Classical Field Theory

Feb 7, 2019

Action Principle

Action: $S = \int_a^b L\,dt \qquad L = T - V$

(See E-L method in Farlow)

$$\delta L = m\dot{x}\,\delta(\dot{x}) - \frac{dV}{dx}\delta x$$

$$= m\dot{x}\,\dot{(\delta x)} - \frac{dV}{dx}\delta x$$

$$= m\left[-\ddot{x}\,\delta x + \frac{d}{dt}(\dot{x}\,\delta x)\right] - \frac{dV}{dx}\delta x$$

$\delta x = 0$ @ $t = a, b$

$$= -m\ddot{x}\,\delta x - \frac{d}{dt}(\dot{x}\,\delta x)\,m - \frac{dV}{dx}\delta x$$

$$\delta S = \int_a^b \delta L\,\cancel{dt} = \ldots = -\int_a^b \left(m\ddot{x} + \frac{dV}{dx}\right)\delta x\,dt = 0 \;\forall\,\delta x$$

So $\quad m\ddot{x} + \dfrac{dV}{dx} = 0$

"t"

$$m\ddot{x} = -\frac{dV}{dx} = -\vec{\nabla}V$$

Claim $\boxed{\text{All fundamental physics obeys least action principle.}}$

To do this relativistically, use $L = \int d^3x\,\mathcal{L}(x)$ → Lagrangian density

$$\boxed{S = \int L\,dt = \int \mathcal{L}(x)\,d^4x} \longrightarrow \text{In Sean Carroll's}$$

In flat spacetime $g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} + & & & \\ & - & & \\ & & - & \\ & & & - \end{pmatrix}$

In Carroll's book, $\eta_{\mu\nu} = \begin{pmatrix} - & & & \\ & + & & \\ & & + & \\ & & & + \end{pmatrix}$

Fields $\longrightarrow$ Scalar field

"Everything is fields in field theory"

$\rightarrow$ has 1 component, 1 degree of freedom

simplest case $\rightarrow$ moving field in 1D

spin 0

$$\boxed{\phi(x) \sim e^{-i\vec{k}\vec{x}}}$$

$k^\mu = (k^0, \vec{k})$

$x^\mu = (t, \vec{x})$

$$k \cdot x = k_\mu x^\mu = \eta_{\mu\nu} k^\nu x^\mu$$

$(\hbar = c = 1)$

$\rightarrow [E] = [\omega] = [k^0]$

So $\phi(x) \sim e^{-i\vec{k}\vec{x}} = e^{-ik_\mu x^\mu}$

$$= \exp\left[-ik^0 t + i\vec{k}\cdot\vec{x}\right]$$

$$= e^{-i\omega t} e^{i\vec{k}\cdot\vec{x}}$$

Massive scalar fields $\quad E^2 = m^2 + \vec{p}^{\,2}$

So $(k^0)^2 = m^2 + (\vec{k})^2$

$$\underbrace{(k^0)^2 - (\vec{k})^2}_{} = m^2$$

$$\boxed{k^\mu k_\mu = m^2} \rightarrow \text{massive particle}$$

$k \rightarrow$ wave number

$$\boxed{k^\mu k_\mu = 0} \rightarrow \text{massless obeys this.}$$

harmonic osc.

How does this motivate Lagrangian density for a scalar field $\phi(x)$

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2} \rightarrow \text{in S.C. book}$$

Action $S = \int \mathcal{L} d^4x$ w.r.t $\phi \rightarrow \delta\phi$

$$\delta S = \int \underbrace{\delta \mathcal{L}}_{\rightarrow 0} d^4 x$$

Do $\delta\mathcal{L}$ w.r.t ϕ

$$\delta\mathcal{L} = \frac{1}{2}(\partial_\mu \delta\phi)(\partial^\mu\phi) + \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\delta\phi) - m^2\phi\delta\phi$$

same

$$= (\partial_\mu \delta\phi)(\partial^\mu\phi) - m^2\phi\delta\phi$$

..int by parts $= -(\partial_\mu\partial^\mu\phi)\delta\phi - m^2\phi\delta\phi$

(dropping total derivative)

Call $\partial_\mu\partial^\mu \equiv \Box \rightarrow$ d'Alembertian

$$= \partial_0\partial^0 + \partial_j\partial^j = \frac{d^2}{dt^2} - \vec{\nabla}^2$$

So $\delta\mathcal{L} = -(\Box + m^2)\phi\,\delta\phi$

Solution $\rightarrow \delta\mathcal{L} = 0 \ \forall\ \delta\phi$

So Klein-Gordon Eqn

$$\boxed{(\Box + m^2)\phi = 0}$$

What are the solutions to $\overset{\phi(x)}{}$ $(\Box + m^2)\,\phi(x) = 0$

- try $\phi(x) = e^{-ik_\mu x^\mu} = e^{-ik_\alpha x^\alpha}$

$$\partial_\mu \phi = -i\,\partial_\mu (k_\alpha x^\alpha)\, e^{-ik_\beta x^\beta}$$

$$= -i\, k_\alpha\, \partial_\mu x^\alpha\, e^{-i\vec{k}\cdot\vec{x}}$$

$$\underbrace{\partial^\mu = \eta^{\mu\nu}\partial_\nu}_{\text{only flat spacetime @ this pt}}$$

$$= -i k_\alpha \delta^\alpha_\mu \phi = -ik_\mu \phi$$

$$\partial^\mu \partial_\mu \phi = (-i)^2 k_\mu k^\mu \phi$$

so $\Box\phi = -k_\mu k^\mu \phi = \cancel{+}\,m^2\phi$ (required)

so it's a solution as long as $\boxed{k_\mu k^\mu = m^2}$

massive particles

Vector fields (spin 1) (S.C book as well)

Instead of $\phi \longrightarrow A_\mu \rightarrow$ vector field (photon)

Lagrangian density $\rightarrow$ $\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu}$

$j^\mu = (\rho, \vec{J})$, $A^\mu = (V, \vec{A}) \longrightarrow$ vector potential

$$\boxed{\begin{aligned}\vec{E} &= -\vec{\nabla}V \\ \vec{B} &= \vec{\nabla}\times\vec{A}\end{aligned}} \quad \text{(static)}$$

Full $\boxed{\begin{aligned}\vec{E} &= -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} \\ \vec{B} &= \vec{\nabla}\times\vec{A}\end{aligned}} \longrightarrow$ static + dynamic

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ +E^1 & 0 & -B^3 & -B^2 \\ +E^2 & B^3 & 0 & -B^1 \\ +E^3 & B^2 & B^1 & 0 \end{pmatrix}$$

↳ E-M stress tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

With def :⟶ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Hence $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$ holds

This identity yields $\left\{ \begin{array}{l} \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} = 0 \end{array} \right\}$

The remaining Maxwell eqns come from varying the action $\delta S = 0$ w.r.t $A_\mu(x)$

$$\delta(j^\mu A_\mu) = j^\mu \delta A_\mu$$

and

$$\delta\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) \text{ w.r.t } A_\mu$$

well

$$F^{\mu\nu}F_{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$= 2\partial^\mu A^\nu \partial_\mu A_\nu - 2\partial_\mu A_\nu \partial^\nu A^\mu$$

So

$$\delta\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) = -\frac{1}{2}\delta\left(\partial^\mu A^\nu \partial_\nu A_\mu - \partial_\mu A_\nu \partial^\nu A^\mu\right)$$

$$= \cdots -\frac{1}{2}\Big[(\partial_\mu \delta A_\nu)\partial^\mu A^\nu + \partial_\mu A_\nu(\partial^\mu(\delta A^\nu))$$

$$- \underbrace{\partial_\mu(\delta A_\nu)\partial^\nu A^\mu}_{} - \underbrace{\partial_\mu A_\nu \partial^\nu(\delta A^\nu)}_{}\Big]$$

↑ product rule

same after indexing ...

$$= \partial_\mu A_\nu (\partial^\nu \delta A^\mu) - (\partial_\mu \delta A_\nu)\partial^\mu A^\nu$$

Want $\delta S = 0 = \int (\quad) \delta A \, d^4x = 0$

$$= (+\partial_\mu \partial^\mu A^\nu)\delta A_\nu + \text{total deriv} - (\partial^\nu \partial_\mu A_\nu)\delta A^\mu$$

$\text{total deriv} \to 0$

low - raise

$$\delta\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) = +\partial_\nu \partial^\nu A^\mu \, \delta A_\mu - (\partial^\nu \partial^\mu A_\nu)\delta A_\mu$$

$$= \left(\Box A^\mu - \partial^\nu \partial^\mu A_\nu\right)\delta A_\mu = 0 \quad \forall \, \delta A_\mu$$

&

$$\boxed{\Box A^\mu - \partial^\nu \partial^\mu A_\nu = \Box A^\mu - \partial_\nu \partial^\mu A^\nu = 0}$$

Can write this as $\partial_\nu F^{\mu\nu} = 0$

with current $-j^\mu A_\mu$ get

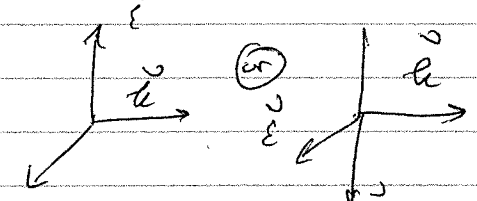
$\boxed{\partial_\nu F^{\mu\nu} = j^\mu}$ → give the remaining Maxwell's eqn

Claim: For free photons (EM waves) then $\rho = 0$, $\vec{J} = 0$

& $j^\mu = 0$ eqn then is just $\boxed{\Box A_\mu - \partial_\mu \partial^\nu A_\nu = 0}$

How many independent EM waves? 2.

2 transverse polarizations.



2 massless modes for photons (2 polarizations)

→ massless $k_\mu k^\mu = 0$ for photons $E = cp$

But A_μ has 4 df (not 2!)

To be relativistic $\rightarrow$ must use Scalars : ϕ
Vectors : A_μ
Tensors : $g_{\mu\nu}$

There must be 2 degrees of freedom in A_μ that don't matter. Have

$$A_\mu = (A_0, A_j) \text{ has too many degrees of freedom}$$

Turns out $\rightarrow$ physical waves have wave eqn

$$\Box A + \dots = 0$$

$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \dots$; gives $e^{-ik_\nu x^\nu}$

Look at $\mu = 0 \rightarrow A_0 \Rightarrow \Box A_0 - \partial_0 \partial^\nu A_\nu = 0$

$$(\partial^0\partial_0 + \partial_j\partial^j) A_0 - \partial_0 \partial^\nu A_\nu = 0$$

time 2nd derivs $= 0$ $\longrightarrow$ $\cancel{\partial^0\partial_0 A_0} + \partial^j \partial_j A_0 - \cancel{\partial_0 \partial^0 A_0} - \partial_0 \partial^j A_j = 0$

$\Rightarrow$ $\boxed{A_0 \text{ is not a propagating mode}}$

$\Rightarrow$ Auxiliary mode (not propagating)

$\rightarrow$ NOT physical.

- This is good, because $(\Box A^\mu + \dots)\, \delta A_\mu$

$$A^0 = A_0 \text{ but } A^j = -A_j$$

If all 4 were allowed to propagate, we will have a <u>bad</u> sign compared to the other 3.

$\Rightarrow$ get a "ghost" mode. "Ghost" has wrong sign KE

. The desire to use $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ was made to eliminate the potential "ghost" mode. Recall scalar: $\mathcal{L} \sim \partial_\mu \phi \partial^\mu \phi$

for A, we might have guessed $\partial_\mu A_\nu \partial^\mu \partial^\nu$ only. But this has

$$\Box A_\mu + \ldots = 0 \quad \forall \text{ four modes} \to \text{"ghost"}$$

$\to$ Use $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ so that it don't allow $\frac{\partial^2}{\partial t^2} A_0$.

. The $\frac{\partial^2}{\partial t^2} A_0$ terms cancel.

Now Start w/ $A_\mu \longrightarrow 4$

Find A_0 is aux $\to 1$

?

$\Big\}$ So that get 2?

$\boxed{\text{GAUGE SYMMETRY}}$ $\to$ gauge mode that can be eliminated

Finally $4-1-1 = 2 \to$ physical.

A_μ aux. gauge symmetry

Look at $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

& transform $\boxed{A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda(x)}$ $\to$ gauge transformation

Then

$$F_{\mu\nu} \to \boxed{\begin{aligned} F'_{\mu\nu} &= \partial_\mu (A_\nu + \partial_\nu \Lambda) - \partial_\nu (A_\mu + \partial_\mu \Lambda) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} \end{aligned}}$$

(9)

Can choose $\Lambda(x)$ ~~to~~ to eliminate A_μ mode leaving 2.

Feb 14 2019

Recap Last week $\rightarrow$ Real Scalar Fields
$\rightarrow$ Vector fields

Note $\Box A_\mu - \partial_\mu \partial^\nu A_\nu = 0 \longleftarrow \delta S = 0$

- A_μ has 4 components $\quad 4$
- $A_0 \rightarrow$ auxiliary $\quad -1$
- gauge field $\quad -1$

$\quad 2$ degrees of freedom

Gauge

Can fix the gauge

$$A_\mu \Rightarrow A_\mu + \partial_\mu \Lambda(x)$$

Pick $\Lambda(x)$ to remove a degree of freedom

Suppose $\underbrace{\partial^\nu A_\nu}_{\text{div}(A)} \neq 0$ $\rightarrow$ Can pick a gauge that sets $\partial^\nu A_\nu \rightarrow 0$

Let $A_\nu \rightarrow A'_\nu = A_\nu + \partial_\nu \Lambda$

then $\partial^\nu A'_\nu \rightarrow \partial^\nu (A_\nu + \partial_\nu \Lambda)$

$$= \partial^\nu A_\nu + \Box \Lambda$$

~~If~~ we pick Λ s.t $\boxed{\Box \Lambda = -\partial^\nu A_\nu}$, then in this gauge

$\partial^\nu A'_\nu = 0$, then drop the prime.

