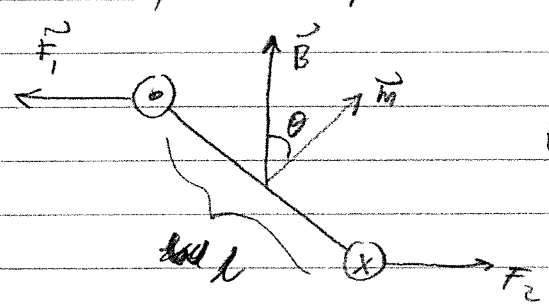
(*) All effects due to ... → spinning electrons (spin)
→ orbiting electrons (L)

Paramagnetism

→ due to spin



Torque on Dipole...



$$F = q(\vec{v} \times \vec{B}) = qvB = IwB$$

$$N = \tau = lF\sin\theta$$
$$= BlIw\sin\theta\,\hat{x}$$
$$= B(Ilw)\sin\theta$$

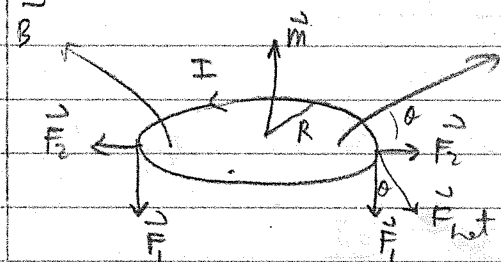
$$\rightarrow \boxed{\vec{\tau} = \vec{m} \times \vec{B}}$$

Electric dipoles $\boxed{\vec{\tau} = \vec{p}\times\vec{E}}$

Net force on dipole: $F = I\left(\oint d\vec{l}\times\vec{B}\right)$

If $\vec{B}$ uniform ... $= I\left(\oint d\vec{l}\right)\times\vec{B} = \vec{0}$

If $\vec{B}$ not uniform ...



$F_1 = F_{net}\cos\theta = IB\cos\theta\, dl$

$\Rightarrow F = (2\pi R)\, IB\cos\theta$

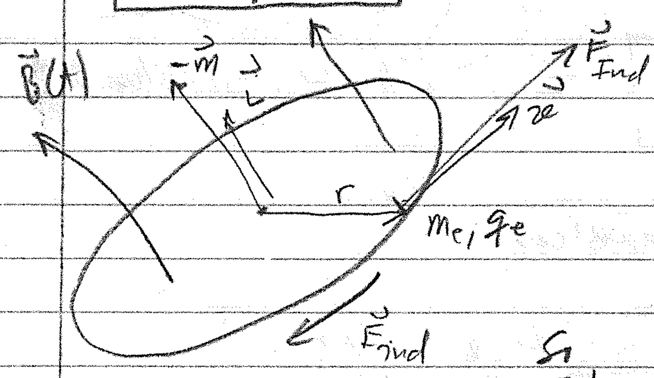
$\vec{\nabla}(I\,\pi r^2) = 2I\pi r$

$\vec{m}\cdot\vec{B} = mB\cos\theta$

$\Rightarrow \boxed{F = \vec{\nabla}(\vec{m}\cdot\vec{B})}$ → potential $\quad \vec{m}\uparrow\uparrow\vec{B} \Rightarrow$ attractive

$\vec{m}\uparrow\downarrow\vec{B} \Rightarrow$ repelsant

$\boxed{\text{Diamagnetism}}$ → due to orbiting electrons ...



Angular momentum $\vec{L} = m_e v r$

Magnetic dipole moment $\vec{m} = IA = I\pi r^2$

Current $I = \dfrac{e}{T} = \dfrac{ev}{2\pi r}$

So,

$$\vec{m} = I(\pi r^2) = q\left(\frac{v}{2\pi r}\right)(\pi r^2) \Rightarrow \boxed{\vec{m} = \frac{qvr}{2} = \frac{q}{2m_e}\vec{L}}$$

$-e$

orbit → $\boxed{\vec{m}_e = \dfrac{-e}{2m_e}\vec{L}}$ → dipole moment for orbital motion (e^-)

For spinning e^-, $\boxed{\vec{m}_e = \dfrac{-e}{m_e}\vec{L}_{spin}}$

$$\oint \vec{E}\cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

$$E(2\pi r) = \frac{-d}{dt}(B\pi r^2) \Rightarrow E = \frac{-r}{2}\frac{\partial B}{\partial t}$$

→ Force on electron: $\boxed{\vec{F} = -q_e\vec{E}}$ ⇒ results in a τ

$$\boxed{\vec{\tau} = \vec{r}\times\vec{F} = rF = -q_e\vec{E}r}$$

Torque

$$\tau = \frac{d\vec{L}}{dt} = -q_e E r \Rightarrow \ = -q_e\left(\frac{-r}{2}\frac{\partial B}{\partial t}\right)r$$

$$\Rightarrow \frac{dL}{dt} = \frac{q_e r^2}{2}\frac{\partial B}{\partial t}$$

$$\Rightarrow \boxed{\Delta\vec{L} = \frac{q_e r^2}{2}\vec{B}}$$

$$\boxed{\Delta\vec{m} = \frac{-q_e}{2m_e}\Delta\vec{L}}$$

So $\boxed{\Delta\vec{m} = \frac{-q_e^2 r^2}{4m_e}\vec{B}}$ ⇒ $\vec{m}\downarrow\uparrow\vec{B}$ ~ repellent

Diamagnetism → repellent force ...

Nov 21, 2019

$\boxed{\text{MAGNETIZATION} \sim \text{BOUND CURRENT}}$

magnetization

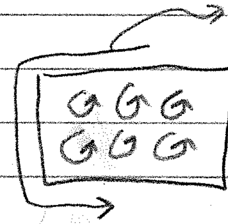
$\vec{m}$: dipole moment per atom
N : atoms per unit volume
$\vec{M}$: $N\vec{m}$ = mag dipole moment per unit volume

→ Polarization $\vec{P} = N\vec{p}$. For uniformly magnetized obj

$$\vec{M} = \vec{m} \Big/ \tfrac{4}{3}\pi R^3$$

$\boxed{\text{Bound currents}}$

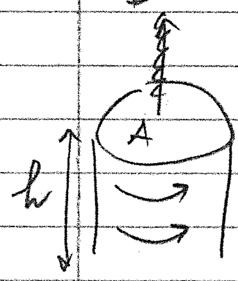
① <u>Uniformly magnetized matter ...</u>



→ surface current ...

→ inner currents cancel out.

