Introductory Topics in Complex Analysis

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Contents

1 de Moivre's Formula

$$(\cos\theta + \sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{1}$$

2 Roots & Things

All roots of $z = r_0 e^{i\theta}$ are of the form

$$z_r = r_0^{1/n} \exp\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right) \tag{2}$$

where $k = 0, 1, 2, \ldots$

3 Regions of the Complex Plane

♠ The ϵ-neighborhood of z_0 is the set of points

$$\mathcal{B}_\epsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \epsilon\}. \tag{3}$$

♠ The deleted ϵ-neighborhood (nbh) of z_0 is the set

$$\mathcal{B}_\epsilon(z_0) \setminus \{z_0\} = \{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}. \tag{4}$$

♠ z_0 is an interior point of $S \subset \mathbb{C}$ if some ϵ-nbh is completely contained in S, i.e.,

$$\exists \mathcal{B}_\epsilon(z_0) \text{ s.t. } \mathcal{B}_\epsilon(z_0) \subset S. \tag{5}$$

♠ z_0 is an exterior point of S if $\exists \mathcal{B}_\epsilon(z_0)$ which does not intersect S.

♠ If z_0 is neither an interior nor an exterior point of S then it is called a boundary point of S. The set of boundary points of S is called the boundary of S.

♠ z_0 is a boundary point of $S \iff \forall \epsilon > 0, \mathcal{B}_\epsilon(z_0)$ contains at least one point in S and at least one point in S^c.

♠ A set $\mathcal{O}$ is called open if it contains none of its boundary points.

♠ A set C is called closed if it contains all of its boundary points.

♠ The closure of a set S is the set $\text{cl}(S) = S \cup \partial S$.

♠ Let $\mathcal{O} \subset \mathbb{C}$. $\mathcal{O}$ is open $\iff \forall z \in \mathcal{O}, \exists \epsilon > 0, \mathcal{B}_\epsilon(z) \subset \mathcal{O}$.

♠ A set S is called path connected if $\forall z_1, z_2 \in S$, there exists a continuous function $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma(0) = z_1, \gamma(1) = z_2$ and $\gamma(t) \in S \forall t \in [0,1]$.

♠ A set S is bounded if $\exists R > 0$ such that $S \subset \mathcal{B}_R(0)$.

♠ A point z_0 is called an accumulation point of a set S if $\forall \epsilon > 0$,

$$\mathcal{B}_\epsilon(z_0) \setminus \{z_0\} \cap S \neq \emptyset, \tag{6}$$

i.e. every deleted nbh of z_0 contains at least an element of S.

♠ A set is closed if and only if it contains all of its accumulation points.

4 Limits

♠ Let f be a function defined on some punctured nbh of z_0. We say that the limit of f is w_0 as z approaches z_0 and write

$$\lim_{z \to z_0} f(z) = w_0 \tag{7}$$

if $\forall \epsilon > 0, \exists \delta > 0$ such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta \tag{8}$$

for $z \in \text{dom}(f)$.

♠ **Proposition:** Limits are unique.

Proof. Assume that

$$\begin{aligned} \lim_{z \to z_0} f(z) &= w_0 \\ \lim_{z \to z_0} f(z) &= w_1. \end{aligned} \tag{9}$$

Given $\epsilon > 0$, choose $\delta_0, \delta_1 > 0$ such that

$$\begin{aligned} |f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta_0 \\ |f(z) - w_1| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta_1. \end{aligned} \tag{10}$$

Consider $\delta = \min\{\delta_0, \delta_1\}$. Then, we have for some z such that $0 < |z - z_0| < \delta$,

$$|f(z) - w_0| < \epsilon \text{ and } |f(z) - w_1| < \epsilon. \tag{11}$$

For this particular z,

$$\begin{aligned} |w_0 - w_1| &= |f(z) - w_0 - f(z) + w_1| \\ &\leq |f(z) - w_0| + |f(z) - w_1| \\ &< \epsilon + \epsilon \\ &= 2\epsilon. \end{aligned} \tag{12}$$

So, for any $\epsilon > 0, |w_1 - w_0| < 2\epsilon$. This means $w_0 = w_1$. □

5 Limits obtained via an admissible path

If $\lim_{z \to z_0} f(z) = w_0$, then given any continuous function γ satisfying

1. $\gamma : [0,1] \to \mathbb{R}^2 \equiv \mathbb{C}$ is continuous
2. $\gamma(t) \neq z_0 \forall t > 0$, $\gamma(t) \in \mathrm{dom}(f) \forall t > 0$
3. $\gamma(0) = z_0$

then $\lim_{t \to 0^+} f(\gamma(t)) = w_0$. Any path satisfying the three conditions above is said to be admissible for f near z_0, or simply admissible.

6 Existence of Limits

If given any two admissible paths γ_0, γ_1 we have

$$\lim_{t \to 0^+} f(\gamma_0(t)) \neq \lim_{t \to 0^+} f(\gamma_1(t)) \tag{13}$$

then $\lim_{z \to z_0} f(z)$ does not exist.

7 Connect to multi-variable calculus

Suppose that $f(z) = u(x,y) + iv(x,y)$ and $z_0 = x_0 + iy_0$. Then

$$\lim_{z \to z_0} f(z) = w_0 = a_0 + ib_0 \iff \begin{cases} \lim_{(x,y) \to (x_0,y_0)} u(x,y) = a_0 \\ \lim_{(x,y) \to (x_0,y_0)} v(x,y) = b_0 \end{cases} \tag{14}$$

8 Limit facts

Suppose that $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} F(z) = W_0$, then

1. $\lim_{z \to z_0} f(z) + F(z) = w_0 + W_0$.
2. $\lim_{z \to z_0} f(z)F(z) = w_0 W_0$.
3. If $W_0 \neq 0$ then $\lim_{z \to z_0} f(z)/F(z) = w_0/W_0$.

Proof. We will prove the second statement. Let $z_0 = x_0 + iy_0$ and $f(z) = u + iv$ and $F(z) = U + iV$. Then

$$f(z)F(z) = (uU - vV) + i(uV + vU). \tag{15}$$

Since the limits of f, F at z_0 are given, we have

$$\begin{aligned}\lim_{(x,y)\to(x_0,y_0)} u &= u_0 \\ \lim_{(x,y)\to(x_0,y_0)} v &= U_0 \\ \lim_{(x,y)\to(x_0,y_0)} U &= v_0 \\ \lim_{(x,y)\to(x_0,y_0)} V &= V_0. \end{aligned} \tag{16}$$

Applying to the algebra of limits for $\mathbb{R}^2 \to \mathbb{R}$, we have

$$\lim_{(x,y)\to(x_0,y_0)} (uU - vV) = u_0U_0 - v_0V_0 = \mathrm{Re}(w_0W_0). \tag{17}$$

Similarly,

$$\lim_{(x,y)\to(x_0,y_0)} (uV + vU) = u_0V_0 + v_0U_0 = \mathrm{Im}(w_0W_0). \tag{18}$$

So, by the previous theorem, $\lim_{z\to z_0} f(z)F(z) = w_0W_0$. □

9 ϵ-neighborhood of ∞

♠ Given $\epsilon > 0$, we call the set $\mathcal{B}_\epsilon(\infty) = \{z \in \mathbb{C} : |z| > 1\epsilon\}$ the ϵ-nbh of ∞.