In the fixed gauge, EOM: $\begin{cases} \Box A_\mu = 0 \\ \partial^\nu A_\nu = 0 \rightarrow \text{Lorentz gauge} \end{cases}$

To solve, assume $A_\mu = \underset{\substack{\uparrow \\ \text{polarization vector.}}}{\varepsilon_\mu} e^{-i\tilde{k}\cdot\tilde{x}}$

- $\Box A_\mu = 0 \Rightarrow$ find that $\underline{k_\mu k^\mu = 0 \text{ must hold}}$

  $\rightarrow$ massless vector field (good)

- With $\partial^\nu A_\nu = 0 \rightarrow$ removes 1 degree of freedom that we chose

$$\Box\Lambda = -\partial^\nu A_\nu$$

But this doesn't completely fix Λ $\rightarrow$ have a residual gauge freedom $\rightarrow$ Can use to set $A_0 = 0$

Residual gauge transform $\rightarrow A_\mu \mapsto A_\mu + \partial_\mu \Lambda$

but with $\Box\Lambda = 0 \rightarrow \partial^\nu A_\nu = 0$ alone

Look at $A_0 = \varepsilon_0 e^{-i\tilde{k}\cdot\tilde{x}}$ with $k_\mu k^\mu = 0$

- If we pick $\Lambda = \lambda e^{-i\tilde{k}\cdot\tilde{x}}$ then $\Box\Lambda \propto k_\mu k^\mu = 0$

Then $A_0 \rightarrow \underset{\substack{\downarrow\, e^{-ikx}}}{A_0} + \underset{\substack{\downarrow\, e^{-ikx}}}{\partial_0 \Lambda} \rightarrow 0$

$$\varepsilon_0 - i\lambda k_0 = 0$$

pick $\lambda = \dfrac{\varepsilon_0}{ik_0}$ then $A_0 = 0 \Rightarrow \partial^\nu A_\nu = \partial^0 \overset{0}{A_0} + \overset{0}{\underbrace{\tilde{\nabla}\cdot\tilde{A}}} = 0$

Complete gauge fixing $\begin{cases} A_0 = 0 \\ \vec{\nabla}\cdot\vec{A} = 0 \end{cases} \rightarrow$ Coulomb

Now have $A_\mu \rightarrow 4$ df

$$\vec{\nabla}\cdot\vec{A} \rightarrow -1 \text{ df}$$
$$A_0 \rightarrow -\cancel{1} \text{ df}$$

$$2 \text{ df} \Rightarrow \text{physical}$$

Now $A_\mu = \varepsilon_\mu e^{-i\vec{k}\cdot\vec{x}}$

$\varepsilon_\mu = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$

$A_0 = 0 \Rightarrow$ lets $\varepsilon_0 = 0$

$\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \vec{k}\cdot\vec{A} = 0$

Consider γ's moving in the z direction

then $k^\mu = (k, 0, 0, k)$ since $k_\mu k^\mu = 0$

So $\vec{k}\cdot\vec{A} \sim \vec{k}\cdot\vec{\varepsilon} = k\varepsilon^3 = 0$

So, no longitudinal components

So $\varepsilon_\mu = (0, \varepsilon_1, \varepsilon_2, 0)$

So $A_\mu = \varepsilon_\mu e^{-i\vec{k}\cdot\vec{x}} = (0, \varepsilon_1, \varepsilon_2, 0)\, e^{-ik\circ x}$ ↗ 4-vector.

So the independent modes can be chosen as $\begin{cases} \varepsilon_\mu^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ \varepsilon_\mu^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \end{pmatrix} \end{cases}$

→ 2 physical modes $\left\{ \begin{array}{l} A_\mu^{(1)} = \varepsilon_\mu^{(1)} e^{-ik\cdot x} \\ A_\mu^{(2)} = \varepsilon_\mu^{(2)} e^{-ik\cdot x} \end{array} \right\}$

→ 2 massless ($q_\mu q^\mu = 0$) <u>transverse modes</u>

■ Why are photons massless? → $\boxed{\text{Because of gauge symmetry}}$

Suppose

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A^\mu A_\mu$$

Under gauge transform → $A_\mu \to A_\mu + \partial_\mu \Lambda$

Have $\left\{ \begin{array}{l} F_{\mu\nu} \to F_{\mu\nu} \\ A_\mu A^\mu \to (A_\mu + \partial_\mu \Lambda)(A^\mu + \partial^\mu \Lambda) \\ \qquad \neq A_\mu A^\mu \end{array} \right.$

→ no longer gauge invariant

→ massive vectors do not have <u>gauge symmetry</u>

Can't add $m^2 A_\mu A^\mu$ if want gauge invariance

$\boxed{\text{massless} \Leftrightarrow \text{gauge invariance}}$

$\boxed{\text{Complex Scalars}}$

→ have 2 degrees of freedom

$$\phi = \phi_1 + i\phi_2$$

$$\phi^* = \phi_1 - i\phi_2$$

or just use ϕ & ϕ^* as 2 df

Could write down

$$\mathcal{L} = (\partial_\mu \phi)(\partial_\mu \phi)^* - m^2 \phi\phi^*$$

can vary w.r.t. to ϕ & ϕ^*

Does this have symmetry? → Yes! (phase symmetry)

$$\text{if } \phi \Rightarrow \phi e^{i\alpha} \;\; (\alpha \to \text{constant})$$

so $|\phi|^2 = \phi^*\phi \longrightarrow \phi^*\phi$

likewise $(\partial_\mu \phi)(\partial_\mu \phi)^* \longrightarrow (\partial_\mu \phi)(\partial_\mu \phi)^*$

since $|e^{i\alpha}|^2 = 1$

→ this is a symmetry of $\mathcal{L}$. global $U(1)$ sym.

$\boxed{\text{Group Theory}}$

(1) Consider N-dim complex vector,

$z = (z_1 \ldots z_N)^T$. Norm $= \sum_{i=1}^{N} z_i^* z_i = z^\dagger z$

"unitary"

A gauge transformation U that takes $|z|^2 \to |z^2|$

is called a $U(N)$ gauge transform

if $z \to Uz$, then $|z|^2 \to (Uz)^\dagger (Uz)$

$$(Uz)^\dagger (Uz) \to \underbrace{z^\dagger U^\dagger U z}_{} = |z|^2$$

true if $U^\dagger U = 1$

mean that $U^\dagger = U^{-1}$ $\to$ Unitary matrix

<u>Note</u> $e^{i\alpha}$ is a 1-by-1 Unitary matrix

<u>So</u> complex scalars have an $U(1)$ gauge invariance

Special case is when also $\det U = 1$

Special unitary group $SU(N)$

Consider instead N-dim real vector,

$X = (x_1 \ldots x_N) \to$ real

$|x| = \vec{x} \cdot \vec{x} = \sum_{i=1}^{N} x_i x_i$

A transformation $O: x \to Ox$ which leave

fix $\vec{x}^2$ alone $\to$ <u>orthogonal transformation</u>

group $\to O(N)$

$|x|^2 = x^T x \to (Ox)^T (Ox) = x^T O^T O x = x^T x$

true if $O^T O \equiv 1 \to$ <u>orthogonal matrix</u>

Special group → $SO(N)$ $\det O = 1$

rotations → $O(3)$

Lorentz group → $x^\mu = (x^0 \dots x^3) \Rightarrow x^2 = x_\mu x^\mu$

$$= (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

a LT keeps $|x^2|$ unchanged

$$x \to \Lambda x \quad \Lambda^T \Lambda = 1$$

Lorentz group $SO(3,1)$

- The theory $\mathcal{L} = |\partial_\mu \phi|^2 - m^2 |\phi|^2$ has a global $U(1)$ sym → $\alpha =$ constant

$$\phi \to U\phi = e^{i\alpha}\phi$$

<u>Local</u> $U(1)$ transform → $\alpha = \alpha(x)$

$\phi \to e^{i\alpha(x)}\phi$ different locally

Is this a symmetry? → <u>No</u>.

<u>But</u> if $\phi \to e^{i\alpha(x)}\phi$ then $\partial_\mu \phi \to \partial_\mu (e^{i\alpha(x)}\phi)$

$$= i(\partial_\mu \alpha)\phi + e^{i\alpha}\partial_\mu \phi$$

$(\partial_\mu \phi)(\partial_\mu \phi)^* \to \times$ $(\partial_\mu \phi)(\partial^\mu \phi)$ no local $U(1)$

<u>But</u> can "<u>fix</u>" derivative → Introducing ⟶

Gauge-covariant derivative

$$\boxed{D_\mu = \partial_\mu + iqA_\mu}$$

→ charge — coupling

← gauge field

gauge fields have symmetry $A_\mu = A_\mu + \partial_\mu \Lambda$

pick $\Lambda = \frac{-\alpha(x)}{q} \rightarrow A_\mu \rightarrow A_\mu - \frac{1}{q}\partial_\mu \alpha$

with both $U(1)$
$$\begin{cases} \phi \rightarrow \phi e^{-i\alpha(x)} \\ A_\mu \rightarrow A_\mu - \frac{1}{q}\partial_\mu \alpha \end{cases}$$

then

$$D_\mu \phi = (\partial_\mu + iqA_\mu)\phi$$

$$\Rightarrow \left(\partial_\mu \cancel{\phi} + iq\left(A_\mu - \frac{1}{q}\partial_\mu \alpha\right)\right) e^{i\alpha(x)}\phi$$

$$\Rightarrow \cancel{\partial_\mu} e^{i\alpha}\partial_\mu\phi + i\cancel{(\partial_\mu\alpha)}e^{i\alpha}\phi$$

$$+ iqA_\mu e^{i\alpha}\phi - i\cancel{(\partial_\mu \alpha)} e^{i\alpha}\phi$$

$$= e^{i\alpha}(\partial_\mu \phi + iqA_\mu \phi)$$

$$\boxed{D_\mu \phi \Rightarrow D_\mu \phi e^{i\alpha}}$$

$$\mathcal{L} \quad (D_\mu \phi)(D^\mu_\mu \phi)^* \sim (D_\mu \phi)(D_\mu \phi)^*$$

(good)

Since we add A_μ, can make it dynamical by also adding $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

Full theory

$$\boxed{\mathcal{L} = |D_\mu \phi|^2 - m^2|\phi|^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}}$$

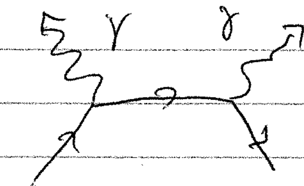
charged scalar field in EM

QED propagating massive scalar $\longrightarrow$

propagating photon $\rightsquigarrow$

$|D_\mu \phi|^2$ contains $A \cdot A\phi^2$

or $A\partial_\mu \phi\phi$



Note $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

$N=1$ $U(1), O(1)$ $e^{i\alpha} e^{i\beta} = e^{i\beta} e^{i\alpha} \rightarrow$ commute $\rightarrow$ abelian

$N=2$ $U(N), SU(N), O(N), N \geq 1 \longrightarrow$ non-abelian groups

Ex strong & weak forces $\rightarrow$ non-abelian

$SU(3)$ $SU(2)$

complicated

Yang-Mills gauge theory

Lie-group, Lie-algebra ...

SPONTANEOUS SYMMETRY BREAKING

↳ mechanism where symmetry still holds dynamically, but the solutions break the symmetry.

$\mathcal{L} \Rightarrow$ has a symmetry. But the solutions break it.

Consider real scalar field ϕ

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi)$$

Suppose that $V(-\phi) = V(\phi)$

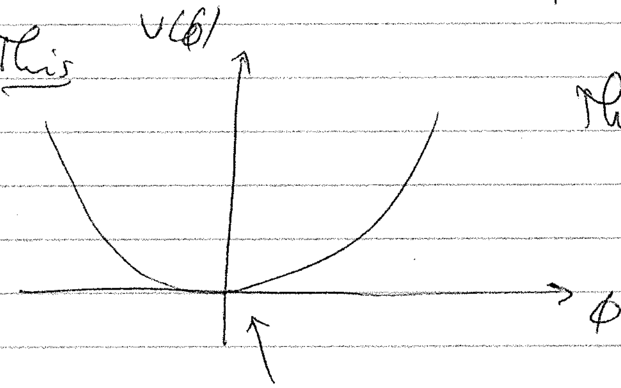
Then $\mathcal{L}$ is invariant under parity transformation.

$$\phi \to -\phi$$

So $\mathcal{L} \to \mathcal{L} \rightarrow$ a symmetry.

Ex $V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{4}\lambda \phi^4 \quad \begin{cases} \lambda > 0 \\ m^2 > 0 \end{cases}$

This



There is a unique minimum

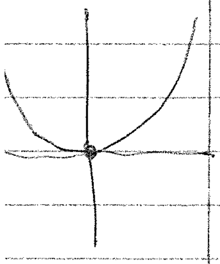
→ ground state of field theory is called the $\boxed{\text{vacuum expectation value}}$

↳ (state of lowest energy)

$$\boxed{\langle \phi \rangle = 0} \quad (\text{VEV})$$

At $\phi = 0$, we still have a symmetry.

~~But now suppose~~ Now, look at small excitation around the vacuum



$$\to \phi = \langle\phi\rangle + \varepsilon = 0 + \varepsilon = \varepsilon$$

In which case, the Lagrangian becomes

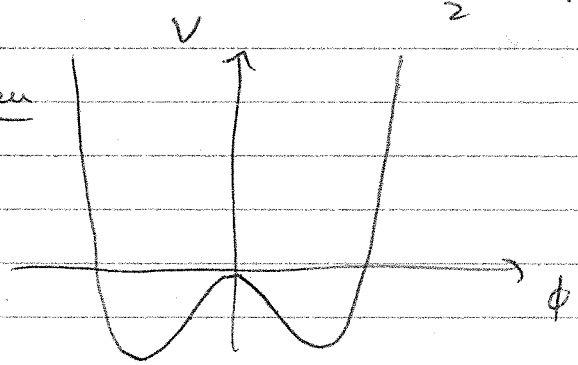
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varepsilon)(\partial^\mu \varepsilon) + \frac{1}{2} m^2 \varepsilon^2 + \cancel{\varepsilon^4}^{\,0}$$

massive particle → real scalar field

But now suppose

Consider $V(\phi) = -\frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4$

Then



2 possible vacuum solutions. which one?

→ Nature spontaneously pick one.

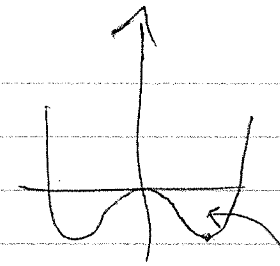
$\mathcal{L}$ still has $\phi \to -\phi$ symmetry

Look at

$$\frac{dV}{d\phi} = m^2\phi + \lambda\phi^3 = 0 \qquad (m^2 < 0)$$

$$\to \langle\phi\rangle = \pm\sqrt{\frac{-m^2}{\lambda}}$$

Suppose it picks the one on the right



$$\langle\phi\rangle = \sqrt{\frac{-m^2}{\lambda}} \equiv v$$

We can shift to a field defined w.r.t to vacuum.

$$\phi' = \phi - \langle\phi\rangle = \phi - v$$

Then $\langle\phi'\rangle = 0$

In terms of ϕ', the $\mathcal{L}$ is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi')(\partial^\mu\phi') - (-m^2)\left[\frac{\phi'^4}{4v^2} + \frac{\phi'^3}{v} + \phi'^2 - \frac{v^2}{4}\right]$$

has no symmetry in terms of ϕ' (parity transform)

Symmetry is hidden.

Look at small excitations about $\langle\phi'\rangle$,

$$\phi' = \langle\phi'\rangle + \varepsilon = 0 + \varepsilon = \varepsilon$$

Plug in...

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varepsilon)(\partial^\mu\varepsilon) - \frac{1}{2}(-2m^2)\,\varepsilon^2$$

again, assuming $m^2 < 0$

$(-2m^2) > 0$

acts as a massive particle scalar with mass

→ By breaking symmetry ... → get massive particle.