* Volume currents → 0

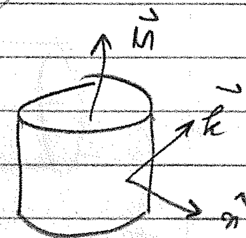


<u>surface current</u>

$k = I/h$. How to relate k & M?

$$M = m/\text{volume} = \frac{m}{Ah} = \frac{I(\text{area})}{Ah} = \frac{I}{h} = k$$

<u>So</u> $\boxed{k = I/h = M}$



Bound current $\boxed{\vec{k} = \vec{M} \times \hat{n}}$

Similar to E: $\boxed{\sigma_b = \vec{P} \cdot \hat{n}}$

② <u>Non-uniform magnetization</u>

↳ non-zero volume current ...

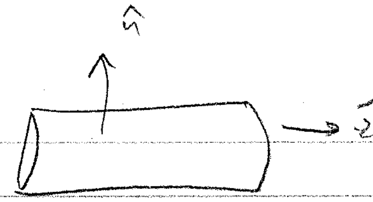
$\boxed{\vec{J}_b = \vec{\nabla} \times \vec{M}}$ → analogous to $\boxed{\rho_b = \vec{\nabla} \cdot \vec{P}}$

⊛ <u>$\vec{B}$ field for magnetized object → amp's Law ...</u>

$$\begin{cases} \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \\ \text{where } I_{enc} = \int (J_f + J_b) \cdot d\vec{a} \end{cases}$$

$$I_{total} = \int (J_f + J_b) \cdot d\vec{a} + \int (k_f + k_B) \cdot d\vec{l}$$

Nov 22, 2019 $\boxed{\text{Ex}}$ 6.12



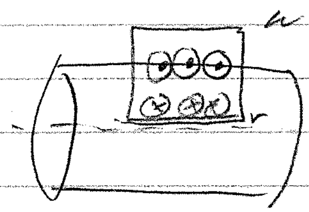
$\vec{M} = M_0 r \hat{z}$

(a) $\vec{J}_b = \nabla \times \vec{M}$, $\vec{K}_b = \vec{M} \times \hat{n}\big|_R$

cylindrical coordinates...

$$\boxed{\begin{aligned} \vec{J}_b &= -\partial_r (M_z)\hat{\phi} = -M_0 \hat{\phi} \\ \vec{K}_b &= \vec{M} \times \hat{r}\big|_R = M_0 R \hat{\phi} \end{aligned}}$$ opposite directions...

(b) Find $\vec{B}$ field.



Ampère's Law $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$

$\vec{B} = \vec{0}$ outside...

Inside: $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}$

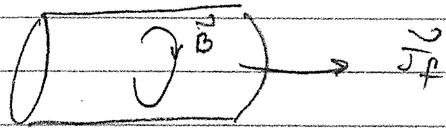
$$B(r)l - \underbrace{B(a)}_{0}l = \mu_0 \left\{ \int \vec{K}_b \cdot d\vec{l} + \int \vec{J}_b \cdot d\vec{A} \right\}$$

$$B(r)\cancel{l} = \mu_0 \{ K_b l - M_0 l (a-r) \}$$
$$= \mu_0 \{ M_0 R \cancel{l} - M_0 \cancel{l} (R-r) \}$$

$$B(r) = \mu_0 M_0 (R - (R-r)) = \mu_0 M_0 r$$

So $\boxed{\vec{B}(r) = \mu_0 M_0 r \hat{z}}$

THE AUXILIARY FIELD



Amp's law $\quad \vec\nabla\times\vec B = \mu_0 J_{total} = \mu_0 (J_f + \underbrace{J_b}_{\vec\nabla\times\vec M})$

$$\rightarrow \boxed{\vec\nabla\times\frac{\vec B}{\mu_0} = \vec J_f + \vec\nabla\times\vec M}$$

$$\boxed{\vec\nabla\times\left(\frac{\vec B}{\mu_0} - \vec M\right) = \vec J_f}$$

Define Auxiliary field ...

$$\boxed{\vec H \equiv \frac{1}{\mu_0}\vec B - \vec M}$$

analogue of displacement field ...

Amp's law for H:

$$\boxed{\vec\nabla\times\vec H = J_{free} \; ; \qquad \oint \vec H\cdot d\vec l = I_{free}}$$

$$\vec\nabla\times\vec B = \mu_0 \vec J_{total} = \mu_0(\vec J_{free} + \vec J_{bound})$$
$$\vec\nabla\times\vec M = J_{bound}$$
$$\vec\nabla\times\vec H = J_{free}$$

Notes on H: (1) units: $[\vec H] = [\vec M] = \frac{[\vec B]}{\mu_0} = \frac{T}{N/A^2} = \boxed{\frac{A}{m}}$

(2) Curl of H... does not uniquely determine H...

Div of H

$$\boxed{\vec\nabla\cdot\vec H = \vec\nabla\cdot\left\{\frac{\vec B}{\mu_0} - \vec M\right\} = -\vec\nabla\cdot\vec M}$$

If $J_f = \vec 0$, then $\vec\nabla\times\vec H = 0$, but $\vec H \neq 0$.

(3) Symmetric systems $\boxed{\vec\nabla\cdot\vec M = 0} \Rightarrow \boxed{\vec\nabla\cdot\vec H = 0}$

$\boxed{\vec\nabla\times\vec H = \vec J_f}$ $\vec H$ uniquely determined

MAGNETIC SUSCEPTIBILITY & PERMEABILITY

⊛ Magnetic susceptibility ... χ_m.

⊛ $\vec{M} \propto \vec{H}$ via $\boxed{\vec{M} = \chi_m \vec{H}}$ → like $\boxed{\vec{P} = \chi_E \vec{E} \epsilon_0}$

↑ true for "linear media"

⊛ Susceptibility

$\chi_m < 0$ for diamagnetic material (M opposes)
$\chi_m > 0$ for paramagnetic material (M amplifies)

$|\chi_m| \simeq 10^{-5}$ for most material - is unitless...