♠ Given $z_0 \in \mathbb{C}$ and f defined on a nbh of z_0, we say that the limit of f as $z \to z_0$ is ∞ and write

$$\lim_{z\to z_0} f = \infty \tag{19}$$

if $\forall\epsilon > 0, \delta > 0$ s.t. $f(z) \in \mathcal{B}_\epsilon(\infty)$ whenever $z \in \mathrm{dom}(f)$ and $z \in \delta-$nbh of z_0, i.e., $\forall\epsilon > 0, \exists\delta > 0$ s.t. $|f(z)| > 1/\epsilon$ whenever $0 < |z - z_0| < \delta$.

♠ Additionally, we say $\lim_{z\to\infty} f(z) = w_0$ for $w_0 \in \mathbb{C}$ if $\forall\epsilon > 0, \exists\delta > 0$ s.t. $f(z)$ lines in the ϵ-nbh of w_0 whenever $z \in$ the δ-nbh of ∞, i.e., $\forall\epsilon > 0, \exists\delta > 0$ s.t. $|f(z) - w_0| < \epsilon$ whenever $|z| > 1/\delta$.

♠ Further, we say that the limit of f as $z \to \infty$ is ∞ if $\forall\epsilon > 0, \exists\mathcal{B}_\delta(\infty)$ s.t. $f(z) \in \mathcal{B}_\epsilon(\infty)$ whenever $z \in \mathcal{B}_\delta(\infty)$.

10 Limit facts involving ∞

Let $z_0, w_0 \in \mathbb{C}$, then

$$\lim_{z\to z_0} f(z) = \infty \iff \lim_{z\to z_0} \frac{1}{f(z)} = 0.$$
$$\lim_{z\to\infty} f(z) = w_0 \iff \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0.$$
$$\lim_{z\to\infty} f(z) = \infty \iff \lim_{z\to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0. \tag{20}$$

Proof. We will prove (3). Suppose that $\lim_{z\to\infty} f(z) = \infty$. Let $\epsilon > 0$ be given. Then $\exists\delta > 0$ s.t. $|f(z)| > 1/\epsilon$ whenever $|z| > 1/\delta$. Then $1/|f(z)| < \epsilon$ whenever $|z| > 1/\delta \iff |w| = 1/|z| < \delta$. Thus, for any $0 < |w| < \delta$, we have that

$$\left|\frac{1}{f(1/w)}\right| = \frac{1}{|f(z)|} < \epsilon \tag{21}$$

as long as $w = 1/z$, i.e., $\forall\epsilon > 0, \exists\delta > 0$ s.t. $|1/f(1/z)| < \epsilon$ whenever $|z| < \delta$. The converse is gotten by reversing the steps. □

11 Continuity & 3 Theorems

♠Let f be defined on a full nbh of z_0. We say that f is continuous at z_0 if the following hold:

1. $\lim_{z\to z_0} f(z)$ exists.
2. $f(z_0)$ exists.
3. $\lim_{z\to z_0} f(z) = f(z_0)$.

♠ Compositions of continuous functions: Suppose that f is continuous at z_0 and g is continuous at $f(z_0) = w_0$ then $g \circ f(z_0)$ is continuous at z_0.

Proof. Let $\epsilon > 0$ be given, then $\exists\gamma > 0$ s.t. $|g(w) - g(w_0)| < \epsilon$ whenever $|w - w_0| < \gamma$. Given this $\gamma, \exists\delta > 0$ s.t. $|f(z) - f(z_0)| < \gamma$ whenever $|z - z_0| < \delta$. So, whenever $|z - z_0| < \delta$, $|f(z) - f(z_0)| < \gamma$ and so $|g(w) - g(w_0)| < \epsilon$. □

♠ If a continuous function is nonzero at a point then it is nonzero near that point: Suppose that f is continuous at z_0 and $|f(z_0)| \neq 0, \exists\delta > 0$ such that $f(z) \neq 0 \forall z \in \mathcal{B}_\delta(z_0)$.

Proof. Choose $\epsilon = |f(z_0)/2| > 0$. Then $\exists\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon = |f(z_0)/2| \forall |z - z_0| < \delta$. Then, for all such z, we have that

$$\begin{aligned} |f(z_0)| &= |f(z_0) + f(z) - f(z)| \\ &\leq |f(z_0) - f(z)| + |f(z)| \\ &\leq \frac{|f(z_0)|}{2} + |f(z)|. \end{aligned} \tag{22}$$

So, $\forall z \in \mathcal{B}_\delta(z_0)$, we have $|f(z_0)|/2 \leq |f(z)|$. $\square$

♠ Continuous functions on a closed and bounded set is bounded: Let R be a closed and bounded subset of the complex plane. Let f be continuous on R. Then $\exists M \geq 0$ such that

$$|f(z)| \leq M \forall z \in R \tag{23}$$

and $\exists z_0 \in R$ at which $|f(z_0)| = M$.

12 Differentiability

♠Let f be defined in a nbh of z_0. The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{24}$$

and it is defined whenever this limit exists. When this limit exists, we say f is differentiable at z_0.

♠ If f is differentiable at z_0, it is continuous at z_0.

Proof. Since the limit of the difference quotient exists,

$$\begin{aligned}
\lim_{z \to z_0} f(z) - f(z_0) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}(z - z_0) \\
&= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} (z - z_0) \\
&= f'(z_0) \cdot 0 \\
&= 0.
\end{aligned} \tag{25}$$

Thus, $\lim_{z \to z_0} f(z) = f(z_0)$, and so f is continuous at z_0. $\square$

13 Differentiability Facts

Let f, g be differentiable at z_0 then

$$\begin{cases} D_z(f+g)(z_0) = f'(z_0) + g'(z_0) \\ D_z cf(z_0) = cf'(z_0) \\ D_z f(z_0)g(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0). \end{cases}$$

If, additionally, $g(z_0) \neq 0$, then f/g is differentiable at z_0 and

$$D_z \frac{f}{g}(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}. \tag{26}$$

Proof. We shall prove the product rule:

$$\begin{aligned}
&\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)g(z_0+\Delta z)-f(z_0)g(z_0)}{\Delta z}\\
=&\lim_{\Delta z\to 0}\frac{1}{\Delta z}\left[(f(z_0+\Delta z)-f(z_0))g(z_0+\Delta z)+f(z_0)g(z_0+\Delta z)-f(z_0)g(z_0)\right]\\
=&\lim_{\Delta z\to 0}\frac{1}{\Delta z}\left[\Delta f g(z_0+\Delta z)+f(z_0)\Delta g\right]\\
=&g(z_0)f'(z_0)+g'(z_0)f(z_0),
\end{aligned} \tag{27}$$

where $g(z_0+\Delta z)$ exists by continuity. □

14 The Chain Rule

Let f be differentiable at z_0 and g be differentiable at $w_0 = f(z_0)$. Then $F(z) = g \circ f(z) = g(f(z))$ is differentiable at z_0 and $F'(z_0) \equiv D_z g \circ f(z_0) = g'(f(z_0))f'(z_0)$.