Verify

$$V = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4$$

Let $\phi = \langle\phi\rangle + \varepsilon$

$$\langle\phi\rangle = \sqrt{\frac{-m^2}{\lambda}} \equiv v \qquad = v + \varepsilon$$

- $\partial_\mu \phi = \partial_\mu \varepsilon$
- $V = \frac{1}{2}m^2(v+\varepsilon)^2 + \frac{1}{4}\lambda(v+\varepsilon)^4$

$$= \frac{1}{2}m^2(v^2 + 2v\varepsilon + \varepsilon^2) + \frac{1}{4}\lambda(v^4 + 4v^3\varepsilon + 6v^2\varepsilon^2 + 4v\varepsilon^3 + \varepsilon^4)$$

keep linear terms $\cong \varepsilon(m^2 v + \lambda v^3) + \varepsilon^2\left(\frac{1}{2}m^2 + \frac{3}{2}\lambda v^2\right) + \cancel{\varepsilon^3(\;)}^{\,0} + \cancel{\varepsilon^4(\;)}^{\,0}$

$$= \varepsilon v\underbrace{(m^2 + \lambda v^2)}_{m^2 - \lambda\frac{m^2}{\lambda} = 0} + \varepsilon^2(-m^2)$$

$$\therefore \boxed{V(\varepsilon) \cong -\varepsilon^2 m^2 = +\frac{1}{2}(-2m^2)\varepsilon^2}$$

↳ "Discrete Symmetry" → this symmetry is a discrete symmetry

↳ not continuous, unlike rotation

Theorem

Goldstone, MIT: In a theory with a continuous sym that is spontaneously broken, then Goldstone's theorem says there will be a massless particle

(Nambu - Goldstone mode)

(NG)

<u>Consider</u> 2 scalar particles

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \cdot \partial^\mu \phi - V(\phi \cdot \phi)$$

- This theory has a global $O(2)$ symmetry (continuous)

$$\phi \to \phi' = R\phi$$

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- θ is continuous, constant (global symmetry)

- Under R $\quad \underbrace{\phi \cdot \phi}_{\phi^T \phi} = \phi'^T \phi' = (R\phi)^T (R\phi)$

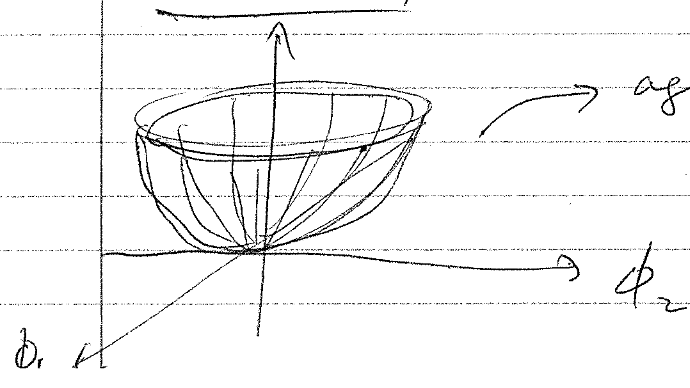
$$= \phi^T R^T R \phi$$

$$= \phi^T \phi$$

$\therefore \quad \mathcal{L} \to \mathcal{L}$ under $O(2)$

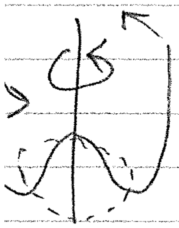
- <u>Suppose</u> $\quad V(\phi \cdot \phi) = \frac{1}{2} m^2 \phi \cdot \phi + \frac{1}{4} \lambda (\phi \cdot \phi)^2$

- <u>If $m^2 > 0$</u>, then

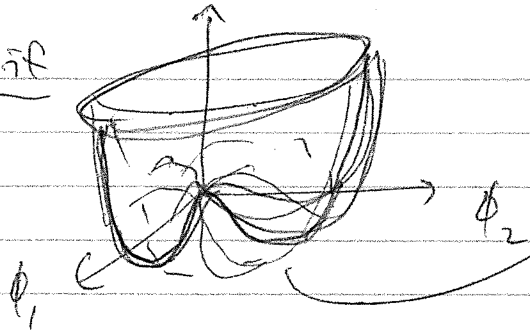


again, has unique vacuum solution.

$$\langle \phi \rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



But if



→ have circle of possible ground state.

→ Nature spontaneously picks a vacuum

Let's pick $\langle \phi \rangle = \begin{pmatrix} v \\ 0 \end{pmatrix}$

Can look at excitations about ϕ $\qquad$ $\langle \phi' \rangle = 0$

$\hookrightarrow \phi' = \phi - \langle \phi \rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} - \begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} \phi_1 - v \\ \phi_2 \end{pmatrix}$

For small excitations

$\hookrightarrow \phi' = \langle \phi' \rangle + \varepsilon = \langle \phi' \rangle + \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} \eta \\ \xi \end{pmatrix}$

Can express $\mathcal{L}$ in terms of these.

Find that → $\begin{cases} \text{one is massless} \to \text{NG mode} \\ \text{the other is massive} \to \text{Higgs particle} \end{cases}$

Feb 28
2019

$\boxed{\text{Continuous global symmetry}}$

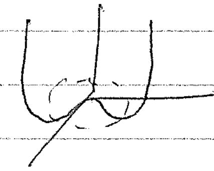
2 scalar fields $\quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad$ Consider $\phi' \longrightarrow R\phi$

where R is a rotation matrix, θ has no x dependent

Look at the $\mathcal{L}$ $\qquad \boxed{\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - V(\phi \cdot \phi)}$

$m^2 < 0 \Rightarrow V(\phi\cdot\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4$

has min at $\langle\phi\rangle^2 = -\frac{m^2}{\lambda} \equiv v^2$



let's pick $\langle\phi\rangle = \begin{pmatrix} v \\ 0 \end{pmatrix}$

Can shift $\phi' = \phi - \langle\phi\rangle$

so that $\langle\phi'\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Excitation round the vacuum $\phi' = \cancel{\langle\phi\rangle} + \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} \eta \\ \xi \end{pmatrix}$

and $\phi = \langle\phi\rangle + \phi' = \begin{pmatrix} v+\eta \\ \xi \end{pmatrix}$

So $\phi\cdot\phi = (v+\eta)^2 + \xi^2$

and $\partial_\mu\phi = \partial_\mu\phi' = \begin{pmatrix} \partial_\mu\eta \\ \partial_\mu\xi \end{pmatrix}$

Can write $\mathcal{L}$ in terms of the excitations, dropping cubic & higher powers

$$V(\phi\cdot\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}\lambda\phi^4$$

$$= \frac{1}{2}m^2\left[(v+\eta)^2 + \xi^2\right] + \frac{1}{4}\lambda\left[(v+\eta)^2 + \xi^2\right]^2$$

$$= \frac{1}{2}m^2\left[v^2 + 2v\eta + \eta^2 + \xi^2\right] + \frac{1}{4}\lambda\left[v^2 + 2v\eta + \eta^2 + \xi^2\right]^2$$

$$= \frac{1}{2}m^2\left[v^2 + 2v\eta + \eta^2 + \xi^2\right] + \frac{1}{4}\lambda\left[4v^3\eta + 4v^2\eta^2 + 2v^2\xi^2 + 2v^2\eta^2 + \dots\right]$$

$$= \eta\left[\frac{1}{2}m^2\cdot 2v + \frac{1}{4}\lambda 4v^3\right] + \eta^2\left[\frac{1}{2}m^2 + \frac{3}{2}\lambda v^2\right]$$

$+ \xi^2\left(m^2/2 + \lambda/2\, v^2 + \dots\right)$

Recall that minimum at $v^2 = -m^2/\lambda$

So $$V(\phi^2) = v\eta\underbrace{(m^2+\lambda v^2)}_{0} + \eta^2\underbrace{\left[\frac{1}{2}m^2 - \frac{3}{2}m^2\right]}_{-m^2} + \frac{1}{2}\varphi^2\underbrace{\left(m^2+(-m^2)\right)}_{0}$$

So to 2nd order, $V(\phi^2) = -m^2\eta^2 = \frac{1}{2}\left(-2m^2\eta^2\right)$

Back to $\mathcal{L}$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi\cdot\phi)$$

$$= \frac{1}{2}\begin{pmatrix}\partial_\mu\eta & \partial_\mu\varphi\end{pmatrix}\begin{pmatrix}\partial^\mu\eta \\ \partial^\mu\varphi\end{pmatrix} - \left[\frac{1}{2}\left(-2m^2\eta^2\right)\right] + \ldots$$

$$= \frac{1}{2}\left[\partial_\mu\eta\,\partial^\mu\eta + \partial_\mu\varphi\,\partial^\mu\varphi\right] - \frac{1}{2}\left(-2m^2\right)\eta^2 + \ldots$$

We started with 2 scalars... $\phi = (\phi_1\ \phi_2)^T$ and wrong sign $m^2<0$
But after spontaneous symmetry breaking around a physical vacuum, we have

$$\mathcal{L} = \underbrace{\frac{1}{2}\partial_\mu\eta\,\partial^\mu\eta - \frac{1}{2}\left(-2m^2\eta^2\right)}_{} + \underbrace{\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi}_{}$$

1 massive scalar η with mass $-2m^2 > 0$

1 massless scalar φ

This is called $\boxed{\text{Higgs Boson}}$

NG mode

$\boxed{\text{Goldstone}}$ → { For every continuous global symmetry that is spontaneously broken you get a massless mode }

Ex EM → U(1) local gauge symmetry → massless photon

But Weak interactions → we'd like to describe these as a gauge theory as well. SU(2) → 3 gauge fields

But the WI is too weak + short ranged
→ Maybe the weak force is carried by 3 massive vector fields.

But can't have both gauge sym and massive terms for the gauge fields.

Note Any interacting massless particle is detectable because it's got a long-range int

↳ NG mode → detectable, but not seen....

#

$\boxed{\text{Cams Higgs + others}}$ Looked Spon. Sym. Breaking of a local gauge theory

→ Found a mechanism where the massless NG modes get "eaten" and the gauge fields acquire mass

$\boxed{\text{Higgs Mechanism}}$

Take Local O(2)

Consider $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ 2 real. And rotation is Local

$$\phi \Rightarrow \phi' = R(x)\phi$$

Take $R(x) = (2\times 2)$ matrix

$$R(x) = \begin{pmatrix} \cos\alpha(x) & -\sin(\alpha(x)) \\ \sin(\alpha(x)) & \cos(\alpha(x)) \end{pmatrix} \qquad \alpha \to \alpha(x) \text{ local}$$

Note $\det(R(x)) = 1 \to SO(2)$ (abelian)

Can write this as $R = e^{i\alpha(x) T} \to$ 2×2 matrix (generator)

For $O(2) \to T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ observe that $T^2 = I$
$T^T = -T$
$T^\dagger = T$

$\to$ (T is Hermitian)

For small α $e^{i\alpha(x)T} = R \simeq I + i\alpha(x)T + \dots$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\alpha(x)\begin{pmatrix} 0 & +i \\ -i & 0 \end{pmatrix}$$

So $R \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\alpha(x) \\ \alpha(x) & 0 \end{pmatrix} + \dots$

$$= \begin{pmatrix} \cos\alpha(x) & -\sin\alpha(x) \\ \sin(\alpha(x)) & \cos\alpha(x) \end{pmatrix}$$ when all powers included

If $\alpha \ll 1$, then $R \simeq \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}$

Now, R has to be orthogonal

$$\left.\begin{aligned} R &\simeq I + i\alpha T \\ R^T &= I + i\alpha T^T = I - i\alpha T \end{aligned}\right\} \to \left\{\begin{aligned} &R \text{ is still} \\ &\text{orthogonal} \end{aligned}\right\}$$

So $R^T R = (I - i\alpha T)(I + i\alpha T) = I + \cancel{\alpha^2}^{\,0} \dots$

Consider $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi - V(\phi \cdot \phi)$

Let $\phi \to \phi' = R(x) \phi$

so $\partial_\mu \phi \to \partial_\mu \phi' = R \partial_\mu \phi + (\partial_\mu R) \phi$ $\neq 0$ for local transform

There's no local symmetry $\Rightarrow$ fix by changing the deriv..

Need to define a gauge-covariant derivative $\boxed{D_\mu = \partial_\mu + ig A_\mu}$

$g \to$ coupling parameter; $A_\mu \to$ function $\Big\}$ like EM

Now $D_\mu = \partial_\mu + ig A_\mu$ has to be 2×2 to act on $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

$$\boxed{D_\mu = I \partial_\mu + T ig A_\mu} \qquad T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

We want $D_\mu \phi \longmapsto D'_\mu \phi' = R D_\mu \phi$ so that

$$D_\mu \phi \cdot D^\mu \phi \Rightarrow (R D_\mu \phi)^T (R D^\mu \phi)$$
$$= (D_\mu \phi)^T R^T R (D^\mu \phi)$$
$$= (D_\mu \phi)^T (D^\mu \phi)$$
$$= (D_\mu \phi) \cdot (D^\mu \phi)$$

How must A_μ transform to get this?

Look at $D'_\mu \phi' = (\partial_\mu + ig A'_\mu) R \phi$

$$= R \partial_\mu \phi + (\partial_\mu R) \phi + ig A'_\mu R \phi$$

set

Let $D_\mu' \phi' = R\partial_\mu \phi + (\partial_\mu R)\phi + ig A_\mu' R\phi = R D_\mu \phi$

$= R(\partial_\mu + ig A_\mu)\phi$

$= R\partial_\mu \phi + R\, ig A_\mu \phi$

So these equal if $ig A_\mu' R\phi = igRA_\mu \phi - \cancel{ig A_\mu' R\phi}$ $(\partial_\mu R)\phi$

So $\frac{ig}{ig} A_\mu' \cancel{R} = RA_\mu - \frac{1}{ig}\partial_\mu R$

So $A_\mu' R = RA_\mu - \frac{1}{ig}\partial_\mu R$

So $A_\mu' R R^{-1} = RA_\mu R^{-1} + \frac{i}{g}(\partial_\mu R) R^{-1}$

So
$$\boxed{A_\mu' = RA_\mu R^{-1} + \frac{i}{g}(\partial_\mu R) R^{-1}}$$

under local $O(2)$ gauge transf $A_\mu \to A_\mu'$

Ex with $R = e^{i\alpha(x)T}$
$R^{-1} = R^T = e^{i\alpha(x)T^T} = e^{-i\alpha(x)T}$ $(T^T = -T)$

So $RR^{-1} = I$

So $A_\mu \to A_\mu' = RA_\mu R^{-1} + \frac{i}{g}(\partial_\mu R)R^{-1}$

$= e^{i\alpha T} A_\mu e^{-i\alpha T} + \frac{i}{g}\left[i(\partial_\mu \alpha) T R\right] R^{-1}$

really, $A_\mu' = A_\mu' T$
fn $A_\mu = A_\mu T$
matrix

Notice $e^{i\alpha T} T = T e^{i\alpha T}$

So $A'_\mu T = A_\mu T e^{i\alpha T} e^{-i\alpha T} - \frac{1}{g}(\partial_\mu \alpha) T$

Take away the T's, the functions obey

$$A'_\mu = A_\mu - \frac{1}{g}(\partial_\mu \alpha)$$

Call $\Lambda(x) = -\frac{\cancel{\partial_\mu}\,\alpha(x)}{g}$

get $\boxed{A'_\mu = A_\mu + \partial_\mu \Lambda(x)}$ → same as $U(1)$ case.

So this is still Maxwell's theory. Note, also get

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to F'_{\mu\nu} = F_{\mu\nu} \quad \text{(g. invariant)}$$

So Need A_μ in D_μ to have Local gauge sym.