⊛ Permeability $\vec{B} = \mu_0 (\vec{H} + \vec{M})$
$= \mu_0 (\vec{H} + \chi_m \vec{H})$ → linear

So $\boxed{\mu \equiv \mu_0 (1 + \chi_m)}$ → $\vec{B} = \mu \vec{H}$

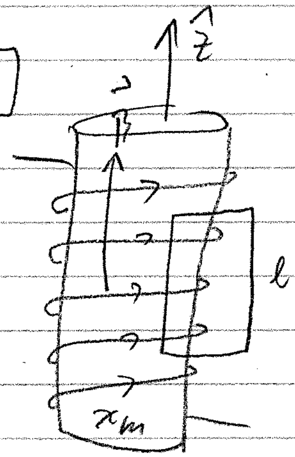
like $\vec{D} = \epsilon \vec{E}$

Relative permeability ... $\boxed{\mu_r = \frac{\mu}{\mu_0} = 1 + \chi_m}$

In vacuum ⇒ $\chi_m = 0$ ⇒ $\boxed{\mu_0 \Rightarrow \text{permeability of free space}}$

Jun 25, 2019

Ex



$$\oint \vec{H} \cdot d\vec{l} = I_{enc}^{free}$$

$$\left\{ \begin{array}{l} \vec{H}_{out} = 0 \hat{z} \\ \vec{H}_{in} = \frac{IN}{l} = In \hat{z} \end{array} \right\}$$

$$\boxed{\begin{array}{l} \vec{B}_{out} = 0 \hat{z} \\ \vec{B}_{in} = \mu_0 (1+\chi_m) \vec{H}_{in} = (1+\chi_m) In \hat{z} \end{array}}$$

$$\boxed{\vec{B}_{in} = \mu_0 (1 + \chi_m) n I \hat{z}}$$

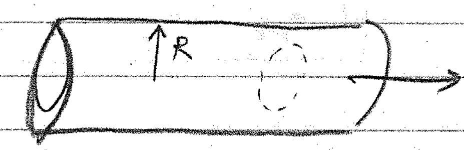
$$\left\{ \begin{array}{l} \text{If paramagnetic then } \chi_m > 0 \Rightarrow \vec{B} \text{ increases (enhanced)} \\ \text{If dia- then } \chi_m < 0 \Rightarrow \vec{B} \text{ reduced} \end{array} \right\}$$

<u>Next</u> $\boxed{\text{Bound surface current}}$

$$\vec{k}_B = \vec{M} \times \hat{n} = \chi_m \vec{H} \times \hat{n} = \chi_m I n\, \hat{z} \times \hat{s} = \chi_m I n \hat{\phi}$$

$$\left\{ \begin{array}{l} \text{If } \chi_m > 0 \Rightarrow \vec{k}_B \uparrow\uparrow \vec{I} \rightarrow \text{enhances } \vec{B} \\ \text{If } \chi_m < 0 \Rightarrow \vec{k}_B \downarrow\uparrow \vec{I} \rightarrow \text{reduces } \vec{B} \end{array} \right\}$$

$\boxed{Ex}$



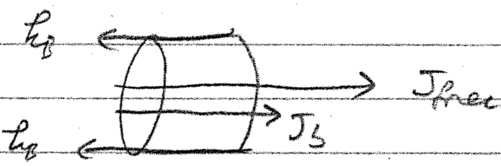
$\vec{J}_{free} = J_0 \hat{z}$ Find $B/H/M$, J_b, k_B.

$$\boxed{\vec{H}_{in} = J_0 \cdot \pi s^2 / 2\pi s\, \hat{\phi} = \frac{J_0 s}{2} \hat{\phi}} \qquad \boxed{\vec{H}_{out} = \frac{J_0 R^2}{2s} \hat{\phi}}$$

$$\boxed{\vec{B}_{in} = \mu_0 (1 + \chi_m) \frac{J_0 s}{2} \hat{\phi}} \qquad \boxed{\vec{B}_{out} = \mu_0 \frac{J_0 R^2}{2s} \hat{\phi}}$$

$$\boxed{\vec{M}_{in} = \chi_m H_{in} = \chi_m \frac{J_0 s}{2} \hat{\phi}} \qquad \boxed{\vec{M}_{out} = 0}$$

$$\boxed{J_b = \vec{\nabla} \times \vec{M} = \chi_m J_0 \hat{z}} \qquad \boxed{k_B = \vec{M} \times \hat{n} \Big|_R = -\frac{\chi_m J_0}{2} \hat{z} R}$$



(when $\chi_m > 0$)

Easy to see that Bound current total =

$$I_{B_{tot}} = \int J_b \cdot da + \int k_B \cdot dl = (\chi_m J_0 \cdot \pi R^2) - \frac{\chi_m J_0}{2} R \cdot (2\pi R) = \boxed{0}$$

$$\rightarrow \boxed{I_{tot} = I = J_0 \pi R^2} \rightarrow \boxed{\vec{B}_{out} = \mu_0 J_0 \pi R^2 \frac{\hat{\phi}}{2\pi s}}$$

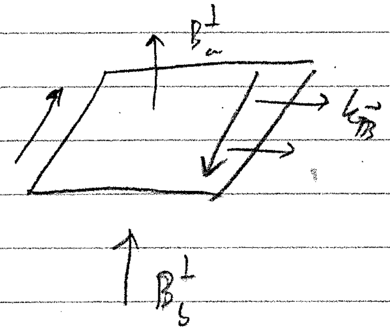
$$\rightarrow \boxed{\vec{B}_{out} = \frac{\mu_0 J_0 R^2}{2s} \hat{\phi}}$$

" II "

BOUNDARY CONDITIONS

$\vec\nabla\cdot\vec B = 0$; $\oint \vec B\cdot d\vec a = 0$

$\vec\nabla\times\vec B = \mu_0 J$; $\oint \vec B\cdot d\vec l = \mu_0 I_{enc}$



$$\boxed{\vec B^\perp_{above} = \vec B^\perp_{below}}$$

$$\boxed{\vec B^\parallel_{above} - \vec B^\parallel_{below} = \mu_0(\vec K\times\hat n)}$$

get same results for aux field H...

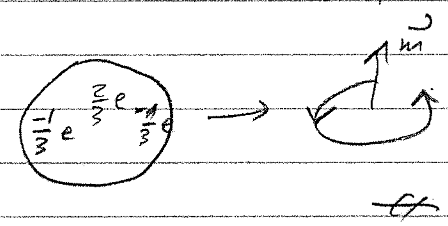
$$\boxed{\vec H^\perp_{above} - \vec H^\perp_{below} = -(\vec M_a^\perp - \vec M_b^\perp)}$$

$$\boxed{\vec H^\parallel_{above} - \vec H^\parallel_{below} = (\vec K_{free}\times\hat n)}$$

$\vec\nabla\times\vec H = -\vec\nabla\times\vec M$

$\vec\nabla\cdot\vec H = J_{free}$

<u>Note</u> neutrons are attracted by $\vec B$ field

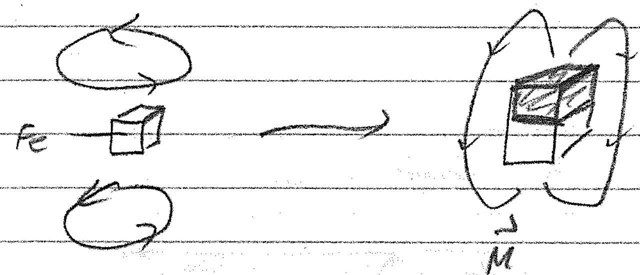


FERROMAGNETISM → due to spin...