Proof. On a nbh of w_0, define $\phi : N \to \mathbb{C}$ by

$$\phi(w) = \begin{cases} \frac{g(w)-g(w_0)}{w-w_0} - g'(w_0) & w \neq w_0 \\ 0 \quad w = w_0 \end{cases} . \tag{28}$$

Observe that because g is differentiable, $\lim_{w\to w_0}\phi(w) = 0$. It follows that ϕ is continuous on its domain. Also, for $w \in N$,

$$(w-w_0)\phi(w) = (g(w)-g(w_0)) - g'(w_0)(w-w_0). \tag{29}$$

Given the continuity of f at z_0, we can choose $\delta > 0$ such that for $z \in \mathcal{B}_\delta(z_0)$ we have $f(z) = w \in N = \mathcal{B}_\epsilon(w_0)$ because

$$|f(z)-f(z_0)| = |w-w_0| < \epsilon \tag{30}$$

whenever $|z - z_0| < \delta$. So, $\forall z \in \mathcal{B}_\delta(z_0)$, we have that $\phi(f(z))$ makes sense. Also, for these values of $z \neq z_0$,

$$\begin{aligned}
\frac{F(z)-F(z_0)}{z-z_0} &= \frac{g(f(z))-g(f(z_0))}{z-z_0}\\
&= \frac{g(w)-g(w_0)}{z-z_0}\\
&= \frac{(w-w_0)\phi(w)+g'(w_0)(w-w_0)}{z-z_0}\\
&= \frac{(f(z)-f(z_0))\phi(f(z))+g'(f(z_0))(f(z)-f(z_0))}{z-z_0}.
\end{aligned} \tag{31}$$

Because $\phi(f(z))$ is continuous, $g'(z_0)$ is simply a constant, and f is differentiable at z_0, we can easily see that

$$\lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = f'(z_0)\phi(f(z_0)) + g'(f(z_0))f'(z_0). \tag{32}$$

But $\phi(f(z_0)) = \phi(w_0) = 0$ by definition, so we have

$$F'(z_0) = g'(f(z_0))f'(z_0). \tag{33}$$

□

15 The Cauchy-Riemann Equations

Let $f(z) = u(x, y) + iv(x, y)$ be defined on a nbh of $z_0 = x_0 + iy_0$. Suppose that

1. u, v have partial derivative on a nbh of z_0.
2. All first order partial derivative are continuous on this nbh of z_0 and the C-R equations:

$$u_x(x_0, y_0) = v_y(x_0, y_0); \quad u_y(x_0, y_0) = -v_x(x_0, y_0) \tag{34}$$

   are satisfied.

Then f is differentiable at z_0 and

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0). \tag{35}$$

Proof. The proof is not that bad, but it is quite technical. So I won't try to reproduce it here. □

16 Analytic Functions: Differentiable on a Ball

♠ A function f is analytic at a point $z \in \mathbb{C}$ if it is differentiable on same nbh f z_0, i.e., at every point in $\mathcal{B}_\delta(z_0)$ for some $\delta > 0$.

♠ f is said to be analytic on an open set $\mathcal{O}$ if it is analytic at each $z \in \mathcal{O}$.

♠ If f is analytic on a set S, we say it is analytic on an open set $\mathcal{O} \subset S$.

♠ Vocabulary: Analytic ≡ Holomorphic.

♠ A function f is said to be entire if it is analytic on $\mathbb{C}$.
♠ If $z_0 \in \mathbb{C}$ is such that f is analytic at every point in a nbh centered at z_0 but not at z_0 (i.e., analytic on $\mathcal{B}_\delta(z_0) \setminus \{z_0\}$) we say z_0 is a singular point for f.

♠ Suppose f, g are analytic on an open set $\mathcal{O}$ then $f \pm g, fg$ are also analytic on $\mathcal{O}$. If $g(z) \neq 0 \forall z \in \mathcal{O}$ then f/g is also analytic on $\mathcal{O}$.

♠ The set of analytic functions on an open set $\mathcal{O}$ form a commutative ring, denoted $\text{Hol}(\mathcal{O})$.

17 Analytic Functions: Familiar, but Weird

Suppose $\mathcal{D}$ is a domain (open, nonempty, path connected) and f is analytic on $\mathcal{D}$. If $f'(z) = 0 \forall z \in \mathcal{D}$ then f is constant on $\mathcal{D}$.

Proof. Given $z_0, z_1 \in \mathcal{D}, \exists$ a path $\gamma(t) : [0, 1] \to \mathcal{D}$ such that $\gamma(0) = z_0, \gamma(1) = z_1$, and γ is a continuous. Next, consider $h(t) = \text{Re}(f \circ \gamma(t)) = u(\gamma(t))$, where $f = u + iv$. By C-R, we have that $f = u + iv$ with u, v both differentiable. And so $h(t)$ is differentiable on $[0, 1]$, and by the mulvar chain rule

$$h'(t) = u_x(\gamma(t))\gamma_1'(t) + u_y(\gamma(t))\gamma_2'(t) \tag{36}$$

with $\gamma(t) = (\gamma_1(t), \gamma_2(t)) \forall t \in [0, 1]$. By MVT, $\exists c \in (0, 1)$ s.t.

$$\begin{aligned} h(1) - h(0) &= h'(c)(1 - 0) \\ &= h'(c) \\ &= u_x(\gamma(c))\gamma_1'(c) + u_y(\gamma(c))\gamma_2'(c) \\ &= u_x(\gamma(c))\gamma_1'(c) - v_x(\gamma(c))\gamma_2'(c) \end{aligned} \tag{37}$$

where the last equality follows from C-R. But we also know that $f' = u_x + iv_x = 0 \iff u_x = v_x = 0$. So $\exists c \in (0, 1)$ such that $h(1) - h(0) = 0 \iff h(1) = h(0)$. With this,

$$\text{Re}(f(z_0)) = \text{Re}(f(\gamma(0))) = h(0) = h(1) = \text{Re}(f(\gamma(1))) = \text{Re}(f(z_1)). \tag{38}$$

Similarly we can show $\text{Im}(f(z_0)) = \text{Im}(f(z_1))$. Therefore, $f(z_0) = f(z_1) \forall z_0, z_1 \in \mathcal{D}$. And so f is constant on $\mathcal{D}$. □

18 Cauchy-Riemann Theorem for Analytic Functions

Let f be a function defined on an open set $\mathcal{O} \subset \mathbb{C}$m then f is analytic on $\mathcal{O}$ if and only if for $f = u + iv$

1. u, v have first-order partial derivatives on all of $\mathcal{O}$.
2. u_x, u_y, v_x, v_y are continuous on all of $\mathcal{O}$.
3. C-R equations are satisfied, i.e., $u_x = v_y, u_y = -v_x$ on all of $\mathcal{O}$.