→ Make A_μ dynamical by adding $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$

So then a local O(2) gauge theory is

$$\boxed{\mathcal{L} = \frac{1}{2}(D_\mu \phi)\cdot(D^\mu \phi) - V(\phi\cdot\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

This is invariant under local O(2)

$$\begin{cases} \phi \to R(x)\phi \\ A_\mu \to R A_\mu R^{-1} + \frac{i}{g}(\partial_\mu R)R^{-1} \end{cases}$$

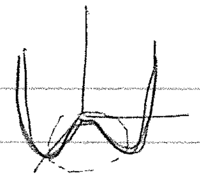
~~D_μ~~ where $D_\mu = \partial_\mu + igA_\mu$

Now

Let $V(\phi\cdot\phi) = \frac{1}{2}m^2 \phi\cdot\phi + \frac{1}{4}\lambda(\phi\cdot\phi)^2$

$\boxed{\text{If } m^2 > 0}$ → no SSB.

$$\left.\begin{array}{l}\text{Have massless } A_\mu \to 2 \text{ modes} \\ \text{And 2 massive scalars } \phi_1, \phi_2\end{array}\right\} \quad 4 \to ?$$

$\boxed{\text{If } m^2 < 0}$ $\rightarrow$  get SSB.

Goldstone theorem says, for a global symmetry that you get one massive Higgs scalar + massless NG mode (2)
(not 4)

$\rightarrow$ Not true for local symmetry.

↳ with SSB of local O(2) can have Higgs mechanism.

Result NG mode gets eaten and $A_\mu \to A'_\mu$, which is massive

left with massive A'_μ ——→ 3
+
massive Higgs scalar → $\underline{1}$
4
(no NG mode)

Higgs Mechanism

if we have $\mathcal{L} = \frac{1}{2} D_\mu \phi \cdot D^\mu \phi - V(\phi \cdot \phi) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$

2nd

$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ and $D_\mu \phi = (\partial_\mu - i g A_\mu)\phi$

↳ $(\mathbb{I}\partial_\mu - i g T A_\mu)\phi$

Sub A'_μ into $F_{\mu\nu}F^{\mu\nu}$

This has local O(2) invariance

$$\begin{cases} \phi \to \phi' = R\phi \\ A_\mu \to A'_\mu = R A_\mu R^{-1} + \frac{i}{g}(\partial_\mu R) R^{-1} \end{cases}$$

with SSB $\rightarrow$ get Higgs mechanism

Let $V(\phi \cdot \phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4$ $\quad (\phi^2 = \phi \cdot \phi)$

if $m^2 < 0$, we have minimum at $\langle\phi\rangle^2 = -\frac{m^2}{\lambda} = v^2$

<u>Pick</u> $\langle\phi\rangle = \begin{pmatrix} 0 \\ v \end{pmatrix} \Rightarrow$ spontaneously breaks $O(2)$

<u>Recall</u> Generator for $O(2)$ is $T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

An infinitesimal gauge transformation is

$$e^{i\alpha T} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

<u>For small α</u> $\longrightarrow \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}$

Now, we can write an arbitrary ϕ in a special way

$$\phi = R^{-1}\phi' = R^{-1}\begin{pmatrix} 0 \\ v+\varepsilon \end{pmatrix} \quad \text{and let } \alpha = \frac{\varphi}{v}$$

$\hookrightarrow$ reparameterization

$$R^{-1} = e^{-i\alpha T} \simeq \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{\varphi}{v} \\ -\frac{\varphi}{v} & 1 \end{pmatrix}$$

this is what we have earlier

<u>So</u> $$\phi = R^{-1}\phi' = \begin{pmatrix} 1 & +\varphi/v \\ -\frac{\varphi}{v} & 1 \end{pmatrix}\begin{pmatrix} 0 \\ v+\varepsilon \end{pmatrix} \simeq \begin{pmatrix} \varphi \\ v+\varepsilon \end{pmatrix}$$

excitation, small

<u>Recall</u> $\langle\phi\rangle + \begin{pmatrix} \varphi \\ \varepsilon \end{pmatrix}$

$\downarrow$

$\begin{pmatrix} 0 \\ v \end{pmatrix}$

<u>Note</u>

This parameterization is called the <u>"unitary gauge"</u>

Can put $\phi = R^{-1}\phi'$ into $\mathcal{L}$.

$$\mathcal{L} = \frac{1}{2} D_\mu (R^{-1}\phi') \cdot D^\mu (R^{-1}\phi') - \cancel{V}\left(\left((R^{-1}\phi')\cdot(R^{-1}\phi')\right)\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Since this is gauge invariant, we can perform a gauge transformation.

$$\phi \to R\phi = (R(R^{-1}\phi')) = \phi' \leftarrow \begin{pmatrix} 0 \\ v+\varepsilon \end{pmatrix}$$

At the same time $A_\mu \longrightarrow A'_\mu$ (new gauge field in the new gauge)

This gives

$$\mathcal{L} = \frac{1}{2} D'_\mu \phi' \cdot D'^\mu \phi' - V(\phi'\cdot\phi') - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu}$$

Let's look at $V(\phi'\cdot\phi') =$ ~~same~~ ?

Note $\phi'\cdot\phi' = (v+\varepsilon)^2$

$$\hookrightarrow V(\phi\cdot\phi) = \frac{1}{2}m^2(v+\varepsilon)^2 + \frac{1}{4}\lambda(v+\varepsilon)^4$$

$$= \frac{1}{2}m^2(v^2+2v\varepsilon+\varepsilon^2) + \frac{1}{4}\lambda(v^2+2v\varepsilon+\varepsilon^2)^2$$

Take quadratic in ε

$$= \frac{1}{2}m^2(v^2+2v\varepsilon+\varepsilon^2) + \frac{1}{4}\lambda(v^4+4v^3\varepsilon+6v^2\varepsilon^2 + 4v\varepsilon^3+\varepsilon^4)$$

$$= \varepsilon(m^2v+\lambda v^3) + \varepsilon^2\left(\frac{1}{2}m^2+\frac{3}{2}\lambda v^2+\dots\right)$$

Recall $v^2 = -\dfrac{m^2}{\lambda}$

$$= \underbrace{\varepsilon(-\lambda v^3+\lambda v^3)}_{0} + \varepsilon^2\left(\frac{1}{2}m^2 - \frac{3}{2}m^2+\dots\right)$$

$$\simeq (-\varepsilon^2 m^2)$$

And so the new potential $V(\phi'\cdot\phi') \simeq -m^2\varepsilon^2 = \frac{1}{2}(-2m^2)\varepsilon^2$

We also need to look at D'_μ ...

$$D'_\mu = (\partial_\mu + ig A'_\mu) \quad \text{where } A'_\mu = T A'_\mu$$

$$= (\mathbb{1}\,\partial_\mu + ig\, T A'_\mu)$$

$$= \begin{pmatrix} \partial_\mu & 0 \\ 0 & \partial_\mu \end{pmatrix} + ig \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} A'_\mu$$

$$= \begin{pmatrix} \partial_\mu & 0 \\ 0 & \partial_\mu \end{pmatrix} + \begin{pmatrix} 0 & -ig A'_\mu \\ ig A'_\mu & 0 \end{pmatrix}$$

So $D'_\mu \phi' = \left[\begin{pmatrix} \partial_\mu & 0 \\ 0 & \partial_\mu \end{pmatrix} + \begin{pmatrix} 0 & -ig A'_\mu \\ ig A_\mu & 0 \end{pmatrix} \right] \begin{pmatrix} 0 \\ v+\varepsilon \end{pmatrix}$

$$= \begin{pmatrix} -g A'_\mu (v+\varepsilon) \\ \partial_\mu \varepsilon \end{pmatrix}$$

So $D'_\mu \phi' \cdot D'^\mu \phi' = \begin{pmatrix} -g A'_\mu (v+\varepsilon) \\ \partial_\mu \varepsilon \end{pmatrix}^T \begin{pmatrix} -g A'^\mu (v+\varepsilon) \\ \partial^\mu \varepsilon \end{pmatrix}$

$$= g^2 A'_\mu A'^\mu (v+\varepsilon)^2 + \partial_\mu \varepsilon \cdot \partial^\mu \varepsilon$$

Then the $\mathcal{L}$ becomes

$$\mathcal{L} = \frac{1}{2} g^2 A'_\mu A'^\mu (v+\varepsilon)^2 + \frac{1}{2} \partial_\mu \varepsilon \cdot \partial^\mu \varepsilon - \frac{1}{2}(-2m^2)\varepsilon^2 - \frac{1}{4} F'_{\mu\nu} F'^{\mu\nu}$$

Expand this out

$$\boxed{\begin{aligned} \mathcal{L} = {} & \frac{1}{2} \partial_\mu \varepsilon \cdot \partial^\mu \varepsilon - \frac{1}{2}(-2m^2)\varepsilon^2 - \frac{1}{4} F'_{\mu\nu} F'^{\mu\nu} \\ & + \frac{g^2 v^2}{2} A'_\mu A'^\mu + \frac{g^2}{2}(2v\varepsilon + \varepsilon^2) A'_\mu A'^\mu + \cdots \end{aligned}}$$

$\boxed{\text{This theory describes}}$

(1) $\boxed{\frac{1}{2}\partial_\mu \varepsilon\, \partial^\mu \varepsilon - \frac{1}{2}(-2m^2)\,\varepsilon^2} \rightarrow$ has $-2m^2 > 0$

↳ massive scalar particle → Higgs boson (scalar, spin 0)

(2) $\boxed{-\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{g^2 v^2}{2}A'_\mu A'^\mu} \rightarrow$ massive vector gauge field

(3) $\boxed{\frac{g^2}{2}(2v\varepsilon + \varepsilon^2)A'_\mu A'^\mu}$ Interaction between ε and A'_μ

Note no massless NG mode (φ is gone)

↳ got eaten resulting in A_μ gains mass → A'_μ

we can count degrees of freedom

↳ Before SSB: $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi \cdot D^\mu\phi - V(\phi\cdot\phi)$

$$\left.\begin{array}{l} \text{massless} \rightarrow A_\mu \rightarrow 2 \\ \phi = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \rightarrow 2 \end{array}\right\} 4 \text{ total}$$

After SSB $\rightarrow \mathcal{L} = \frac{1}{2}\partial_\mu \varepsilon\, \partial^\mu \varepsilon - \frac{1}{2}(-2m^2)\,\varepsilon^2 - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu}$
$+ \frac{g^2 v^2}{2}A'_\mu A'^\mu + \ldots$

$$\left.\begin{array}{l} \text{massive scalar } \varepsilon \rightarrow 1 \\ \text{massive gauge field } A'_\mu \rightarrow 3 \end{array}\right\} 4 \text{ total}$$

Next look at $U(1)$ → similar to $SO(2)$

NOETHER'S THEOREM

Nov 14, 2014

Let the action be

$$S = \int_{\Omega} d^4x \, \mathcal{L}\left(\phi^A(x), \partial_\mu \phi^A(x)\right)$$

Consider infinitesimal spacetime + internal transformation

$$x^\mu \longmapsto x'^\mu = x^\mu + \delta x^\mu$$

$$\phi^A(x) \to \phi'^A(x') = \phi^A(x) + \delta\phi^A(x)$$

So

$$\delta S = \int_{\Omega'} d^4x' \, \mathcal{L}\left(\phi'^A(x'), \partial'_\mu \phi'^A(x')\right) - \int_{\Omega} d^4x \, \mathcal{L}\left(\phi^A(x), \partial_\mu \phi^A(x)\right)$$

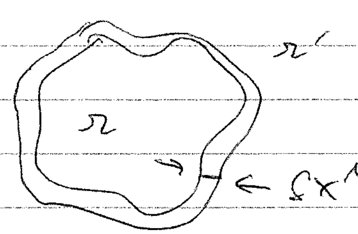
relabel $x' \to x$

But Ω' is the new volume

$$\delta S = \int_{\Omega'} d^4x \, \mathcal{L}\left(\phi'^A(x), \partial_\mu \phi'^A(x)\right) - \int_{\Omega} d^4x \, \mathcal{L}\left(\phi^A(x), \partial_\mu \phi^A(x)\right)$$

$$= \int_{\Omega} d^4x \left[\mathcal{L}\left(\phi'^A(x), \partial_\mu \phi'^A(x)\right) - \mathcal{L}\left(\phi^A(x), \partial_\mu \phi^A(x)\right)\right]$$

$$+ \int_{\Omega'-\Omega} d^4x \, \mathcal{L}\left(\phi'^A(x), \partial_\mu \phi'^A(x)\right)$$



Note
$$\int_{\Omega'-\Omega} d^4x = \int_{\partial\Omega} dS_\lambda \, \delta x^\lambda$$

So
$$\int_{\Omega'-\Omega} d^4x \, \mathcal{L}\left(\phi'^A, \partial_\mu \phi'^A\right) = \int_{\delta\Omega} dS_\lambda \, \delta x^\lambda \, \mathcal{L}\left(\phi'^A, \partial_\mu \phi'^A\right)$$

where to leading order $\delta X^\lambda \mathcal{L}(\phi'^A, \partial_\mu \phi'^A)$ ← old Lag

$$\simeq \delta X^\lambda \mathcal{L}(\phi^A, \partial_\mu \phi^A)$$

Gauss' Law:

$$\int_{\partial\Omega} dS_\lambda\, \delta X^\lambda \mathcal{L}(\phi^A, \partial_\mu \phi^A) = \int_\Omega d^4x\, \partial_\lambda \left(\delta X^\lambda \mathcal{L}(\phi^A, \partial_\mu \phi^A)\right)$$

← divergence

Define

$$\bar\delta f(x) = f'(x) - f(x)$$

$$= \left[f'(x') - f(x)\right] - \left[f'(x') - f'(x)\right]$$

$$= \delta f(x) - \partial_\mu f(x)\, \delta X^\mu$$

where $\delta f(x) = f'(x') - f(x)$

and

$$f'(x') = f'(x+\delta x) = f'(x) + \delta X^\mu \partial_\mu f'(x)$$

$$\simeq f'(x) + \delta X^\mu \partial_\mu f(x)$$

prime is NOT the derivative

$f' \neq \partial_\mu f$

Then

$$\mathcal{L}(\phi'^A(x), \partial_\mu \phi'^A(x)) - \mathcal{L}(\phi^A(x), \partial_\mu \phi^A(x))$$

$$\cancel{= \mathcal{L}(\phi^A(x), \partial_\mu \phi^A(x)) + \bar\delta}$$

$$= \mathcal{L}\left(\phi^A(x) + \bar\delta\phi^A(x), \partial_\mu \phi^A + \bar\delta \partial_\mu \phi^A\right) - \mathcal{L}(\phi^A(x), \partial_\mu \phi^A(x))$$

commute

$$= \mathcal{L}\left(\phi^A(x) + \bar\delta\phi^A(x), \partial_\mu \phi^A + \partial_\mu \bar\delta\phi^A\right) - \mathcal{L}(\phi^A(x), \partial_\mu \phi^A(x))$$

$$\simeq \cancel{\mathcal{L}(\phi^A(x), \partial_\mu \phi^A)} + \frac{\delta\mathcal{L}}{\delta\phi^A}\bar\delta\phi^A + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^A}\partial_\mu\bar\delta\phi^A - \cancel{\mathcal{L}(\phi^A(x), \partial_\mu \phi^A(x))}$$

$$= \underbrace{\left(\frac{\delta \mathcal{L}}{\delta \phi^A} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \right)}_{\text{Euler-Lagrange}} \bar{\delta} \phi^A + \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \cdot \bar{\delta} \phi^A$$

$$+ \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \cdot \partial_\mu \phi^A$$

$$= \underbrace{\left(\frac{\delta \mathcal{L}}{\delta \phi^A} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \right)}_{\text{Euler Lagrange Eqn, 0 on shell}} \bar{\delta} \phi^A + \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \bar{\delta} \phi^A \right)$$