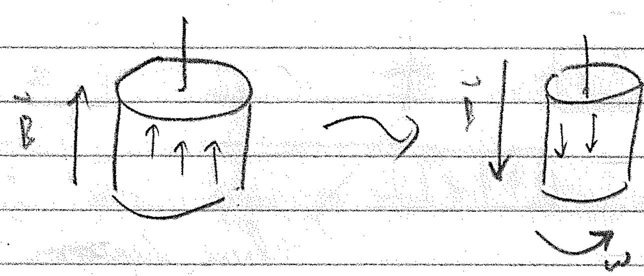
Nov 26, 2019

Spinning e feels a force. when $\vec{B}$ is applied → e^- align

→ when B removed, the aligned e^- causes a $\vec{B}$ field $\parallel \vec{B}$.



Angular momentum ⇒ $\boxed{\vec{m}_{spin} = \frac{q_e}{m_e}\vec{L}_{spin}}$



$D\vec{L}$ ∃ causes torque

→ precess @ ω.

Atomic structure of Fe ... Fe: $[Ar]\,4s^2\,3d^6$

[↑↓] [↑↓ | ↑ | ↑ | ↑ | ↑] → causes paramagnetism

$4s^2$ $3d^6$

But ferromagnetisms are caused by <u>Conductional electrons.</u>
(shared electrons between neighboring atoms)

↳ Looks like $4s^2\,3d^8$

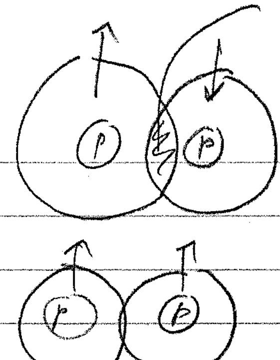
→ Fe shares 2 electrons per atom.

Conduction electrons have aligned spins ... → magnetizes Fe.

<u>Why does this happen?</u> ↑↑↑ -- ↑↑↑ -- ↑↑↑ – aligned, not anti-aligned.

B's overlap

$\uparrow\downarrow$:



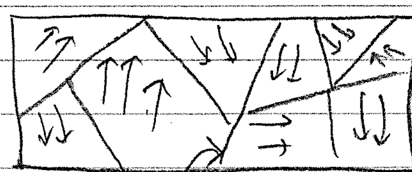
→ more energetic

$\uparrow\uparrow$: → less energetic ... → more favorable

(exchange interaction) ⇒ $\boxed{\text{Fe dipoles align ...}}$

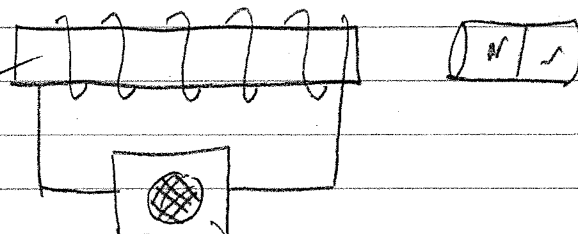
$\boxed{\text{Q: Why isn't Fe always magnetized?}}$

↳ "Domains" :



→ net $\vec{B} = 0$.

Barkhausen Effect



When domain walls break

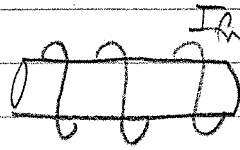
→ $\vec{B}$ jump ⇒ $\frac{\partial B}{\partial t} > 0$

⇒ $\frac{\partial \phi}{\partial t} > 0$

→ EMF → I

→ get sound.

* Make strong Magnet

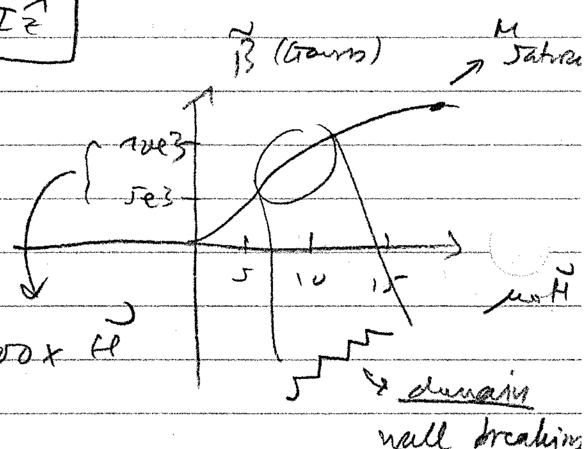


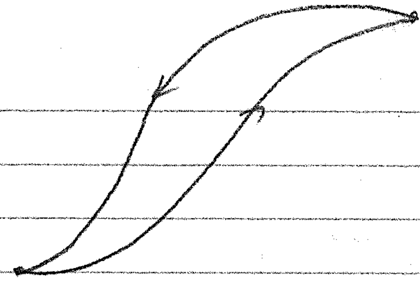
$\oint \vec{H} \cdot d\vec{l} = I_{free}$

↳ $\boxed{\vec{H} = nI\hat{z}}$

$$\boxed{\vec{B} = \mu_0(\vec{H} + \vec{M})}$$

For Iron, $\vec{M} \simeq 1000 \times \vec{H}$.





→ Histeresis Curve

Curie temperature ~> 770°C for Fe
→ spontaneous phase shift
→ all e^- randomize → lose ferromag

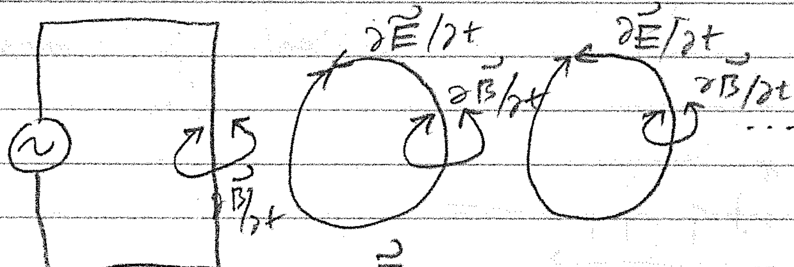
Dec 2, 2019

EM WAVE + RADIATION + RELATIVITY

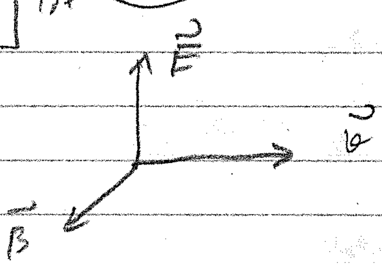
(A) EM Waves

Maxwell's eqns in vacuum w/ no current / charges ...

$$\left\{ \begin{array}{ll} \vec{\nabla}\cdot\vec{E} = 0 & \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t} & \vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t} \end{array} \right\}$$



→ propagates from source ...