19 Analytic Function Facts

♠ Suppose $f, \bar{f}$ are both analytic on $\mathcal{D}$ then f is constant.

Proof. Using the C-R theorem. Suppose that $f = u + iv$ and $\bar{f} = U + iV$ where $u = U, v = -V$. Because $f, \bar{f}$ are both analytic we have

$$\begin{aligned} u_x = v_y; u_y = -v_x \\ U_x = V_y; U_y = -V_x \end{aligned} \tag{39}$$

on all of $\mathcal{D}$. So $u_x = U_x = V_y = -v_y = -u_x \iff u_x = 0$ on all of $\mathcal{D}$. Similarly, $v_x = 0$ on all of $\mathcal{D}$. It follows that $f' = u_x + iv_x = 0$ on all of $\mathcal{D}$. By the previous theorem, we have that f must be constant. □

♠ If $|f(z)| = C \forall z \in \mathcal{D}$ where $\mathcal{D}$ is a domain and f is analytic on $\mathcal{D}$, then f is constant on $\mathcal{D}$.

Proof. If $C = 0$ then the statement is true. If $C \neq 0$, then

$$f(\bar{z})f(z) = |f(z)|^2 = C^2 > 0. \tag{40}$$

Because $f(z) \neq 0 \forall z \in \mathcal{D}$ and is analytic on all of $\mathcal{D}$,

$$f(\bar{z}) = \frac{C^2}{f(z)} \tag{41}$$

is also analytic. This says that both $\bar{f}, f$ are analytic on $\mathcal{D}$. Therefore, f must be constant. □

20 Harmonic Functions

♠ A function U is said to be harmonic on a set $\mathcal{O}$ if

$$\Delta u = u_{xx} + u_{yy} \equiv 0 \tag{42}$$

on $\mathcal{O}$. This equation is called Laplace's equation.

♠ If $f = u + iv$ is analytic in D and u, v are twice differentiable with continuous partials in $\mathcal{D}$ then u, v are harmonic in $\mathcal{D}$.

Proof. By C-R, $u_x = v_y; u_y = -v_x$. So, $u_{xx} = v_{yx} = v_{yx} = u_{yy}$. So $\Delta u = 0$. Similarly, $\Delta v = 0$. □

♠ If $f = u + iv$ is analytic on a domain $\mathcal{D}$ then u, v are harmonic in $\mathcal{D}$.

21 Harmonic Conjugates

Given a harmonic function u on $\mathcal{D}$ and another harmonic function v on $\mathcal{D}$. If u, v satisfy the C-R equations, then we say v is a harmonic conjugate of u. Note that this relation is not symmetric.

♠ A function $f = u + iv$ on a domain $\mathcal{D}$ is analytic if and only if v is a harmonic conjugate of u.

Proof. If f is analytic, then u, v satisfying the C-R equation by C-R theorem. So v is a harmonic conjugate of u. Conversely, if v is a harmonic conjugate of u then C-R hold everywhere in D. By C-R theorem, f is analytic on $\mathcal{D}$. □

22 The Exponential Function

This function is so nice there's nothing to say about it.

23 The Complex Logarithm

♠ In general, for $z = re^{i\theta} \neq 0$.

$$\log(z) = \ln(|z|) + i(\theta + 2\pi n) \tag{43}$$

where $\theta = \arg(z)$.

♠ The principal value of log is given by

$$\text{Log}(z) = \ln(|z|) + i\theta_{-\pi} \tag{44}$$

where $\theta_{-\pi} = \text{Arg}(z) \in (-\pi, \pi]$.

♠ $\text{Log}(z) = \ln(1) + i\pi = i\pi$.

♠ Some properties for complex log don't work the way we expect: e.g. sum of logs is not the same as the log of powers. Tip: double-check everything and use only the "safe" properties.

24 Branches

♠ Given $\alpha \in \mathbb{R}$, define the α-branch of log by

$$\log_\alpha(z) = \ln|z| + i\theta_\alpha \tag{45}$$

where θ_α is the argument of $z \neq 0$ which lives between α and $\alpha + 2\pi$.

♠ $e^{\log_\alpha(z)} = z$, but $\log(e^z) \neq z$ in general.

♠ The $\log_\alpha$ function is not continuous. However, if we cut away the α-branch of log then $\log_\alpha$ is not only continuous but also analytic on this restricted domain.

25 Contours

A contour C is a path/curve with parameterization $z \in C^0([a,b],\mathbb{C})$ where z is differentiable at all but a finite number of points in $[a,b]$. Everywhere else it is continuously differentiable and non-degenerate. In other words, a contour is smooth arcs pieced together.

26 Contour Integrals

Suppose C is a contour with parameterization $z \in C^0([a,b],\mathbb{C})$ and $f : \mathcal{O} \subset \mathbb{C} \to \mathbb{C}$. We define the contour integral of f along $\mathbb{C}$ (direction matters) as

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt. \tag{46}$$

This makes sense because z' exists everywhere except a finite number of points which don't contribute to the integral. In addition, the contour integral is independent of parameterization up to direction of integration.

27 Lemma on Modulus & Contours

Let $w \in C^0([a,b],\mathbb{C})$ then

$$\left|\int_a^b w(t)\,dt\right| \leq \int_a^b |w(t)|\,dt. \tag{47}$$

Proof. This is essentially the triangle inequality. Let

$$r_0 = \left|\int_a^b w\,dt\right|. \tag{48}$$

If $r_0 = 0$ then the statement is obvious. Now suppose $r_0 > 0$. In this case,

$\exists\theta_0 \in \mathbb{R}$ such that

$$\begin{aligned}\int_a^b w\,dt = r_0 e^{i\theta_0} \implies r_0 &= e^{-i\theta_0}\int_a^b w\,dt \\ &= \int_a^b we^{-i\theta_0}\,dt \in \mathbb{R} \\ &= \operatorname{Re}\left(\int_a^b we^{-i\theta_0}\,dt\right) \\ &= \int_a^b \operatorname{Re}\left(we^{-i\theta_0}\right)\,dt. \end{aligned} \tag{49}$$

But

$$\operatorname{Re}\left(we^{-i\theta_0}\right) \leq \left|\operatorname{Re}\left(we^{-i\theta_0}\right)\right| \leq \left|e^{-i\theta_0}w\right| = |w| \forall t \in [a,b]. \tag{50}$$

And so

$$\left|\int_a^b w\,dt\right| = r_0 \leq \int_a^b |w|\,dt. \tag{51}$$

□

28 Bound on Modulus of Contour Integrals

Let C be a contour and let $f : \operatorname{Dom}(f) \to \mathbb{C}$ be piecewise continuous on C. If $|f(z)| \leq M \forall z \in \mathbb{C}$, then

$$\left|\int_C f(z)\,dz\right| \leq M\mathcal{L}(C) \tag{52}$$

where $\mathcal{L}(C)$ is the arclength of C.