Putting things together

$$\delta S = \int_\Omega d^4x \left(\frac{\delta \mathcal{L}}{\delta \phi^A} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \right) \bar{\delta} \phi^A + \int_\Omega d^4x \, \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \bar{\delta} \phi^A \right)$$

$$+ \int_{\Omega' - \Omega} d^4x \, \partial_\mu \left(\delta x^\mu \mathcal{L} \right)$$

$$= \int_\Omega d^4x \left(\frac{\delta \mathcal{L}}{\delta \phi^A} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \right) \bar{\delta} \phi^A + \int_\Omega d^4x \, \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \bar{\delta} \phi^A + \mathcal{L} \delta x^\mu \right]$$

Now, use $\bar{\delta} \phi^A = \delta \phi^A - \partial_\mu \phi^A \delta x^\mu$ then

$$\partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \bar{\delta} \phi^A + \mathcal{L} \delta x^\mu \right) = \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \delta \phi^A - \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \partial_\nu \phi^A \delta x^\nu + \mathcal{L} \delta x^\mu \right]$$

$$= \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \delta \phi^A \right) - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \partial^\nu \phi^A - \eta^{\mu\nu} \mathcal{L} \right) \delta x_\nu$$

Call $\boxed{T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \partial^\nu \phi^A - \eta^{\mu\nu} \mathcal{L}}$ → energy momentum so

$$\delta S = \int_\Omega d^4x \left(\frac{\delta \mathcal{L}}{\delta \phi^A} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \right) \delta \phi^A + \int_\Omega d^4x \, \overset{\partial_\mu}{\left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \delta \phi^A - T^{\mu\nu} \delta x_\nu \right)}$$

Define

$$\boxed{J^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \delta \phi^A - T^{\mu\nu} \delta x_\nu} \leftarrow \text{current}$$

get

$$\delta S = \int_\Omega d^4x \left(\frac{\delta \mathcal{L}}{\delta \phi^A} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \right) \delta \phi^A + \int_\Omega d^4x \, \partial_\mu J^\mu \rightarrow \text{div}(J)$$

Result: if $\phi^A \mapsto \phi^A + \delta\phi^A$
and/or
$x^\mu \mapsto x^\mu + \delta x^\mu$

} are/is a symmetry transform

$\rightarrow \boxed{\delta S = 0}$ Then if also ϕ^A is on shell - obeys eqs of motion, then

$$\boxed{\frac{\delta \mathcal{L}}{\delta \phi^A} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} = 0}$$

$\text{div}(J^\mu) = 0$

These together give $\boxed{\partial_\mu J^\mu = 0}$

This is the Noether's Theorem result

$\rightarrow$ | when a theory has a symmetry & the equations of motion hold you get a <u>conserved</u> quantity |

Consider $\partial_\mu J^\mu = 0$

Integrate over space $\rightarrow$ use $J^\mu = (\rho, \vec{J})$ then,

$$\int d^3x\, \partial_\mu J^\mu = 0$$

$$\int d^3x\, (\partial_0 J^0 + \partial_j J^j) = 0 \longrightarrow \vec{\nabla}\cdot\vec{J}$$

$$\frac{d}{dt}\int d^3x\, J^0 + \int d^3x\, \underbrace{\partial_j J^j}_{} = 0$$

&

$$\frac{d}{dt}\int d^3x\, \underbrace{J^0}_{\rho} + \int d^3x\, \vec{\nabla}\cdot\vec{J} = 0$$

$$\frac{d}{dt}\underbrace{\int d^3x\, \rho}_{Q} + \underbrace{\int_S d\vec{A}\cdot\vec{J}}_{0 \text{ if } \vec{J}\to\vec{0} \text{ on the boundary}} \longrightarrow \text{Gauss's Law.}$$

So $\boxed{\dfrac{dQ}{dt} = 0}$ → conservation of charge.

#

Mar 21, 2019

GRAVITATIONAL ACTIONS

- Flat spacetime $S = \int \mathcal{L}\, d^4x$
- ↳ metric $= \eta_{\mu\nu}$

But in curved space, $g_{\mu\nu} \neq \eta_{\mu\nu}$.

- if $X^\mu \longmapsto X^{\mu'}$ then $dX^{\mu'} = \Lambda^{\mu'}_{\ \nu} X^\nu$

and $[\Lambda^{\mu'}_{\ \nu}] = \dfrac{\partial X^{\mu'}}{\partial X^\nu}$ the Jacobian

and $g_{\mu'\nu'} = \Lambda^{\alpha}_{\ \mu'} \Lambda^{\beta}_{\ \nu'} g_{\alpha\beta}$

volume elements → $d^4x \longrightarrow d^4x' = \left|\dfrac{\partial X^{\mu'}}{\partial X^{\nu}}\right| d^4x$ ← $\det|\Lambda^{\mu'}_{\ \nu}|$

⇒ Need an invariant volume element ⇒ compensate with a factor of

$$g = \det(g_{\mu\nu}) = |g_{\mu\nu}|$$

Since

$$g_{\mu\nu} = \begin{pmatrix} + & & & \\ & - & & \\ & & - & \\ & & & - \end{pmatrix}, \quad \det(g) < 0$$

So $-g > 0$

For the determinants ⇒ $g' = \left|\frac{\partial x}{\partial x'}\right|^{2} g$

So $g' = \left|\frac{\partial x'}{\partial x}\right|^{-2} g$

So

$$\sqrt{-g'} = \left|\frac{\partial x'}{\partial x}\right|^{-1} \sqrt{-g}$$

Then

$$d^4x\,\sqrt{-g} = d^4x' \left|\frac{\partial x'}{\partial x}\right|^{-1} \left|\frac{\partial x'}{\partial x}\right| \sqrt{-g'} = d^4x'\sqrt{-g'}$$

So $\boxed{d^4x\,\sqrt{-g} \text{ is an invariant volume element}}$

If $g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ then $g = -1$ & $\sqrt{-g} = 1$

So, for curved spaces

coord - invariant

$$\boxed{S = \int d^4x\,\sqrt{-g}\,\mathcal{L}}$$

The action for pure gravity (no matter) in GR is $(c=1)$

$$\boxed{S = \int d^4x\,\sqrt{-g}\,\frac{1}{16\pi G} R}$$ ← Einstein-Hilbert action

Here $R^{\mu}{}_{\mu}$ where $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = g^{\lambda\beta} R_{\beta\mu\lambda\nu}$

$\hookrightarrow$ $R = R^{\mu\nu}{}_{\mu\nu} = g^{\mu\nu} R_{\mu\nu}$

Recall

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

where

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)$$

We need to vary $\mathcal{L} = \int d^4x \sqrt{-g}\, R$ with respect to $g_{\mu\nu}$ or $g^{\mu\nu}$

Need to pick either $g_{\mu\nu}$ or $g^{\mu\nu}$ as the fundamental field.

- These obey

$$g^{\mu\nu} g_{\nu\sigma} = \delta^{\nu}_{\sigma} \leftarrow \underline{\text{constant}}.$$

$\hookrightarrow$ So $(\delta g^{\mu\nu}) g_{\nu\sigma} + (g^{\mu\nu})(\delta g_{\nu\sigma}) = 0$

So $\boxed{(\delta g^{\mu\nu})\, g_{\nu\sigma} = -(g^{\mu\nu})(\delta g_{\nu\sigma})}$

Now, multiply with $g^{\sigma\rho}$

$$\hookrightarrow (\delta g^{\mu\nu})\, g_{\nu\sigma} g^{\sigma\rho} = -g^{\mu\nu} g^{\sigma\rho} (\delta g_{\nu\sigma})$$

$$(\delta g^{\mu\rho}) = -g^{\mu\nu} g^{\sigma\rho}\, \delta g_{\nu\sigma}$$

re write $\hookrightarrow$ $\boxed{\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta}\, \delta g_{\alpha\beta}}$

Likewise,

$$\boxed{\delta g_{\mu\nu} = -g_{\mu\alpha} g_{\nu\beta}\, \delta g^{\alpha\beta}}$$

☒ Carroll uses $g^{\mu\nu}$ as fundamental

Look at $\sqrt{-g}\,\mathcal{L} = \sqrt{-g}\,R = \sqrt{-g}\,g^{\mu\nu}R_{\mu\nu}$

So $\delta(\sqrt{-g}\,\mathcal{L}) = \delta\left[\sqrt{-g}\,g^{\mu\nu}R_{\mu\nu}\right]$

$$= (\delta\sqrt{-g})\,g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}\,(\delta g^{\mu\nu})\,R_{\mu\nu} + \sqrt{-g}\,g^{\mu\nu}(\delta R_{\mu\nu})$$

we know this

Want $\delta S = \int d^4x \,(\;)_{\mu\nu}\,\delta g^{\mu\nu} = 0$

so that $(\;)_{\mu\nu} = 0$ should give Einstein's eqn

$\boxed{\text{What is } \delta\sqrt{-g} \text{ in terms of } \delta g^{\mu\nu}?}$

Here $g = \det(g_{\mu\nu})$

There's an identity for matrices → $\boxed{\ln(\det M) = \mathrm{Tr}(\ln M)}$

verify with silly example ~ $M = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}$

then $\det M = abc \to \ln(\det M) = \ln(abc) = \ln(a) + \ln(b) + \ln(c)$

$$= \mathrm{Tr}\begin{pmatrix} \ln a & & \\ & \ln b & \\ & & \ln c \end{pmatrix} = \mathrm{Tr}(\ln M)$$

We can use $M = [g_{\mu\nu}] \to M^{-1} = [g^{\mu\nu}]$

$g = \det(g_{\mu\nu}) = \det(M)$

So, vary the identity, get

$$\frac{1}{\det M}\,\delta\det M = \mathrm{Tr}(M^{-1}\delta M)$$

a number

So $\frac{1}{g}(\delta g) = \mathrm{Tr}(g^{\mu\nu}\delta g_{\mu\nu}) = g^{\mu\nu}\delta g_{\mu\nu}$

So $\boxed{\delta g = g\,g^{\mu\nu}\delta g_{\mu\nu}} = \boxed{-g\,g_{\mu\nu}\delta g^{\mu\nu}}$

Then $\delta\sqrt{-g} = \delta(-g)^{1/2} = \frac{1}{2}(-g)^{-\frac{1}{2}}\,\delta(-g) = \frac{-1}{2}\frac{1}{\sqrt{-g}}\,\delta g$

$= \frac{-1}{2}\frac{1}{\sqrt{-g}}\cdot(-g)\,g_{\mu\nu}\,\delta g^{\mu\nu}$

$$\boxed{\delta\sqrt{-g} = \frac{-1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu} = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\,\delta g_{\mu\nu}}$$

Now, what about δR ? , i.e. what is $(\delta R_{\mu\nu})$?

Recall $\delta(\sqrt{-g}R) = (\delta\sqrt{-g})\,\underbrace{g^{\mu\nu}R_{\mu\nu}}_{R} + \sqrt{-g}\,\delta g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}\,g^{\mu\nu}\,\delta R_{\mu\nu}$

$= -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}$

So $\delta(\sqrt{-g}\,R) = \sqrt{-g}\,\underbrace{\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nv}R\right)}\,\delta g^{\mu\nu} + \sqrt{-g}\,g^{\mu\nu}\,\delta R_{\mu\nu}$

Einstein tensor $\to$ $G_{\mu\nu}$. Recall Einstein eq: $G_{\mu\nu} = 8\pi G\,T_{\mu\nu}$

For no matter $\longrightarrow$ $T_{\mu\nu} = 0$ so $G_{\mu\nu} = 0$

Turns out that $\boxed{\int d^4x\,\sqrt{-g}\,g^{\mu\nu}\,\delta R_{\mu\nu} = 0}$ (page 162, Carroll)

(derive this by int. by parts)

Notice it is NOT that $\delta R_{\mu\nu} = 0$

Remember $\to$ keep everything in an integral ...

With

$$S = \int d^4x\,\sqrt{-g}\,\frac{1}{16\pi G}\,R$$

then

$$(16\pi G)\,\delta S = \int d^4x\,\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} = 0$$

So $\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0}$ Einstein equations with no matter... (pure gravity) $(\Lambda = 0)$

With $\Lambda \neq 0$ and no matter

$$S = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,\left(R - 2\Lambda\right)$$

verify this (easy)

get $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0$

Now, how do we add matter?

matter terms

$$S = \int d^4x\,\sqrt{-g}\left(\frac{1}{16\pi G}(R-2\Lambda) + \mathcal{L}_{matter}\right)$$

most simple scalar field $\mathcal{L} \overset{?}{=} \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$

But really, we have

$$\mathcal{L} \overset{?}{=} \frac{1}{2}(\partial_\mu\phi)\,g^{\mu\nu}(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2$$

notice that there's some interaction ...

Recall

For scalars $D_\mu\phi = \partial_\mu\phi = \phi_{;\mu} = \phi_{,\mu} = \nabla_\mu\phi$

If we vary this action, we also need to include

$$\delta\left(\sqrt{-g}\left[\overset{?}{=}\frac{1}{2}(\partial_\mu\phi)\,g^{\mu\nu}(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2\right]\right)$$

How does this give $T_{\mu\nu}$? By definition!

⇒ Any matter fields add extra stuff to vary wrt $g^{\mu\nu}$ in $\sqrt{-g}\,\mathcal{L}_M$

Simply define

$$\delta S = \delta S_g + \delta S_M = \int d^4x\,\sqrt{-g}\,\frac{1}{16\pi G}(R-2\Lambda) + \int d^4x\,\sqrt{-g}\,\mathcal{L}_M$$

$$\delta S_M = \int d^4x \; \delta(\sqrt{-g}\,\mathcal{L}_M)$$

Define $$\boxed{\delta\left(+\sqrt{-g}\,\mathcal{L}_M\right) \equiv -\frac{1}{2}\sqrt{-g}\,T_{\mu\nu}\,\delta g^{\mu\nu}}$$

or $$\boxed{T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left(+\sqrt{-g}\,\mathcal{L}_M\right)}$$

With that, $(\Lambda = 0)$

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{16\pi G}R + \mathcal{L}_M\right]$$

$$\Rightarrow \quad \delta S = \int d^4x \left\{\frac{\sqrt{-g}}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + \delta\left(\sqrt{-g}\,\mathcal{L}_M\right)\right\}$$

$$= \int d^4x \left[\frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) - \frac{1}{2}T_{\mu\nu}\right]\sqrt{-g}\,\delta g^{\mu\nu} = 0$$

Therefore

$$\frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = \frac{1}{2}T_{\mu\nu}$$

so

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\,T_{\mu\nu}}$$

If $\mathcal{L}_M = \frac{-1}{4}F^{\mu\nu}F_{\mu\nu} \rightarrow \sqrt{-g}\,\mathcal{L}_M = \frac{-1}{4}\sqrt{-g}\,F_{\mu\nu}\,g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}$

and we can show

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Vary wrt $g^{\mu\nu}$ to get $T^{\mu\nu}$ from this (verify!)

$$\boxed{T_{\mu\nu} = F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta}F^{\alpha\beta}}$$

How do we know this is actually energy-momentum?