Accelerating charge
⟹ EM radiation, which has energy, momentum

To decouple E & B curl equations ...

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{E}}) - \nabla^2\vec{E} = -\partial_t(\vec{\nabla}\times\vec{B})$$

0

$$\Rightarrow \boxed{\vec{\nabla}^2\vec{E} = \partial_t(\vec{\nabla}\times\vec{B}) = \mu_0\varepsilon_0\,\partial_t^2 E}$$

So $\boxed{\vec{\nabla}^2 \vec{E} = \frac{1}{c^2} \partial_t^2 \vec{E}}$ similarly $\boxed{\vec{\nabla}^2 \vec{B} = \frac{1}{c^2} \partial_t^2 \vec{B}}$

Wave equations... $\boxed{\nabla^2 \vec{F} = \frac{1}{v^2} \partial_t^2 \vec{F}}$

In 3D...

$$\partial_x^2 E_x + \partial_y^2 E_y + \partial_z^2 E_z - \mu_0 \varepsilon_0 \partial_t^2 E = 0$$

Properties of E, M waves

(1) speed of propagation is $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

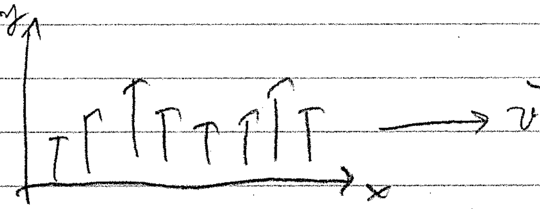
(2) Propagating $\vec{E}$ & $\vec{B}$ fields are transverse waves.

Suppose field is only in $\vec{x}$... then

$$\vec{\nabla} \cdot \vec{E} = 0 = \partial_x E_x + \underbrace{\cancel{\partial_y E_y}}_{0} + \underbrace{\cancel{\partial_z E_z}}_{0}$$

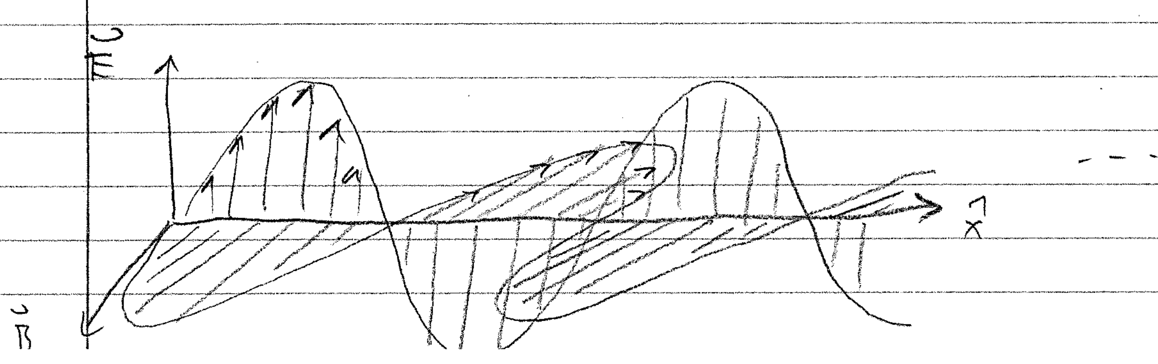
$\Rightarrow \partial_x E_x = 0$ as well. $\Rightarrow$ E amplitude doesn't vary in direction of propagation.

$\boxed{Ex}$



same holds for $\vec{B}$ as well since $\nabla \cdot \vec{B} = 0$.

(3) B points $\perp$ to E field (due to curl relationship)

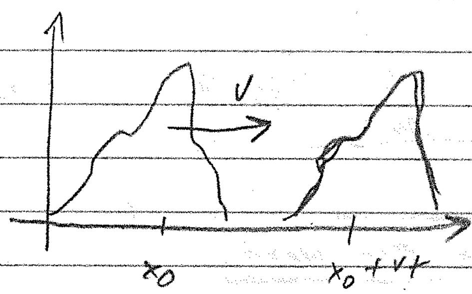


4. Propagation direction is given by right hand rule $\tilde{E} \times \tilde{B}$

5. $\tilde{E}$ & $\tilde{B}$ fields are in phase ...

(B) **Solutions to wave eqn**

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \rightarrow \underline{\text{Soln}} \quad f(x,t) = g(x - vt)$$



Most general solution.

$$f(x,t) = A\cos\left[k(x - vt) + \delta\right]$$

<u>Terminology</u> :
- A: amplitude
- λ: wavelength
- k: wave number
- δ: phase
- T: period
- ν: linear freq
- ω: angular freq

$\boxed{\omega = 2\pi\nu}$

$$\boxed{\lambda = 2\pi/k \qquad T = \frac{\lambda}{v} \qquad \nu = \frac{1}{T}}$$

$\delta \in [0, 2\pi]$. If $\delta = 0$, max displacement @ $x = 0$, $t = 0$

$\boxed{\delta/k} \rightarrow$ distance to max displacement from $x = 0$, $t = 0$

$$\boxed{f(x,t) = A\cos(kx - \omega t + \delta)} \qquad \omega = kv$$

Direction switch $A\cos(kx + \omega t + \delta)$

Ex
$$\begin{cases} \vec{E} = E_0 \sin(x - vt)\,\hat{y}, \\ \vec{B} = B_0 \sin(x - vt)\,\hat{z}. \end{cases}$$

Maxwell ... $\rightarrow$ $\boxed{E_0 = \overset{c}{v} B_0}$

$\rightarrow$ $\boxed{E_0 = c B_0}$

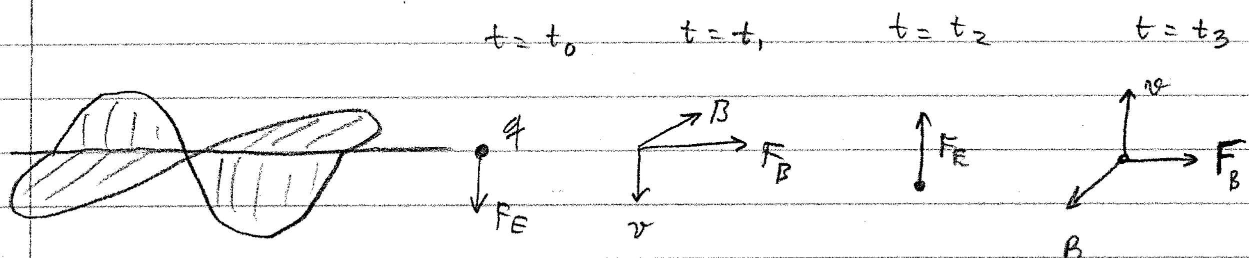
And so, $\boxed{B_0 = \frac{1}{c} E_0 = \frac{k}{\omega} E_0 \Rightarrow \frac{k}{\omega}(\hat{x} \times \vec{E}) = \vec{B}}$