Proof. This result follows from the previous lemma. Let $z(t) : [a,b] \to \mathbb{C}$ be a parameterization, then

$$\begin{aligned}\left|\int_C f\,dz\right| &= \left|\int_a^b f(z(t))z'(t)\,dt\right| \\ &\leq \int_a^b |f(z(t))z'(t)|\,dt \\ &\leq \int_a^b |f(z(t))||z'(t)|\,dt \\ &\leq M\int_a^b |z'(t)|\,dt \\ &= M\mathcal{L}(C). \end{aligned} \tag{53}$$

□

29 TFAE

Let f be continuous on $\mathcal{D}$. The following are equivalent (TFAE):

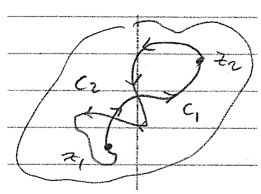
1. $f(z)$ has an antiderivative $F(z)$ throughout $\mathcal{D}$.
2. Given any $z_1, z_2 \in \mathcal{D}$ and contours $C_1, C_2 \subset \mathcal{D}$ both going from z_1 to z_2,
$$\oint_{C_1} f(z)\, dz = \oint_{C_2} f(z)\, dz. \tag{54}$$
In other words, the integral is independent of contour.
3. Given any close contour $C \subset \mathcal{D}$,
$$\int_C f(z)\, dz = 0. \tag{55}$$

In the case that one (and hence every) condition is satisfied, we have that for any $z_1, z_2 \in \mathcal{D}$ and contour C from $z_1 \to z_2 \subset \mathcal{D}$,

$$\int_C f(z)\, dz = F(z_2) - F(z_1) \tag{56}$$

where F's existence is guaranteed by (1).

Proof. ($2 \iff 3$) Suppose (2) is valid and let C be a closed contour in $\mathcal{D}$. Then C contains 2 points z_1, z_2 and we can divide C into 2 pieces $C_1 + C_2$ where $C_1 : z_1 \to z_2$ and $C_2 : z_2 \to z_1$.



Note that by reversing the direction of C_2, we ave both C_1 and $-C_2$ go from z_1 to z_2 and stay inside of $\mathcal{D}$. Thus,

$$\oint_C f\, dz = \int_{C_1} f\, dz - \int_{-C_2} f\, dz. \tag{57}$$

By (2), we have that

$$\int_{C_1} f\, dz = \int_{C_2} f\, dz. \tag{58}$$

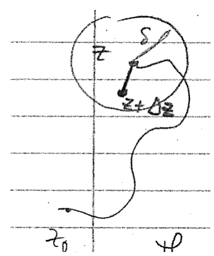
This means

$$\oint_C f(z)\, dz = 0. \tag{59}$$

So (2) $\implies$ (3).

Now, assume (3) is true and let $z_0, z_1 \in \mathcal{D}$. Let $C_1, C_2 \subset \mathcal{D}$ be contours going from z_0 to z_1. We observe that $C := C_1 - C_2$ is a s.c.c. in $\mathcal{D}$. So by (3),

$$0 = \oint_C f\,dz = \int_{C_1 - C_2} f\,dz = \int_{C_1} f\,dz - \int_{C_2} f\,dz. \tag{60}$$



(1 $\iff$ 2) Assume (1) is true. Let $z_0, z_1 \in \mathcal{D}$ and let C be a contour from $z_0 \to z_1$, i.e., $C : z(t) \in C([a,b], \mathbb{C})$ piecewise differentiable, $z(a) = z_0$ and $z(b) = z_1$. As F is an antiderivative of f, for all $t \in [a,b]$ for which $z'(t)$ exists the chain rule gives

$$\frac{d}{dt} F(z(t)) = F'(z(t))z'(t) = f(z(t))z'(t). \tag{61}$$

So,

$$\oint_C f\,dz = \sum_{k=1}^{n} \int_{a_k}^{b_k} f(z(t))z'(t)\,dt = \sum_{k=1}^{n} \int_{a_k}^{b_k} \frac{d}{dt} F(z(t))\,dt \tag{62}$$

where a_k, b_k are points at which z fails to be differentiable, $a_1 = a, b_n = b$. By the fundamental theorem of calculus,

$$\begin{aligned}\oint_C f\,dz &= \sum_{k=1}^{n} \int_{a_k}^{b_k} \frac{d}{dt} F(z(t))\,dt \\ &= \sum_{k=1}^{n} F(z(b_k)) - F(z(a_k)) \\ &= F(b) - F(a) = F(z_1) - F(z_0). \end{aligned} \tag{63}$$

So, given any 2 contours $C_1, C_2 \in\subset \mathcal{D}$ from $z_0 \to z_1$, we have

$$\int_{C_1} f\,dz = F(z_1) - F(z_0) = \int_{C_2} f\,dz. \tag{64}$$

Now, assume (2) is true. We need to construct an antiderivative F. Let $z_0 \in \mathcal{D}$ and define $F : \mathcal{D} \to \mathbb{C}$ by

$$F(z) = \int_{C_z} f(w)\,dw \tag{65}$$

where C_z is a contour from $z_0 \to z_1$. Since $\mathcal{D}$ is a domain, it is a path connected, and so for each z, a path C_z exists. By (2) this is not dependent on the choice of contour C_z. So F is well-defined. We wish to show that $F(z)$ is differentiable and its derivative is f.

Let $z \in\subset \mathcal{D}$ and choose $\epsilon > 0$. Given th continuity of f, let δ be chosen so that

1. $$|f(w) - f(z)| < \frac{\epsilon}{2} \forall |w - z| < \delta \tag{66}$$

2. $\mathcal{B}_\delta(z) \subset \mathcal{D}$ (or $\mathcal{D}$ is open.)