$T_{00} \sim (\vec{E}^2 + \vec{B}^2) \sim$ energy density

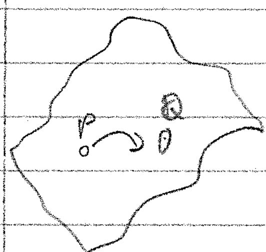
$T_{0j} \sim$ Poynting vector...

April 11, 2019

DIFFEOMORPHISMS

A diffeomorphism is a mapping of one manifold to another.

In GR ⇒ mapping of spacetime to itself.



Carroll's book describes all the math...

P is at x^μ
Q is at $x^\mu + \xi^\mu$

then $x^\mu \to x^\mu + \xi^\mu$ under a diffeomorphism

<u>Want to know</u> → How scalars, vectors, tensors change under diff. + how it's a symmetry of GR

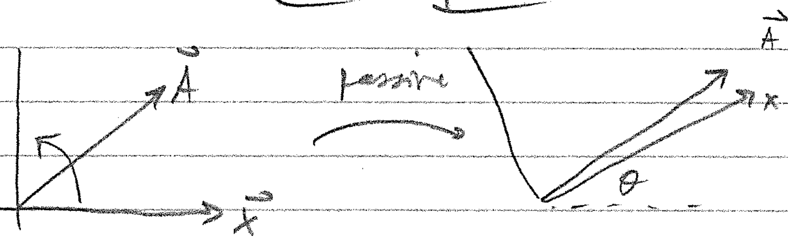
Changes in tensors are given by the Lie derivative

We'll look at the "passive" version of coord. transformation.

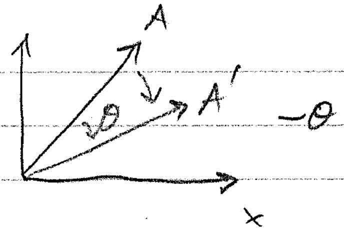
↳ Coord. transf can be <u>active</u> or <u>passive</u>.

$\boxed{\text{Rotations}}$

"passive"



"Active"



See that the new component of $\vec{A}$ in X' frame under a passive transformation are the same as the components $A^{i'}$ of $\vec{A}'$ under an active transformation

Symmetries always involve active transformations. With unbroken symmetries these are inverses of passive transformations

Even though passive transformations are just observer changes, we can still use them to find the form of " transformation"

In GR, diffeo is an active transform: $A^\mu \to A'^\mu$ under moving or translating $x^\mu \to x^\mu + \xi^\mu$

The passive version is a general transformation

$$x^\mu \to x^{\mu'}(x) = x^\mu - \xi^\mu \qquad (-\text{ because inverse})$$

We can use the inverse general coord. transf to find the form of Lie derivative

$\mathcal{L}_\xi \to$ denotes a Lie deriv using ξ^μ (4 transformations)

$$\begin{cases} A_\mu \to A_\mu + \mathcal{L}_\xi A_\mu & \text{under diffs} \\ A^\mu \to A^\mu + \mathcal{L}_\xi A^\mu \end{cases}$$

If $\delta S = 0$ under diffs, then the theory is diffeo. invariant $\Rightarrow$ GR is diffeomorphism invariant

2 Consider an infinitesimal general coord. transform:

$$x^{\mu'} = x^{\mu'}(x) = x^\mu - \xi^\mu$$

Jacob. matrix

Now, a vector under GCT obeys $A^{\mu'}(x') = X^{\mu'}_{\ \nu} A^\nu(x)$

$$X^{\mu'}_{\ \nu} = \frac{\partial x^{\mu'}}{\partial x^\nu} = \frac{\partial}{\partial x^\nu}\left(x^\mu - \xi^\mu\right) = \delta^\mu_\nu - \partial_\nu \xi^\mu$$

Then

$$A^{\mu'}(x') = \left(\delta^\mu_\nu - \partial_\nu \xi^\mu\right) A^\nu(x) =$$

$x' = x - \xi$ + Taylor expand

$$A^{\mu'}(x-\xi) \cong A^{\mu'}(x) - \xi^\nu \partial_\nu A^{\mu'}(x) + \dots$$

Then we $\xi^\nu \partial_\nu A^{\mu'}(x) \simeq \xi^\nu \partial_\nu A^\mu(x)$ + 2nd order...

So $A^{\mu'}(x') = A^{\mu'}(x) - \xi^\nu \partial_\nu A^\mu(x) = A^\mu(x) - (\partial_\nu \xi^\mu) A^\nu(x)$

So $\boxed{A^{\mu'}(x) = A^\mu(x) - (\partial_\nu \xi^\mu) A^\nu(x) + \xi^\nu (\partial_\nu A^\mu(x))}$

This gives the same result as for the active diff.

$$A^\mu(x) \to A^{*\mu'}(x) = A^\mu(x) + \mathcal{L}_\xi A^\mu(x)$$

So we get $\boxed{\mathcal{L}_\xi A^\mu(x) = -(\partial_\nu \xi^\mu) A^\nu(x) + \xi^\nu (\partial_\nu A^\mu(x))}$

Lie derivative of a contravariant vector.

(this is without parallel transport...)

3 For a scalar $\phi(x)$, under $x^{\mu'} = x^\mu - \xi^\mu$

we know that $\phi(x) = \phi'(x') = \phi'(x-\xi) \simeq \phi'(x) - \xi^\nu \partial_\nu \phi'(x)$

$$\simeq \phi'(x) - \xi^\nu \partial_\nu \phi(x)$$

So $\boxed{\phi'(x) = \phi(x) + \xi^\nu \partial_\nu \phi(x)}$

Under a diff, (active) $\phi(x) \to \phi'(x') = \phi(x) + \mathcal{L}_\xi \phi(x)$
$(x^\mu \to x^\mu + \xi^\mu)$

So $\boxed{\mathcal{L}_\xi \phi(x) = \xi^\nu \partial_\nu \phi(x)}$ $\rightsquigarrow$ Lie deriv of a scalar...

- For a covariant vector, $A_\mu(x)$

$$A_{\mu'}(x') = \Xi_{\mu'}^{\ \nu} A_\nu(x) \quad , \quad \Xi_{\mu'}^{\ \nu} \text{ is the inverse of } \Xi^{\mu'}_{\ \nu} = \delta^\mu_\nu - \partial_\nu \xi^\nu$$

We can verify that $\boxed{\Xi_{\mu'}^{\ \nu} = \delta_\mu^\nu + \partial_\mu \xi^\nu}$

by multiplying the two...

$$\Xi_{\mu'}^{\ \nu} \Xi^{\mu'}_{\ \gamma} \stackrel{?}{=} \delta^\nu_\gamma \ ?$$

$$(\delta^\nu_\mu + \partial_\mu \xi^\nu)(\delta^\mu_\gamma - \partial_\gamma \xi^\mu) = \delta^\nu_\gamma - \cancel{\partial_\gamma \xi^\nu} + \cancel{\partial_\gamma \xi^\nu} - 0$$

$$= \delta^\nu_\gamma \quad \text{works!}$$

So then

$$A_{\mu'}(x') = \Xi_{\mu'}^{\ \nu} A_\nu(x)$$

$$= (\delta^\nu_\mu + \partial_\mu \xi^\nu) A_\nu(x) = A_\mu(x) + (\partial_\mu \xi^\nu) A_\nu(x)$$

For $x' = x - \xi$, $A_{\mu'}(x-\xi) \simeq A_{\mu'}(x) - \xi^\nu \partial_\nu A_{\mu'}(x)$

$$\simeq A_{\mu'}(x) - \xi^\nu \partial_\nu A_\mu(x) + \cdots$$

Then,

$$\boxed{A_{\mu'}(x) = A_\mu(x) + (\partial_\mu \xi^\nu) A_\nu(x) + \xi^\nu \partial_\nu A_\mu(x)}$$

Claim, this is the same as $A_\mu(x) \to A_{\mu'}(x) = A_\mu(x) + \mathcal{L}_\xi A_\mu(x)$
when $x^\mu \to x^\mu + \xi^\mu$

So $\boxed{\mathcal{L}_\xi A_\mu(x) = (\partial_\mu \xi^\nu) A_\nu(x) + \xi^\nu \partial_\nu A_\mu(x)}$ $\rightsquigarrow$ Lie deriv of cov. vector

Given these, can guess the form for a tensor, say $T^{\mu\nu}{}_{\sigma}$

$$\boxed{\mathcal{L}_\xi T^{\mu\nu}{}_{\sigma} = -(\partial_\alpha \xi^\mu) T^{\alpha\nu}{}_{\sigma} - (\partial_\alpha \xi^\nu) T^{\mu\alpha}{}_{\sigma} + (\partial_\sigma \xi^\alpha) T^{\mu\nu}{}_{\alpha} + \xi^\alpha \partial_\alpha T^{\mu\nu}{}_{\sigma}}$$

See Carroll's book on page p.434 to get the full mathematical rigor...
But he uses V^μ for ξ^μ.

Exercise Show that the same formulas hold for covariant derivatives everywhere in place of ∂_μ

① Show $\mathcal{L}_\xi \phi = \xi^\alpha \partial_\alpha \phi = \xi^\alpha D_\alpha \phi$

where $D_\alpha \phi = \phi_{;\alpha} = \phi_{,\alpha} = \partial_\alpha \phi$ (done)

② $\mathcal{L}_\xi A^\mu = -(\partial_\alpha \xi^\mu) A^\alpha + \xi^\alpha \partial_\alpha A^\mu$

$= -(D_\alpha \xi^\mu) A^\alpha + \xi^\alpha (D_\alpha A^\mu)$ (connections...)

③ $\mathcal{L}_\xi A_\mu = (\partial_\mu \xi^\alpha) A_\alpha + \xi^\alpha (\partial_\alpha A_\mu)$

$= (D_\mu \xi^\alpha) A_\alpha + \xi^\alpha (D_\alpha A_\mu)$

So Lie derivs of tensors are tensors.

Now, let's look at $g_{\mu\nu}$. Under a diff. $g_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}$

where $\mathcal{L}_\xi g_{\mu\nu} = (D_\mu \xi^\alpha) g_{\alpha\nu} + (D_\nu \xi^\alpha) g_{\mu\alpha} + \xi^\alpha D_\alpha g_{\mu\nu}$

Recall, $D_\lambda g_{\mu\nu} = 0$ (metric tensor is cov. constant)

So $\mathcal{L}_\xi g_{\mu\nu} = D_\mu (g_{\alpha\nu} \xi^\alpha) + D_\nu (g_{\mu\alpha} \xi^\alpha) + 0$

$$= \boxed{D_\mu \xi_\nu + D_\nu \xi_\mu}$$

So, under a diff, $\boxed{g_{\mu\nu} \to g_{\mu\nu} + D_\mu \xi_\nu + D_\nu \xi_\mu}$

However, using partial derivatives, ...

$$\mathcal{L}_\xi g_{\mu\nu} = (\partial_\mu \xi^\alpha) g_{\alpha\nu} + (\partial_\nu \xi^\alpha) g_{\mu\alpha} + \xi^\alpha \partial_\alpha g_{\mu\nu}$$ does not simplify ...

$\to \partial_\lambda g_{\mu\nu} \neq 0 \to$ doesn't simplify nicely.

Next $\to$ See how diffs are a symmetry of GR. What does it mean to break diffeomorphism?

April 18, 2019 $\boxed{\text{Spacetime Symmetry}}$

~~Global~~ Want to consider:

1. Global LTs in Minkowski space (no gravity)
2. Diffeomorphisms in curved spacetime (with gravity)
3. Local LTs in curved spacetime. (w/ gravity)

I. Global LTs in Minkowski Space

You know LT's are coords transform $\to$ $X^\mu \to X^{\mu'}$

$$X^{\mu'} = \Lambda^{\mu'}{}_\nu X^\nu \quad \text{where } \Lambda^{\mu'}{}_\nu \text{ are constants...}$$

Vectors: $V_{\mu'} = \Lambda_{\mu'}{}^\nu V_\nu$; $V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu$

- We typically write $\Lambda^{\mu'}_\nu$ with indices on top of another... Have inverses ...

$$\Lambda^{\mu'}{}_\nu \Lambda^\nu{}_{\alpha'} = \delta^{\mu'}_{\alpha'} = \delta^\mu_\alpha = \Lambda^\mu{}_{\nu'} \Lambda^{\nu'}{}_\alpha .$$

- Also $\eta_{\mu'\nu'} = \Lambda^\alpha{}_{\mu'} \Lambda^\beta{}_{\nu'} \eta_{\alpha\beta}$

But - All this is the passive point of view.

Now, we want to look at the active LT's where x^μ doesn't change.
(no prime indices...)

↳ Now we distinguish $\Lambda_\mu{}^\nu$ vs. $\Lambda^\mu{}_\nu$ these are inverses.
These obey

$$\Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta = \delta_\beta^\alpha$$

and

$$\Lambda^\alpha{}_\mu \Lambda_\beta{}^\mu = \delta_\beta^\alpha \ldots$$

- Also must have $\eta_{\mu\nu} \to \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta \eta_{\alpha\beta} = \eta_{\mu\nu}$

↳ Minkowski metric is unchanged.

- To consider infinitesimal LT's

$$\begin{cases} \Lambda_\mu{}^\nu = \delta_\mu^\nu + \varepsilon_\mu{}^\nu \\ \Lambda^\mu{}_\nu = \delta^\mu_\nu + \varepsilon^\mu{}_\nu \end{cases}$$

where $\varepsilon_\mu{}^\nu \rightsquigarrow$ small + constant. and $(\varepsilon_\mu{}^\nu)^2 = 0$

Check: $\Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta = (\delta_\mu^\alpha + \varepsilon_\mu{}^\alpha)(\delta^\mu_\nu + \varepsilon^\mu{}_\nu)$

$$= \delta^\alpha_\nu + \varepsilon^\alpha{}_\nu + \varepsilon_\nu{}^\alpha + \cancelto{0}{\phi}$$

$$= \delta^\alpha_\nu \text{, provided that } \varepsilon^\alpha{}_\nu = -\varepsilon_\nu{}^\alpha$$

↳ provided that $\varepsilon^\alpha{}_\beta = -\varepsilon_\beta{}^\alpha$

Check $\Lambda^\mu{}_\alpha \Lambda_\nu{}^\alpha = (\delta^\mu_\alpha + \varepsilon^\mu{}_\alpha)(\delta_\nu^\alpha + \varepsilon_\nu{}^\alpha)$

$$= \delta^\mu_\nu + \varepsilon^\mu{}_\nu + \varepsilon_\nu{}^\mu + \cancelto{0}{\phi}$$

$$= \delta^\mu_\nu \text{ since } \varepsilon^\mu{}_\nu = -\varepsilon_\nu{}^\mu \qquad \text{O}$$

Check

$$\eta_{\mu\nu} \overset{?}{=} \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta \eta_{\alpha\beta} = \eta_{\mu\nu} + \varepsilon_\mu{}^\alpha \eta_{\alpha\nu} + \varepsilon_\nu{}^\beta \eta_{\mu\beta} = \eta_{\mu\nu} + \varepsilon_{\mu\nu} + \varepsilon_{\nu\mu}$$

$$= \eta_{\mu\nu}$$

So Minkowski metric is unchanged if $\boxed{\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}}$

The parameters $\varepsilon_{\mu\nu}$ are anti symmetric and 4-dimensional...

$$[\varepsilon_{\mu\nu}] = \begin{pmatrix} 0 & \varepsilon_{01} & & \\ & 0 & & \\ - & & 0 & \\ & & & 0 \end{pmatrix}$$

only 6 independent component.