Dec 3, 2019 ③ ENERGY ~ MOMENTUM IN EM WAVE

$\vec{k}$: propagation vector

$\hat{n}$: polarization vector (points in $\vec{E}$ direction)

$$\boxed{\vec{B} = \frac{k}{\omega}(\hat{k} \times \vec{E}) = \frac{1}{c}(\hat{k} \times \vec{E})}$$



Energy

$$W_E = \frac{\varepsilon_0}{2} \int E^2 \, d\tau$$

$$\begin{cases} \dfrac{W_E}{V} = \dfrac{\varepsilon_0}{2} E^2 \\ \dfrac{W_B}{V} = \dfrac{1}{2\mu_0} B^2 \end{cases}$$

So, total energy per unit volume:

$$\boxed{u_{EM} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)} \rightarrow \text{energy density of radiation...}$$

$$\frac{\partial u_{EM}}{\partial t} = \frac{1}{2}\partial_t\left(\varepsilon E^2 + \frac{\vec{B}^2}{\mu_0}\right)$$

$$= \frac{1}{2}\left(\varepsilon_0 \cdot 2\vec{E}\,\partial_t\vec{E} + \frac{1}{\mu_0}2\vec{B}\cdot\partial_t\vec{B}\right)$$

$$= \varepsilon_0\frac{\partial\vec{E}}{\partial t}\cdot\vec{E} + \frac{1}{\mu_0}\vec{B}\cdot\partial_t\vec{B}.$$

$\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$

$\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\,\partial\vec{E}/\partial t$

$$= \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\cdot\vec{E} - \frac{1}{\mu_0}(\vec{\nabla}\times\vec{E})\cdot\vec{B}$$

$$= \frac{1}{\mu_0}\vec{\nabla}\cdot\left(\vec{B}\times\vec{E}\right)$$

So

$$\boxed{\frac{\partial u_{EM}}{\partial t} = \frac{-1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B})}$$

$/\!/\ \vec{k}$

Define Poynting vector: $\boxed{\vec{S} \equiv \frac{\vec{E}\times\vec{B}}{\mu_0}}$ $\rightsquigarrow$ energy flux density / power density

Energy per unit area per unit time: $\boxed{\frac{\partial u_{EM}}{\partial t} = -\vec{\nabla}\cdot\vec{S}}$

$\boxed{\text{units of } \vec{S}}$

$$[S] \sim \frac{1}{\mu_0}(E)(E/c) \sim \varepsilon_0 E^2 c.$$

$$= \left[\frac{\text{energy}}{\text{volume}}\right]\cdot\left[\frac{\text{distance}}{\text{time}}\right] = \frac{J}{m^3}\cdot\frac{m}{s} = \frac{J}{m^2}\cdot\frac{1}{s}$$

$$= W/m^2$$

So $\boxed{[S] = W/m^2}$ "Energy flux density"

$$\boxed{\text{Work} = \int_V \vec{\nabla}\cdot\vec{S}\,d\tau = \oint_{\partial V}\vec{S}\cdot d\vec{A}}$$

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Momentum

For photon: $p = \frac{E}{c}$. Momentum density : $\boxed{g = \frac{u_{EM}}{c}}$

momentum : $\vec{g} = \frac{1}{c^2}\vec{S}$

or $\boxed{\vec{g} = \varepsilon_0 (\vec{E} \times \vec{B})}$

Time average

$$E^2 = E_0^2 \cos(kx - \omega t + \delta)$$

$$\langle E^2 \rangle = E_0^2 \langle \cos^2(t) \rangle = \frac{E_0^2}{2}$$

$$\frac{1}{T}\int_0^T \cos(t)\, dt$$

So $\boxed{\langle E^2 \rangle = \frac{E_0^2}{2}}$ similarly. $\boxed{\langle B^2 \rangle = \frac{B_0^2}{2}}$,

Time - averaged quantities $\boxed{\langle u \rangle = \frac{1}{2}\varepsilon_0 E^2}$ → energy density (per unit volume)

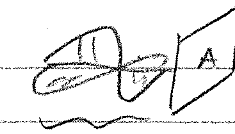
$\boxed{\langle \vec{S} \rangle = \frac{1}{2} c\varepsilon_0 E_0^2 \hat{k}}$ → energy flux density (E/area · time).

$\boxed{\langle \vec{g} \rangle = \frac{1}{2c}\varepsilon_0 E_0^2 \hat{k}}$ → momentum density ...

Time - average of Poynting vector → Intensity

$\boxed{I \equiv \langle S \rangle = \frac{1}{2} c\varepsilon_0 E_0^2}$ → average power per unit area ...

Radiation Pressure



For *perfect absorber*: $\Delta p = \langle \vec{g} \rangle (\text{area})(\text{length})$ $l = \Delta t \cdot c$

$$\text{Pressure} = P = \frac{\text{Force}}{\text{area}} = \frac{\text{momentum}}{\text{area} \cdot \text{time}}$$

$$P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \langle \vec{g} \rangle \cdot c$$

So $\boxed{P = \frac{I}{c} = \frac{1}{2}\varepsilon_0 E_0^2}$

Dec 5, 2019

EM Waves in Matter

Maxwell's Eqn in Matter w/ no free charge & current ...
→ Linear media:

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$$
$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu\varepsilon \partial_t E$$

Propagation speed: $\boxed{v = 1/\sqrt{\varepsilon\mu} = \frac{c}{n}, \quad n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}}$

index of refraction

Light travels slower in matter

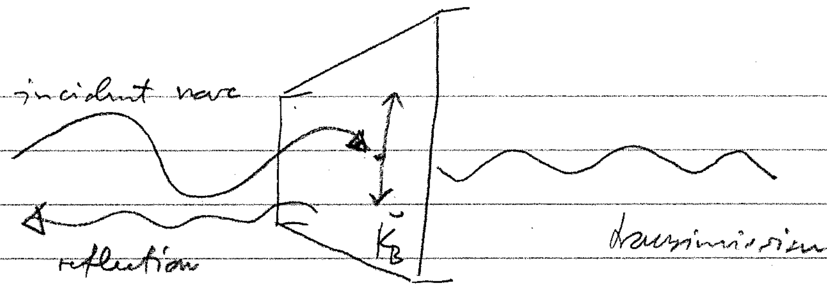
Properties

① Energy Density $\boxed{u_{EM} = \frac{1}{2}\left(\varepsilon E^2 + \frac{1}{\mu}B^2\right)}$