Given a $\Delta z \in \mathbb{C}$ such that $\Delta z < \delta$, we consider a path $C_{z,\Delta z}$ defined by $w(t) = z + t\Delta z$, $t \in [0, 1]$. We have that $C_z + C_{z,\Delta z}$ is a contour in $\mathcal{D}$ from $z_0 \to z + \Delta z$. Then,

$$\begin{aligned}
\frac{1}{\Delta z}\left(F(z+\Delta z) - F(z)\right) &= \frac{1}{\Delta z}\left(\int_{C_z + C_{z,\Delta z}} f(w)\,dw - \int_{C_z} f(w)\,dw\right) \\
&= \frac{1}{\Delta z}\int_{C_{z,\Delta z}} f(w)\,dw \\
&= \frac{1}{\Delta z}\int_0^1 f(z + t\Delta z)(z + t\Delta z)'\,dt \\
&= \int_0^1 f(z + t\Delta z)\,dt. \qquad (67)
\end{aligned}$$

So, for $|\Delta z| < \delta$,

$$\begin{aligned}
\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| &= \left|\int_0^1 f(z + t\Delta z)\,dt - f(z)\right| \\
&= \left|\int_0^1 \left[f(z + t\Delta z) - f(z)\right]\,dt\right| \\
&\leq \int_0^1 |f(z + t\Delta z) - f(z)|\,dt \\
&\leq \int_0^1 \frac{\epsilon}{2}\,dt \\
&\leq \frac{\epsilon}{2} \\
&< \epsilon \qquad (68)
\end{aligned}$$

by choice of δ. So, we have shown that given $z \in \mathcal{D}$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| < \epsilon \tag{69}$$

whenever $|\Delta z| < \delta$. So, F is differentiable at z and $F'(z) = f(z)$. □

30 Cauchy-Goursat Theorem

Suppose that C is a simple closed contour and f is analytic on the interior of C and all points of C then

$$\oint_C f(z)\,dz = 0. \tag{70}$$

Proof. The proof involves slicing the interior of C into squares and partial squares. I won't try to reproduce it here. □

31 Simply-connected domain

A domain $\mathcal{D}$ is called simply-connected if every simple closed contour $C \subset \mathcal{D}$ contains only points of $\mathcal{D}$ and its interior, i.e., every simple closed contour is contractible to a point.

32 Multiply-connected domain

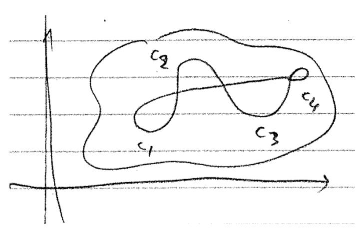
A multiply-connected domain $\mathcal{D}$ is a dmain which is not simply-connected. (very imaginative)

33 Cauchy-Goursat Theorem for simply-connected domain

Let $\mathcal{D}$ be a simply connected domain. f is analytic in $\mathcal{D}$. For all closed contour $C \subset \mathcal{D}$,

$$\oint_C f(z)\,dz = 0. \tag{71}$$

Proof. Notice that we C need not be simple. Consider the figure



Let C be a closed contour in $\mathcal{D}$ with a finite number of self-intersections. Given that C only has n interactions, we can split C into a finite number m

of simple closed contour C_j. Also, given $\mathcal{D}$ is simply connected, the interior of each C_j lives in $\mathcal{D}$. By the previous theorem, we have

$$\oint_{C_j} f(z)\,dz = 0 \forall j = 1, 2, 3, \cdots \implies \oint_C f(z)\,dz = \oint_{\sum C_j} f(z)\,dz = 0. \tag{72}$$

□

34 Corollary to Cauchy-Goursat for simply-connected domain

If f is analytic on a simply connected domain in $\mathcal{D}$ then f has an antiderivative F everywhere in $\mathcal{D}$.

Proof. TFAE. □

35 Cauchy-Goursat Theorem for multiply-connected regions

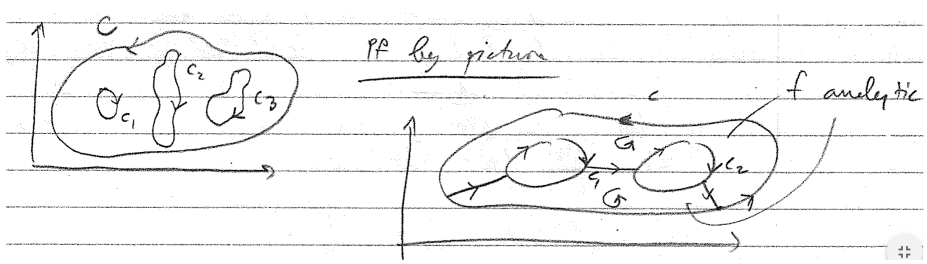
Suppose that

1. C is a s.c.c.(+).
2. C_j, $j = 1, 2, \ldots, n$ are s.c.c.(-), all disjoint and all live in the interior of C.

If f is analytic on $C, C_j \forall j$ and the region between C, C_j (enclosed by C but outside of C_j) then

$$\oint_C f(z)\,dz + \sum_{j=1}^{n} \oint_{C_j} f(z)\,dz = 0. \tag{73}$$

Proof. The proof follows from the this figure



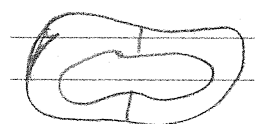
□

36 Principle of Path Deformation (Corollary to Cauchy-Goursat)

Let C_1 and C_2 be simple closed curves and C_2 encloses C_1. Both are (+) oriented. Then if f is analytic on the region between C_1, C_2 then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz. \tag{74}$$

Proof. Consider the following suggestive figure:

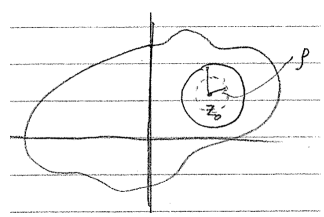


□

37 Cauchy's Integral Formula

Let C be a s.c.c.(+) and let f be analytic on C and its interior. If z_0 lives interior to C then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz. \tag{75}$$



Proof. Let $\delta < 1$ be small enough such that $|z - z_0| < \delta$ so that C encloses z. Since the quotient $f(z)/(z - z_0)$ is analytic in the region exterior to $\mathcal{B}_\delta(z_0)$ and interior to C, we have that

$$\oint_C \frac{f(z)}{z - z_0}\,dz = \oint_{C_\rho} \frac{f(z)}{z - z_0}\,dz \tag{76}$$

where $\rho < \delta$ and C_ρ is a (+) circle centered at z_0 of radius ρ. The equality is guaranteed by the principle of deformation of path.