$\varepsilon_{0j} = -\varepsilon_{j0} \rightsquigarrow 3$ → infinitesimal boosts

$\varepsilon_{jk} = -\varepsilon_{kj} \rightsquigarrow 3$ → infinitesimal rotations...

To summarize how things transform under infinitesimal LTs...

- Scalars $\phi \to \phi$
- Coordinate x^μ don't change. d^4x doesn't change.
- Minkowski metric $\eta_{\mu\nu} \to \eta_{\mu\nu}$ unchanged.

But all dynamical vectors and tensors change.

$$\begin{cases} A_\mu \to A_\mu + \varepsilon_\mu{}^\nu A_\nu \\ A^\mu \to A^\mu + \varepsilon^\mu{}_\nu A^\nu \end{cases}$$

For a tensor $\tau^{\mu\nu}{}_\lambda \to \tau^{\mu\nu}{}_\lambda + \varepsilon^\mu{}_\alpha \tau^{\alpha\nu}{}_\lambda + \varepsilon^\nu{}_\alpha \tau^{\mu\alpha}{}_\lambda + \varepsilon_\lambda{}^\alpha \tau^{\mu\nu}{}_\alpha$

Now, when will $S = \int d^4x\, \mathcal{L}$ be invariant $(\delta S = 0)$ under global LT's?

If $\mathcal{L}$ is a scalar function then under a global LT, $\mathcal{L} \to \mathcal{L}$ so then $S \to S$ or $\delta S = 0$, which says it's a symmetry.

Ex Is $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$ a scalar under LTs?

Under LTs.. $\phi \to \phi$, $\phi^2 \to \phi^2$, and so $\frac{1}{2} m^2 \phi^2 \to \frac{1}{2} m^2 \phi^2$

Now
$$\begin{cases} \partial_\mu \phi \to \partial_\mu \phi + \varepsilon_\mu{}^\alpha \partial_\alpha \phi & \text{(covariant rule)} \\ \partial^\mu \phi \to \partial^\mu \phi + \varepsilon^\mu{}_\alpha \partial^\alpha \phi \end{cases}$$

So
$$(\partial_\mu \phi)(\partial^\mu \phi) = (\varepsilon_\mu{}^\alpha \partial_\alpha \phi + \partial_\mu \phi)(\varepsilon^\mu{}_\alpha \partial^\alpha \phi + \partial^\mu \phi)$$
$$= \cancel{\mathcal{O}(\varepsilon^2)} + (\partial_\mu \phi)(\partial^\mu \phi) + (\partial_\mu \phi)(\varepsilon^\mu{}_\alpha \partial^\alpha \phi) + (\partial^\mu \phi)(\varepsilon_\mu{}^\alpha \partial_\alpha \phi)$$

$$= (\partial_\mu \phi)(\partial^\mu \phi) + \varepsilon^{\mu\alpha}(\partial_\alpha \phi)(\partial_\mu \phi) + \varepsilon^{\mu\alpha}(\partial_\alpha \phi)(\partial_\mu \phi)$$

$$= \cancel{(\partial_\mu \phi)(\partial^\mu \phi) + \varepsilon^{\mu\alpha}(\partial_\alpha \phi)(\partial_\mu \phi) - \varepsilon^{\alpha\mu}(\partial_\mu \phi)(\partial_\alpha \phi)}$$

Now, $\varepsilon^{\mu\alpha}(\partial_\alpha \phi)(\partial_\mu \phi) = -\varepsilon^{\alpha\mu}(\partial_\mu \phi)(\partial_\alpha \phi)$
$= -\varepsilon^{\alpha\mu}(\partial_\alpha \phi)(\partial_\mu \phi)$

So $\varepsilon^{\mu\alpha}(\partial_\alpha \phi)(\partial_\mu \phi) = 0$

So $(\partial_\mu \phi)(\partial^\mu \phi) \to (\partial_\mu \phi)(\partial^\mu \phi)$

Exercise

$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + m^2 A_\mu A^\mu. \text{ Show this is a scalar under global LTs..}}$

Use that $F_{\mu\nu}$ is a tensor..

$$F_{\mu\nu} \to F_{\mu\nu} + \varepsilon_\mu{}^\alpha F_{\alpha\nu} + \varepsilon_\nu{}^\alpha F_{\mu\alpha}$$

Likewise for $A_\mu A^\mu$.. Show $\mathcal{L} \to \mathcal{L}$

2 $\boxed{\text{Diffeomorphism in Curved Spacetime}}$

$$S = \int d^4x \sqrt{-g}\, \mathcal{L}$$

Under diffeomorphism with ξ^μ, scalars & tensors transf with changes given by the Lie deriv...

Scalars : $\phi \to \phi + \xi^\alpha \partial_\alpha \phi$

or $\phi \to \phi + \xi^\alpha D_\alpha \phi$

(cov) Vectors $A_\mu \to A_\mu + \xi\, (\partial_\mu \xi^\alpha) A_\alpha + \xi^\alpha \partial_\alpha A_\mu$

or $A_\mu \to A_\mu + (D_\mu \xi^\alpha) A_\alpha + \xi^\alpha D_\alpha A_\mu$

(contr) Vectors $A^\mu \to A^\mu - (\partial_\alpha \xi^\mu) A^\alpha + \xi^\alpha \partial_\alpha A^\mu$

or $A^\mu \to A^\mu - (D_\alpha \xi^\mu) A^\alpha + \xi^\alpha D_\alpha A^\mu$

Tensors

$$T^\mu{}_\nu \to T^\mu{}_\nu - (D_\alpha \xi^\mu) T^\alpha{}_\nu + (D_\nu \xi^\alpha) T^\mu{}_\alpha + \xi^\alpha D_\alpha T^\mu{}_\nu$$

Now

$$g_{\mu\nu} \to g_{\mu\nu} + D_\mu \xi_\nu + D_\nu \xi_\mu$$

So this means $\sqrt{-g}$ also transforms... We can find identities...

$$\boxed{\Gamma^\mu_{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g})}$$ $\leadsto$ show this...

With this we can show that

$$\boxed{D_\mu V^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, V^\mu)}$$ $\leadsto$ show this

divergence

With these, the Lie deriv of $\sqrt{-g}$ can be found...

$$\boxed{\sqrt{-g} \longrightarrow \sqrt{-g} + \partial_\alpha\left(\sqrt{-g}\,\xi^\alpha\right)}$$

This says $\boxed{\mathcal{L}_\xi \sqrt{-g} = \partial_\alpha\left(\sqrt{-g}\,\xi^\alpha\right) = \sqrt{-g}\, D_\mu \xi^\mu}$

If $\mathcal{L}$ is a scalar under diffs, then

$$\mathcal{L} \longrightarrow \mathcal{L} + \xi^\alpha \partial_\alpha \mathcal{L}$$

But what about S? where $S = \int d^4x\, \mathcal{L}\sqrt{-g}$?

Under diffs? $S \to S + \xi^\alpha \partial_\alpha S$

$$S \to \underset{\substack{\uparrow \\ \text{don't chge}}}{\int d^4x} \left[\sqrt{-g} + \partial_\alpha\left(\sqrt{-g}\,\xi^\alpha\right)\right]\left(\mathcal{L} + \xi^\alpha \partial_\alpha \mathcal{L}\right)$$

$$= \int d^4x \sqrt{-g}\,\mathcal{L} + \int d^4x \left[\sqrt{-g}\,\xi^\alpha \partial_\alpha \mathcal{L} + \partial_\alpha\left(\sqrt{-g}\,\xi^\alpha\right)\mathcal{L}\right]$$

$$+ \cancelto{0}{\int d^4x\, \xi^2}$$

$$= S + \int d^4x\, \partial_\alpha\left(\sqrt{-g}\,\xi^\alpha \mathcal{L}\right)$$

So $S \to S + \underbrace{\int d^4x\, \partial_\alpha\left(\sqrt{-g}\,\xi^\alpha \mathcal{L}\right)}$ $\longrightarrow$ use 4D ~~Stokes's~~... Gauss'

$$S \to S + \left[\int_{\text{3D surface}} d^3x\; \hat{n}_\alpha \left(\sqrt{-g}\,\xi^\alpha \mathcal{L}\right)\right]$$

Push 3D surface to ∞ where $\xi^\alpha = 0$ So $S \to S$ And $\boxed{\delta S = 0}$

And action is unchanged under diffs $\rightarrow$ sym of GR...

Exercise

1 **Show** $\mathcal{L} = \frac{1}{2}(D_\mu\phi)(D^\mu\phi) - \frac{1}{2}m^2\phi^2$ is a scalar under diffs. $\mathcal{L} \to \mathcal{L} + \xi^\alpha D_\alpha \mathcal{L}$

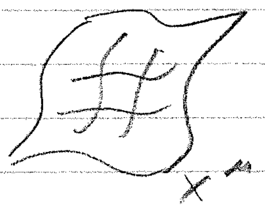
know ϕ^2 is a scalar...

$$D_\mu\phi = \cancel{\partial_\mu\phi} \to D_\mu\phi + (D_\mu\xi^\alpha)D_\alpha\phi + \xi^\alpha D_\alpha D_\mu\phi$$

tensors ← $D^\mu\phi \to \dots$

Verify that $(D_\mu\phi)(D^\mu\phi) \to (D_\mu\phi)(D^\mu\phi) + \xi^\alpha D_\alpha(\text{itself})$

April 25, 2019 $\boxed{\text{LOCAL LORENTZ TRANSFORMATION}}$



$$S = \int \sqrt{-g}\, d^4x \left[\frac{1}{16\pi G}R + \mathcal{L}_M(\phi, A_\mu, \dots, g_{\mu\nu})\right]$$

~> diff. inv.

Where's Lorentz symmetry?
→ local symmetries in local frames.

Local Lorentz frames exist at every point
→ Could coordinate transform into a local Lorentz frame at point P.

$$g_{\mu\nu} \to g_{\mu'\nu'} = X^\alpha_{\mu'} X^\beta_{\nu'} g_{\alpha\beta} = \eta_{\mu'\nu'} \quad \text{at point P}$$

Can even get frames where $\Gamma^{\lambda'}_{\mu'\nu'} = 0$ at point P.

But, if there's curvature, then $R^{\lambda'}{}_{\mu'\nu'\sigma'} \neq 0$ at P still holds, since this depends on $\partial\Gamma$

But there's another way to go without using a coordinate transform.
→ Use $\boxed{\text{Vierbeins}}$

$\boxed{\text{Introduce}}$ e^μ_a ~> vector on spacetime → local vector in tangent space that metric η_{ab} so that $\boxed{g_{\mu\nu} = e_\mu^{\ a} e_\nu^{\ b} \eta_{ab}}$ on change of basis.

$a: 0,1,2,3$ & 4-component...

- We can redefine GR in terms of vierbeins → make them dynamical. Any tensor can be written as

$$A_\mu = e_\mu^{\ a} A_a \rightsquigarrow \text{component in local Lorentz basis} \ldots$$

- There's an inverse Vierbein $e^\mu_{\ a}$ such that

$$e^\mu_{\ a} e_\nu^{\ a} = \delta^\mu_\nu$$

$$e^\mu_{\ a} e_\mu^{\ b} = \delta^a_b$$

- Then $g^{\mu\nu} = e^\mu_{\ a} e^\nu_{\ b} \eta^{ab}$

- Can verify that $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$

- Take determinant of $g_{\mu\nu} = e_\mu^{\ a} e_\nu^{\ b} \eta_{ab} = e_\mu^{\ a} \eta_{ab} e_\nu^{\ b}$

$$g = e\,(\underbrace{\det \eta}_{-1})\,e$$

where $g \rightsquigarrow \det g$
$e \rightsquigarrow \det e$ & 4×4 not necessarily symmetric

$$\Rightarrow -g = e^2 \Rightarrow \boxed{\sqrt{-g} = e}$$

- So,
$$\boxed{\begin{aligned} S &= \int \sqrt{-g}\, d^4x\, \mathcal{L}(g_{\mu\nu}, A_\mu, \ldots) \\ &= \int e\, d^4x\, \mathcal{L}(e_\mu^{\ a}, A_a \ldots) \end{aligned}}$$

→ no μ, ν
What is this?

- Fermions ⇒ spinors ψ, spin-$\frac{1}{2}$ ⇒ $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

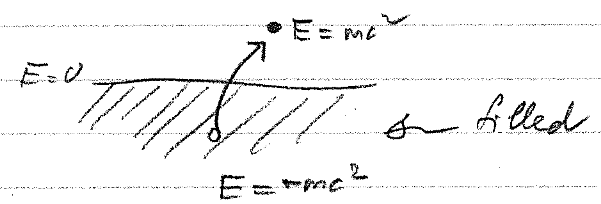
→ there's no vector / tensor representation for Fermions.
they are <u>spinors</u>

There's no problem dealing w/ them in SR.
→ SR has spinor representation under the Lorentz group

<u>But</u> GR has no representation for these under diffeomorphisms...

$\boxed{\text{Dirac SR theory}}$ → <u>relativistic quantum theory</u>
↳ needs 4-component spinors $\psi = \begin{pmatrix} \vdots \end{pmatrix}$ for Lorentz symmetry
spin-$\frac{1}{2}$ ⇒ need 2 components
other 2 ⇒ anti-particles...

$$E^2 = c^2 p^2 + m^2 c^4 \quad \text{has } E = \pm \text{ solutions...}$$



missing $E<0$ is like $+mc^2$ ⇒ Dirac proposed positrons.

Then, QED ⇒ QFT for charged particles & photons...
contains Dirac matrices γ^a → 4×4

<u>SR</u> $\left\{ \gamma^a = \begin{pmatrix} 4 \times 4 \end{pmatrix} \right.$ where a is a spatial index under LTs

Dirac Lagrangian : $\boxed{\mathcal{L} = i(\partial_\mu \bar{\psi})\gamma^\mu \psi}$

mix of vectors & spinors.

But in GR, can't represent γ^μ or ψ. The fix is to use Vierbeins, because in local Lorentz frames, there are representation of these objects. $e^{\mu}{}_{a}\gamma^{a} \rightsquigarrow$ vector in spacetime.

Can include fermions in GR using vierbeins.

There's no unique local Lorentz basis at any point P.
$\Rightarrow$ can rotate or boost.