② Poynting vector $\boxed{S = \frac{1}{\mu}(\vec{E} \times \vec{B})}$

③ Intensity $\boxed{I = \frac{1}{2}\varepsilon v E_0^2}$

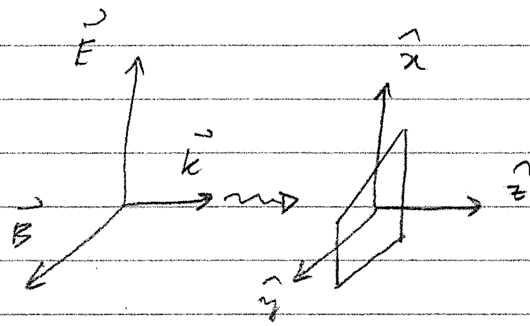
Across a boundary



$\boxed{\text{Boundary condition}}$

$$\boxed{\begin{array}{l} \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel} \; ; \; \vec{B}_1^{\perp} = \vec{B}_2^{\perp} \longrightarrow \text{continuous} \\ \varepsilon_1 \vec{E}_1^{\perp} = \varepsilon_2 E_2^{\perp} \; ; \; \frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_2^{\parallel} \longrightarrow \text{discontinuous} \end{array}}$$

Consider normal incident



- Incident wave:

$$\begin{cases} \vec{E}_I(z,t) = E_{0,I} \cos(k_1 z - wt)\,\hat{x} \\ \vec{B}_I(z,t) = \frac{1}{v_1} E_{0,I} \cos(k_1 z - wt)\,\hat{y} \end{cases}$$

- Reflected wave

$-\hat{y}$ since $\vec{k} \to -\vec{k}$

$$\begin{cases} \vec{E}_R(z,t) = E_{0,R} \cos(-k_1 z - wt)\,\hat{x} \\ \vec{B}_R(z,t) = \frac{1}{v_1} E_{0,R} \cos(-k_1 z - wt)\,\hat{y}\,(-1) \end{cases}$$

- Transmitted wave

$$\begin{cases} \vec{E}_T(z,t) = E_{0,T} \cos(k_2 z - wt)\,\hat{x} \\ \vec{B}_T(z,t) = \frac{1}{v_2} E_{0,T} \cos(k_2 z - wt)\,\hat{y} \end{cases}$$

From BC: $E_1^{\parallel} = E_2^{\parallel}$ & $\frac{B_1^{\parallel}}{\mu_1} = \frac{B_2^{\parallel}}{\mu_2}$. We get

$\boxed{E_{0,I} + E_{0,R} = E_{0,T}}$ and $\boxed{\frac{1}{\mu_1}\left(\frac{E_{0,I}}{v_1} - \frac{E_{0,R}}{v_1}\right) = \frac{1}{\mu_2}\left(\frac{E_{0,T}}{v_2}\right)}$

Simplifying 2nd eqn $\boxed{E_{0,I} - E_{0,R} = E_{0,T}\,\beta}$

where $\boxed{\beta = \dfrac{\mu_1 v_1}{\mu_2 v_2} = \dfrac{\mu_1 n_2}{\mu_2 n_1}}$

In most substances...

$$\mu \sim \mu_0 \; ; \; \beta \sim \frac{v_1}{v_2} \sim \frac{n_2}{n_1}$$

So

$$\boxed{E_{0,R} = \left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_{0,I}}$$

$$\boxed{E_{0,T} = \left|\frac{2n_1}{n_1 + n_2}\right| E_{0,I}}$$

Normal incidence

Fraction of Energy reflected / transmitted $\quad I = \frac{1}{2}\varepsilon v E_0^2$

Reflection : $R = \dfrac{I_R}{I_I} = \dfrac{E_{0,R}^2}{E_{0,I}^2} = \left(\dfrac{n_1 - n_2}{n_1 + n_2}\right)^2$ → reflection coef

Transmission : $T = \dfrac{I_T}{I_I} \sim \dfrac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\dfrac{E_{0,T}}{E_{0,I}}\right)^2 = \dfrac{4n_1^2}{(n_1+n_2)^2}\cdot\dfrac{n_2}{n_1} = \dfrac{4n_1 n_2}{(n_1+n_2)^2}$

transmission coef...

$\boxed{R + T = 1}$

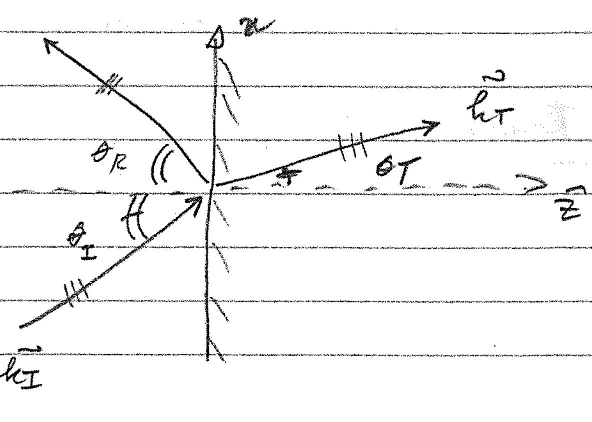
{conservation of energy}

$\boxed{Ex}$ Light from air to glass...

$$\begin{cases} n_{air} \simeq 1 \\ n_{glass} \simeq 1.5 \end{cases}$$

$R = 0.04 \; ; \; T = 0.96$ ⇝ glass transparent...