Now, consider

$$
\begin{aligned}
\mathcal{E} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0} - f(z_0) \\
&= \frac{1}{2\pi i}\oint_{C_\rho} \frac{f(z)}{z-z_0} - \frac{f(z_0)}{2\pi i}\oint_{C_\rho} \frac{1}{z-z_0}\,dz \\
&= \frac{1}{2\pi i}\left(\oint_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right).
\end{aligned}
\tag{77}
$$

Given that $f(z)$ is continuous at z_0, $\forall \epsilon > 0, \exists \rho > 0$ s.t. $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < 2\rho < \delta$. Since $|z - z_0| = \rho < 2\rho$ on C_ρ, we have

$$
\left|\frac{f(z)-f(z_0)}{z-z_0}\right| = \frac{1}{\rho}|f(z)-f(z_0)| < \frac{\epsilon}{\rho} \text{ on } C_\rho. \tag{78}
$$

So,

$$
|\mathcal{E}| \leq \frac{1}{2\pi}\frac{\epsilon}{\rho}\mathcal{L}(C_\rho) = \epsilon. \tag{79}
$$

So, given any $\epsilon > 0$, $|\mathcal{E}| \leq \epsilon$. This says that

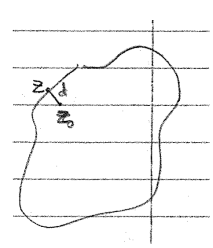
$$
\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = f(z_0). \tag{80}
$$

□

38 Cauchy's Integral Formula for First-Order Derivative

Let C s.c.c.(+) and let f be analytic on the interior of C and on C. Then if $z_0 \in \text{int}(C)$ then

$$
f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz. \tag{81}
$$



Proof. Let $M = \max|f(z)|$ where $z \in C$. Given $z_0 \in \text{int}(C)$, let $d = \min|z - z_0| > 0$ where $z \in C$. Let $h = \Delta z$ is such that $|h| = |\Delta z| < d$. Using Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz. \tag{82}$$

Because $|h| < d$, $z_0 + h \in \text{int}(C)$. So,

$$f(z_0 + h) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - (z_0 + h)}\, dz. \tag{83}$$

Now, observe that

$$\begin{aligned}
\mathcal{E} &= \frac{f(z_0 + h) - f(z_0)}{h} - \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2}\, dz \\
&= \frac{1}{h}\frac{1}{2\pi i} \oint_C \frac{f(z)}{z - (z_0 + h)}\, dz - \frac{1}{h}\frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz - \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2}\, dz \\
&= \dots \\
&= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} \frac{h}{z - (z_0 + h)}\, dz
\end{aligned} \tag{84}$$

for all $z \in \text{int}(C), d \leq |z - z_0|$. So,

$$\frac{1}{|z - z_0|^2} \leq \frac{1}{d^2}. \tag{85}$$

Also, $0 \leq d - |h| \leq |z - (z_0 + h)| \forall |h| < d$. So for all $z \in C$, whenever $|h| < d$,

$$\left| \frac{f(z)}{(z - z_0)^2} \frac{h}{z - (z_0 + h)} \right| \leq \frac{M|h|}{d^2(d - |h|)}. \tag{86}$$

So, whenever $|h| < d$, we have

$$|\mathcal{E}| \leq \frac{1}{2\pi} \frac{M|h|}{d^2(d - |h|)} \mathcal{L}(C) = \frac{M|h|}{2\pi d^2(d - |h|)} \mathcal{L}(C). \tag{87}$$

Let $\epsilon > 0$ be given and choose

$$\delta = \min\left[\frac{d}{2}, \frac{\pi d^3}{M\mathcal{L}(C)}\right] \tag{88}$$

then whenever $|h| < \delta \leq \frac{d}{2} < d$,

$$\frac{1}{d - |h|} \leq \frac{1}{d/2}. \tag{89}$$

With this,

$$\mathcal{E} \leq \frac{M|h|}{2\pi d^3/2} \mathcal{L}(C) < \frac{M\mathcal{L}(C)}{\pi d^3} \frac{\pi d^3 \epsilon}{M\mathcal{L}(C)} = \epsilon. \tag{90}$$

So,

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} \, dz. \tag{91}$$

□

39 Cauchy's Integral Formula for Higher-Order Derivatives

Let C be s.c.c.(+) and f analytic on C and its interior. Then $\forall z_0 \in \text{int}(C)$, and $n \in \mathbb{N}$, f is n-times differentiable at z_0 and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz. \tag{92}$$

40 Analyticity of Derivatives

If f is analytic at z_0 then f has derivatives of all orders which are also analytic at z_0.

Proof. We simply applying the preceding theorem. □

41 Analyticity of Derivatives on a Domain

If $\mathcal{D}$ is a domain and f is analytic on $\mathcal{D}$ then f has derivatives of all orders and each derivative is analytic on $\mathcal{D}$. This means f is infinitely differentiable on $\mathcal{D}$.

42 Infinite Differentiability

Let $f(z) = u(x, y) + iv(x, y)$ be analytic at $z_0 = (x_0, y_0)$. Then u, v have continuous partial derivatives of all orders at z_0. Further, if $f = u + iv$ is analytic on $\mathcal{D}$, then u, v are infinitely differentiable in $\mathcal{D}$, i.e., $u, v \in C^\infty(\mathcal{D})$.

Proof. The proof follows from Cauchy-Riemann theorem and equations. □

43 Hörmander's Theorem

If u is harmonic in a domain $\mathcal{D}$ then u is smooth $\iff u \in C^\infty(\mathcal{D})$.

Proof. If u is harmonic then u has a harmonic conjugate v. Then $f = u + iv$ is analytic, etc. □

44 Morera's Theorem

Let f be continuous on $\mathcal{D}$. If for all closed $C \subset \mathcal{D}$,

$$\oint_C f(z)\,dz = 0, \tag{93}$$

then f is analytic on $\mathcal{D}$.

Proof. The proof follows from TFAE. By TFAE, f has an antiderivative F throughout $\mathcal{D}$. But F is analytic because $f' = F$. This means F's derivatives are analytic throughout $\mathcal{D}$ as well. So, f is analytic throughout $\mathcal{D}$. □

45 Cauchy's Inequality

Let f be analytic on and inside a (+) circle C with center z_0 and radius R. Let $M_R = \max\left[|f(z)|\right], z \in C_R$. Then $\forall n \in \mathbb{N}$,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M_R}{R^n}. \tag{94}$$

Proof. This follows from Cauchy's integral formula and the triangle inequality:

$$\begin{aligned}\left|f^{(n)}(z_0)\right| &= \left|\frac{n!}{2\pi i}\oint_{C_R}\frac{f(z)}{(z-z_0)^{n+1}}\,dz\right| \\ &\leq \frac{n!}{2\pi}\frac{M_R}{R^{n+1}}(2\pi R) \\ &= \frac{n!M_R}{R^n}. \end{aligned} \tag{95}$$

□

46 Liouville's Theorem

If f is bounded and entire and f is constant.