And so, $$\boxed{e_{\mu}{}^{a} \longrightarrow e_{\mu}{}^{a} + \varepsilon^{a}{}_{b}\, e_{\mu}{}^{b}}$$

$\Rightarrow$ a vector under LLT's.

Also a vector under diffs...

$$\boxed{e_{\mu}{}^{a} \longrightarrow e_{\mu}{}^{a} + (D_{\mu}\xi^{\alpha})e_{\alpha}{}^{a} + \xi^{\alpha}D_{\alpha}e_{\mu}{}^{a}}$$

With these, $g_{\mu\nu} = e_{\mu}{}^{a}e_{\nu}{}^{b}\eta_{ab}$ transforms as ...

under a diff,
$$g_{\mu\nu} \rightarrow \left(e_{\mu}{}^{a} + (\partial_{\mu}\xi^{\alpha})e_{\alpha}{}^{a} + \xi^{\alpha}\partial_{\alpha}e_{\mu}{}^{a}\right)$$
$$\times\left(e_{\nu}{}^{b} + (\partial_{\nu}\xi^{\alpha})e_{\alpha}{}^{b} + \xi^{\alpha}\partial_{\alpha}e_{\nu}{}^{b}\right)$$
$$\times\left(\eta_{ab} + \xi^{\alpha}\cancelto{0}{\partial_{\alpha}\eta_{ab}}\right)$$

Expand this out, to first order

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + (\partial_{\mu}\xi^{\alpha})e_{\alpha}{}^{a}e_{\nu}{}^{b}\eta_{ab} + \xi^{\alpha}(\partial_{\alpha}e_{\mu}{}^{a})e_{\nu}{}^{b}\eta_{ab}$$
$$+ e_{\mu}{}^{a}(\partial_{\nu}\xi^{\alpha})e_{\alpha}{}^{b}\eta_{ab} + e_{\mu}{}^{a}\xi^{\alpha}(\partial_{\alpha}e_{\nu}{}^{b})\eta_{ab}$$

$$= g_{\mu\nu} + (\partial_{\mu}\xi^{\alpha})g_{\alpha\nu} + (\partial_{\nu}\xi^{\alpha})g_{\mu\alpha} + \xi^{\alpha}\partial_{\alpha}\left(e_{\mu}{}^{a}e_{\nu}{}^{b}\eta_{ab}\right)$$
$$= g_{\mu\nu} + (\partial_{\mu}\xi^{\alpha})g_{\alpha\nu} + (\partial_{\nu}\xi^{\alpha})g_{\mu\alpha} + \xi^{\alpha}\partial_{\alpha}g_{\mu\nu}$$

So $g_{\mu\nu} \rightarrow g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}$ as expected

- But under a local Lorentz transformation,

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$$

$$\rightarrow (e_\mu{}^a + \varepsilon^a{}_c e_\mu{}^c)(e_\nu{}^b + \varepsilon^b{}_c e_\nu{}^c)\eta_{ab}$$

$$= g_{\mu\nu} + \varepsilon^a{}_c e_\mu{}^c e_\nu{}^b \eta_{ab} + e_\mu{}^a \varepsilon^b{}_c e_\nu{}^c \eta_{ab} + \dots$$

$$= \cancel{g_{\mu\nu} + \varepsilon_{bc} e_\mu{}^c e_\nu{}^b \eta_{ab} + e_\mu{}^b \varepsilon^a{}_c e_\nu{}^c \eta_{ba}}$$

$$= g_{\mu\nu} + \varepsilon_{bc} e_\mu{}^c e_\nu{}^b + e_\mu{}^a \varepsilon_{ac} e_\nu{}^c \quad a \leftrightarrow b$$

$$= g_{\mu\nu} + \varepsilon_{bc} e_\mu{}^c e_\nu{}^b + e_\mu{}^b \varepsilon_{bc} e_\nu{}^c$$

$$= g_{\mu\nu} + \varepsilon_{bc} \underbrace{(e_\mu{}^c e_\nu{}^b + e_\mu{}^b e_\nu{}^c)}_{\text{symmetric in } b,c \ \rightarrow \ \text{same thing}}$$

$$= g_{\mu\nu} + \varepsilon_{bc}(e_\mu{}^b e_\nu{}^c + e_\mu{}^b e_\nu{}^c)$$

But since $\varepsilon_{bc} = -\varepsilon_{cb}$ (LLT)

So $g_{\mu\nu} \rightarrow g_{\mu\nu}$ under LLT

↳ $g_{\mu\nu}$ invariant under LLTs. ✓

Note: $g_{\mu\nu}$ has 10 independent components $g_{\mu\nu} = g_{\nu\mu}$

$e_\mu{}^a$ has 16 indep. components

$$e_\mu{}^a \neq e^a{}_\mu .$$

The extra 6 components are Lorentz degrees of freedom.

Can make 6 LLT's. with $\varepsilon_{ab} = -\varepsilon_{ba}$

↳ Can gauge away 6 using LLT's → giving 10.

- Any $\mathcal{L}$ made out of scalars

$$A_\mu A^\mu = (e_\mu{}^a A_a)(e^\mu{}_b A^b)$$

$$\rightarrow (e_\mu{}^a + \varepsilon^a{}_c e_\mu{}^c)(A_a + \varepsilon_a{}^c A_c)(e^\mu{}_b + \varepsilon_b{}^c e^\mu{}_c)(A^b + \varepsilon^b{}_c A^c)$$

$$= A_\mu A^\mu + \cancel{\varepsilon^a{}_c e_\mu{}^c A_a e^\mu{}_b A^b + e_\mu{}^a \varepsilon_a{}^c A_c e^\mu{}_b A^b}$$

$$+ e_\mu{}^a A_a \varepsilon_b{}^c e^\mu{}_c A^b + e_\mu{}^a A_a e^\mu{}_b \varepsilon^b{}_c A^c$$

Now, $\varepsilon_{ac}(e_\mu{}^c A^a e^\mu{}_b A^b + e_\mu{}^a A^c e^\mu{}_b A^b)$ same here...

$$= \varepsilon_{ca}(e_\mu{}^a A^c e^\mu{}_b A^b + e_\mu{}^c A^a e^\mu{}_b A^b)$$

$$= 0 \quad (\text{since } \varepsilon_{ac} = -\varepsilon_{ca})$$

So, if $S = \int \sqrt{-g}\, d^4x\, \underbrace{\mathcal{L}(g_{\mu\nu}, \ldots \Phi, A_\mu)}_{\text{scalar}\ldots}$

does nothing under Local LT's.

→ but Fermions are not allowed.

With vierbeins, Fermions are allowed → can add Fermions and do LLT

$$S = \int e\, d^4x\, \mathcal{L}(\Phi, A_a, e_\mu{}^c, \Psi \ldots)$$

$\mathcal{L}$ is a scalar under both diff and LLTs.

But there's more to the story... what abt derivatives?

$D_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda$

what about $D_\nu e_\mu{}^a$ ← not a tensor + not tensor under both LLTs and diff unless we change the def.

→ need 2nd type of connection...

$$\boxed{D_\nu e_\mu{}^a = \partial_\nu e_\mu{}^a - \Gamma^\lambda_{\nu\mu} e + \underbrace{\omega_\nu{}^a{}_b}_{\text{spin connection...}} e_\mu{}^b}$$

$\omega_\nu{}^a{}_b$ makes $D_\nu e_\mu{}^a$ is a tensor...

$\omega_\nu{}^a{}_b \rightsquigarrow \omega_\nu{}^{ab} = -\omega_\nu{}^{ba}$

↳ there are $4 \times 6 = 24$ ind components

Riemann

$R^\lambda{}_{\mu\nu\sigma} = \Gamma\Gamma + \partial\Gamma + \partial\Gamma$

adding $\omega_\mu{}^a{}_b$ → gives torsion.

Riemann space ⟶ Riemann-Cartan space...

STANDARD MODEL EXTENSION

→ Global Lorentz invariance – in Minkowski spacetime

$$S = \int d^4x \left[\frac{1}{2} D_\mu \phi D^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]$$

where $D_\mu = \partial_\mu - iqA_\mu$

$$\to -\frac{iq}{2} A_\mu \partial^\mu \phi \ldots$$

→ Lorentz invariant.

How can we break Lorentz invariance here?
→ Usually, $\mathcal{L}$ is a scalar → gives observer independence
→ can make active / passive LTs.

physical — not physical ...

The SME point of view is that observer independence is fundamental.
→ must maintain the passive LT's.
But the active transformations might break.

☒ If $\mathcal{L}$ isn't a scalar then the physics depends on coor. frame

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu$$

Then $\delta \mathcal{L} = \left(\Box A_\mu + \partial_\mu \partial^\nu A_\nu + 1 \right) \delta A^\mu$

only in one frame. But if we go to a different frame ...

$$\mathcal{L} = -\frac{1}{4} F'_{\mu\nu} F'^{\mu\nu} + \Lambda_\mu{}^\alpha A_\alpha$$

→ different ...

$$\delta\mathcal{L} = \left(\Box A_{\mu'} - \partial_{\mu'}\partial^{\nu'} A_{\nu'} + \underbrace{\Lambda_{\mu'}{}^{\alpha}\,\delta^{\mu'}_{\alpha}}_{\neq 1} \right) \delta A^{\mu'}$$

⇒ We need $\mathcal{L}$ to be a scalar

⇒ invariant under the passive LT's...

How do we break the active LT without breaking the passive one?

⇒ Using fixed background fields ⇝ called SME coeffs...

SME coeffs ⇝ a_μ, b_μ, $c_{\mu\nu}$, $H_{\mu\nu}$, ...

Ex $\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + a_\mu A^\mu}$ dynamical.

fixed background

■ Under passive LT's, $A^\mu \to A^{\mu'} = \Lambda^{\mu'}{}_{\alpha} A^\alpha$

$a_\mu \to a_{\mu'} = \Lambda_{\mu'}{}^{\alpha} a_\alpha$

So $a_\mu A^\mu = a_{\mu'} A^{\mu'}$ ⇝ maintain observer independence

→ $\boxed{\mathcal{L} \to \mathcal{L}}$

■ But under active LTs

$$\begin{cases} a_\mu \to a_\mu & \text{(fixed background)} \\ A_\mu \to A_\mu + \varepsilon_\mu{}^\nu A_\nu & \rightsquigarrow \text{infinitesimal LT's} \end{cases}$$

⇒ $\mathcal{L} \not\to \mathcal{L}$ under the active broken LTs...

Terminology with fixed background

↳ calls active transformation with a_μ unchanged

↳ $\boxed{\text{particle transformation}}$

calls passive transf $\rightsquigarrow$ $\boxed{\text{Observer transformation}}$

$\rightsquigarrow$ remain unbroken

For different fields $\rightsquigarrow$ can have different coefficients...

→ electrons

→ photons

→ protons...

SME → framework containing all of these...

$\boxed{\text{Why break Lorentz invariance?}}$

↳ the original motivation was from string theory.

Idea: → spontaneous Lorentz violation might occur...

Can also look at models with spontaneous Lorentz breaking

Ex $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu D^\mu\phi + \ldots + \frac{1}{2}K(A_\mu A^\mu - a^2)^2$

where $V(x^2) = \frac{1}{2}K(x^2 - a^2)^2$, $x^2 = A_\mu A^\mu$

So $\frac{dV}{dx^2} = K(x^2 - a^2) = 0 \Rightarrow x^2 = +a^2 = A_\mu A^\mu$

→ this does not allow $A_\mu A^\mu = 0$ at minimum

↳ $(A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2 = a^2 \rightsquigarrow$ hyperbola...

Spontaneously pick a timelike ... $A_\mu = (a, 0, 0, 0)$

$\langle A_\mu \rangle \equiv$ vacuum expectation value (vev)

spacetime index

$\Rightarrow$ spontaneously breaks Lorentz invariance.

$\Rightarrow$ $\boxed{\text{SME coeffs can originate like this .. Call ..}}$

$a_\mu = \langle A_\mu \rangle$ (vev)

<u>Then</u>, $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A^\mu \partial_\mu \phi + \ldots + \ldots$

$A^\mu = a^\mu + \varepsilon^\mu$

(vev) excitation (variation around vev)

<u>So</u> $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + a^\mu \partial_\mu \phi + \ldots$

SME coefficient

SME includes all such possible interaction ...

$$\mathcal{L} = \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\text{usual}} + \underbrace{\ldots\ldots}_{\text{all SME coupling}}$$

$\boxed{\text{Minimal SME}}$ $\rightsquigarrow$ just include leading effects

$a_\mu, b_\mu, c_{\mu\nu}, H_{\mu\nu}, (k_F)_{\mu\nu\lambda\sigma}, \ldots$

Experiments put <u>bounds</u> on these.

Can also <u>EXPLICITLY</u> break Lorentz invariance ...

↳ just put terms into Lagrangian ... → these coeffs are very small ... 10^{-32}

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (k_F)_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}}$$

fixed, explicitly breaks Lorentz symmetry if added to Lagrangian

Or, can say it originates from spontaneous Lorentz sym. breaking.

→ no difference here ...

$\boxed{\text{With gravity}}$

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{16\pi G}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \underbrace{(k_F)_{\mu\nu\alpha\beta}}_{\text{breaks diffs}}\overbrace{F^{\mu\nu}F^{\alpha\beta}}^{\text{does not break GC. Inv.}}\right]$$

Then, Einstein Eqns have the form:

$$\boxed{G^{\mu\nu} = 8\pi G T^{\mu\nu}}$$

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\,\mathcal{L}_m)}{\delta g_{\mu\nu}}$$

$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (k_F)_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}$

where $\boxed{\mathcal{L}_M = \mathcal{L}_M^{(LI)} + \mathcal{L}_M^{(LV)}}$

In gravity, we have geometric identities ...

$\boxed{\text{Bianchi}}$ ⇒ untraced ⤳ $\boxed{D_\mu G^{\mu\nu} = 0}$

↳ always <u>true</u>, where D_μ is cov derivative in GR

$\boxed{\text{Can split}}$ $T^{\mu\nu} = T^{\mu\nu}_{(LI)} + T^{\mu\nu}_{(LV)}$

<u>Consistency</u> requires

$$D_\mu T^{\mu\nu} = 0$$

works well with dynamical fields $\rightarrow$ $\boxed{D_\mu T^{\mu\nu}_{(LI)} = 0}$

<u>But</u> $D_\mu T^{\mu\nu}_{(LV)} \stackrel{?}{=} 0$. If it doesn't hold

$\rightarrow$ inconsistent theory ...

But with spontaneous sym. breaking $\rightarrow$ there's no problem.

Usual SME point of view $\rightarrow$ LV with gravity has to be spontaneous symmetry breaking.

$\boxed{\text{But}}$

In massive gravity $\rightarrow$ graviton field $g_{\mu\nu}$ has a mass

- But can't use $g_{\mu\nu} g^{\mu\nu} = 4$ $\rightarrow$ ~~fixed~~ fixed

Massive gravity theories use a background $\overline{k}_{\mu\nu}$

then $\mathcal{L} = \ldots \sim (g^{\mu\nu} \overline{k}_{\mu\nu})^2 + \ldots$

mass term ...

(Read the <u>doc</u>)