$\boxed{\text{Reflection at an oblique angle}}$



$E_I = E_{0,I} \cos(\tilde{k}_I \cdot \tilde{r} - wt)$ $\tilde{r} = \tilde{x} + \tilde{z}$

$B_I = \frac{1}{v_1}(\tilde{k}_I \times \tilde{E}_I)$

$E_R = E_{0,R} \cos(\tilde{k}_R \cdot \tilde{r} - wt)$

$B_R = \frac{1}{v_1} E_{0,R} (\tilde{k}_R \times E_R)$

$E_T = E_{0,T} \cos(\tilde{k}_T \cdot \tilde{r} - wt)$

$B_T = \frac{1}{v_2}(\tilde{k}_T \times \tilde{E}_T)$

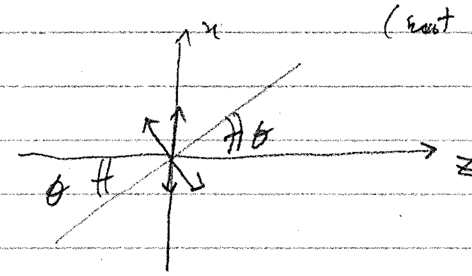
At boundary:

$\tilde{r} = \tilde{x} + \tilde{z} = \tilde{x}$

and so

$\tilde{k} \cdot \tilde{r} = \tilde{k} \cdot \tilde{x} = kx \sin\theta$

(not $\cos\theta$)



And so, at boundary...

$$E_{0,I} \cos(\tilde{k}_I \cdot \tilde{x} - wt) - E_{0,R} \cos(\tilde{k}_R \cdot \tilde{x} - wt) = E_{0,T} \cos(\tilde{k}_T \cdot \tilde{x} - wt)$$

This true if $k_I x \sin\theta_I = k_R x \sin\theta_R = k_T x \sin\theta_T \quad \forall x$

$$\Rightarrow \boxed{k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T}$$

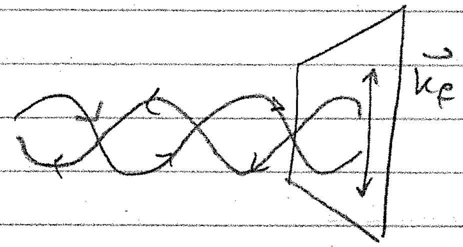
Now, $k_I \sin\theta_I = k_R \sin\theta_R$, $k_i = \frac{w n_i}{c}$ (same

Reflection So $n_I \sin\theta_I = n_R \sin\theta_R$

$$\Rightarrow \underbrace{1 = \frac{n_I}{n_R}}_{\text{same medium}} = \frac{\sin\theta_R}{\sin\theta_T} \Rightarrow \boxed{\theta_R = \theta_T} \rightsquigarrow 2^{nd} \text{ law of optics.}$$

Transmission $\Rightarrow \frac{n_T}{n_I} = \frac{\sin\theta_I}{\sin\theta_T} \Rightarrow \boxed{\frac{n_1}{n_2} = \frac{\sin\theta_2}{\sin\theta_1}} \rightsquigarrow 3^{rd}$ law of optics (Snell's law)

If incident on perfect conductor



$E = 0$; $B = 0$ ~> perfect conductor.

→ $\boxed{\text{Perfect reflection}}$

Dec 6, 2014

FOUNDATION OF RELATIVITY

Wave eq.

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

Galilean transformation ⇒ $x' = x - vt$
$t' = t$

$$\frac{\partial x'}{\partial x} = 1, \quad \frac{\partial t'}{\partial t} = 1$$

$$\frac{\partial x'}{\partial t} = -v, \quad \frac{\partial t'}{\partial x} = 0$$

Under this transformation ... w/ chain rule

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \dots$$

with $E(x', t') = E(x'(x,t), t'(x,t))$

To get ... $\frac{\partial E}{\partial x} = \frac{\partial E}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial x}$ and so on...

$$= \frac{\partial E}{\partial x'}$$

$$\Rightarrow \boxed{\frac{\partial^2 E}{\partial x^2} = \frac{\partial^2 E}{\partial x'^2}}$$

$$\frac{\partial E}{\partial t} = \frac{\partial E}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial x} = -v\frac{\partial E}{\partial x'} + \frac{\partial E}{\partial t'}$$

next, $\frac{\partial}{\partial t}\left(\frac{\partial E}{\partial t}\right) = \cdots \Rightarrow \boxed{\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial t'^2} - 2v\frac{\partial^2 E}{\partial x' \partial t'} + v^2 \frac{\partial^2 E}{\partial x'^2}}$

with this,

$$\boxed{\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2 E}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t'^2} - 2\frac{v}{c}\frac{\partial^2 E}{\partial x' \partial t'} = 0}$$

⇝ whose solution is not a wave ...

wave equ not invariant under galilean transf.
But invariant under Lorentz transformation.

Hendrick Lorentz (1892)

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} x \\ ct \end{pmatrix} \qquad \begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned}$$

where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \geq 1$.

Under this transformation:

$$\boxed{\frac{\partial^2 E}{\partial x'^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t'^2}}$$ to keep invariance

Consequence: (1) Speed of light is constant ..
(2) $V_{MAX} = c$ (due to addition of velocity)

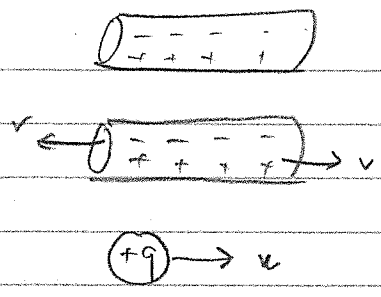
$$\boxed{u' = \frac{u+v}{1+\frac{uv}{c^2}}}$$

(3) Time dilation
(4) length contraction

(5) Magnetism via Relativity ...

F^0

F

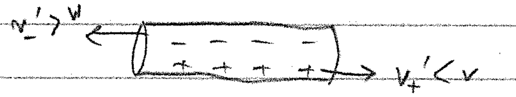


$$\lambda_0 = \frac{dq}{dx_0}$$

$$\lambda = \frac{dq}{dx} \qquad I = 2\lambda v$$

$$F = q\vec{v}\times\vec{B}\,, \quad \boxed{F_B = qu\left(\frac{\mu_0 I}{2\pi s}\right)}$$

In particle frame...



$$v_-' \neq -v_+' \Rightarrow \lambda_-' \neq \lambda_+' \Rightarrow \text{net charge} \rightarrow F_E \neq 0.$$

(+q) at rest

$$\lambda_-' = \frac{dq}{dx}\gamma_- = \lambda_0\gamma_-$$

$$\lambda_+' = \frac{dq}{dx}\gamma_+ = \lambda_0\gamma_+$$

$$\rightarrow \lambda_{tot} = -\lambda_0(-\gamma_- + \gamma_+) = \frac{-2\lambda_0 uv}{c^2\left(1-\frac{v^2}{c^2}\right)}$$

Electric Force

$$F = qE = \frac{\lambda_{tot}}{2\pi\varepsilon_0 s}\cdot q = \frac{q}{2\pi\varepsilon_0 s}\cdot\frac{-2\lambda uv}{c^2(1-v^2/c^2)}$$

$$= \frac{-q}{\pi\varepsilon_0 c^2}\left(\frac{\lambda uv}{s}\right)$$

Now, $I = 2\lambda v$

$$c = 1/\sqrt{\mu_0\varepsilon_0} \Rightarrow \boxed{F = -qu\frac{\mu_0 I}{2\pi s}}$$