Proof. Let $M \geq 0$ for which $|f(z)| \leq M \forall z \in \mathbb{C}$. Given any $z_0 \in \mathbb{C}$, f is analytic on every neighborhood of z_0 and so $\forall R > 0$,

$$|f'(z_0)| \leq \frac{1!M_R}{R} \tag{96}$$

where $M_R = \max|f(z)| \leq M$ where $z \in C_R(z_0)$. So, for any $z_0 \in \mathbb{C}$, $R > 0$,

$$|f'(z_0)| \leq \frac{M}{R}. \tag{97}$$

This shows $f'(z_0) = 0 \forall z_0 \in \mathbb{C}$. So, f is constant because $\mathbb{C}$ is a domain. □

47 The Fundamental Theorem of Algebra

If $P(z)$ is a non constant polynomial, i.e.,

$$P(z) = a_0 + a_1 z^1 + \cdots + a_n z^n \tag{98}$$

where $a_n \neq 0, n = \deg(P)$, then $\exists z_0 \in \mathbb{C}$ at which $P(z_0) = 0$.

Proof. Let

$$w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z} \tag{99}$$

and note that

$$P(z) = (w + a_n) z^n. \tag{100}$$

We observe that z^k from $k \in \{1, 2, 3, \dots\}$ has $1/z^k \to 0$ has $z \to \infty$. So, given $\epsilon = |a_n|/2$, there exists $R > 0$ for which

$$|w| \leq \frac{|a_n|}{2} \forall |z| > R. \tag{101}$$

So, for $|z| > R$,

$$|w + a_n| \geq ||w| - |a_n|| = |a_n| - |w| \geq \frac{|a_n|}{2}. \tag{102}$$

So,

$$\left| \frac{1}{P(z)} \right| = \frac{1}{|w + a_n||z^n|} \leq \frac{2}{|a_n|} \frac{1}{|z^n|} \leq \frac{2}{|a_n|} \frac{1}{R^n} \tag{103}$$

where $|z| > R$. Now, suppose that $P(z) \neq 0 \forall z \in \mathbb{C}$ to get a contradiction. Since $P(z)$ is never vanishes, $f(z) = 1/P(z)$ is entire. Since, in particular, $f(z)$ is continuous, it is bounded on all closed bounded set. So, $\exists M > 0$ such that $|f(z)| \leq M \forall z, |z| \leq R$. So, by what we've just shown

$$\left| \frac{1}{P(z)} \right| \leq \max \left[M, \frac{2}{|a_n| R^n} \right]. \tag{104}$$

So, we have $f(z)$ is bounded and entire. By Liouville's theorem, $1/P(z)$ must be constant. This is a contradiction. □

48 Corollary to The Fundamental Theorem of Algebra

If $P(z)$ has degree n, then there exists $c \in \mathbb{C}$ and $z_1, z_2, \dots, z_n \in \mathbb{C}$ such that

$$P(z) = c(z - z_1) \dots (z - z_n). \tag{105}$$

49 The Maximum Modulus Principle 1

Suppose that an analytic function f has $|f(z)|$ maximized at z_0 in some nbh $\mathcal{B}_\epsilon(z_0)$ for some $\epsilon > 0$. Then $f(z)$ is constant on $\mathcal{B}_\epsilon(z_0)$.

Proof. Take $0 < \rho < \epsilon$ and by invoking Cauchy's integral formula, we have

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\oint_{C_\rho} \frac{f(z)}{z - z_0}\,dz \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + \rho e^{it})}{z_0 + \rho e^{it} - z_0} i\rho e^{it}\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{it})\,dt. \end{aligned} \tag{106}$$

So

$$\begin{aligned} |f(z_0)| &= \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0 + \rho e^{it})\,dt\right| \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\left|f(z_0 + \rho e^{it})\right|}_{\le |f(z_0)|}\,dt \\ &\le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,dt = |f(z_0)|. \end{aligned} \tag{107}$$

This says

$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0 + \rho e^{it})\right| dt \tag{108}$$

so

$$\frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(z_0)| - \left|f(z_0 + \rho e^{it})\right|}_{\ge 0}\,dt. \tag{109}$$

This says $\forall t \in [0, 2\pi]$ and $\forall \rho < \epsilon$

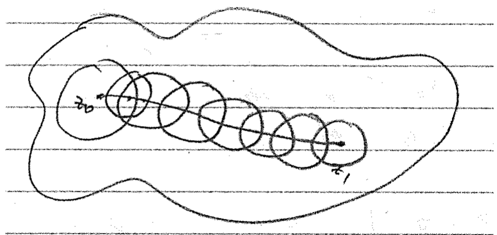
$$|f(z_0)| = \left|f(z_0 + \rho e^{it})\right|. \tag{110}$$

This is true for all $\rho < \epsilon$, so $|f(z)| = |f(z_0)|$ for all $z \in \mathcal{B}_\epsilon(z_0)$. □

50 The Maximum Modulus Principle 2

Let f be analytic and non-constant on a domain $\mathcal{D}$ (open and connected), then $|f(z)|$ cannot be maximized in $\mathcal{D}$.

Proof. Assume to reach a contradiction that f is maximized at $z_0 \in \mathcal{D}$. Let $z_1 \in \mathcal{D}$ be arbitrary. Then by the following figure



we get a contradiction, using the maximum modulus principle 1, as desired.

□

51 Convergence of Sequences

Consider a sequence $\{z_n\} = (z_0, z_1, \dots)$ of complex numbers. Write $\{z_n\} \in \mathbb{C}$. We say that the sequence converges if $\exists z \in \mathbb{C}$ for which the following holds: $\forall \epsilon > 0, \exists N = N_\epsilon \in \mathbb{N}$ s.t.

$$|z - z_n| < \epsilon \, \forall n \geq N. \tag{111}$$

In this sense, we also say that $\{z_n\}$ converges to z and call z the limit of the sequence:

$$z = \lim_{n \to \infty} z_n. \tag{112}$$

52 Real and Imaginary parts of a convergent sequence

Let $z_n = x_n + iy_n$ be a sequence, then $z_n \to z = x + iy$ if and only if $x_n \to x$ and $y_n \to y$ in the sense of real numbers.

53 Cauchy sequences

A sequence $\{z_n\}$ is called a Cauchy sequence if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that

$$|z_n - z_m| < \epsilon \, \forall n, m \geq N. \tag{113}$$

54 Cauchy and Convergence

A sequence is convergent if and only if it is Cauchy.

55 Series

Consider a sequence $\{z_n\}_{n=0}^{\infty}$ and the series formed with the sequential elements as its terms:

$$\sum_{n=0}^{\infty} z_k = z_0 + z_1 + z_2 + \dots \tag{114}$$

where, a priori, we don't assume they add to anything. This series convergences if $\{S_N\}$ where

$$S_N = \sum_{n=0}^{N} z_k \tag{115}$$

is a convergent sequence, i.e.,

$$S = \lim_{N\to\infty} S_N \tag{116}$$

exists.

56 Convergence of Series

♠ Given $z_n = x_n + iy_n$ then $\sum z_n$ converges to $x + iy \iff \sum x_n \to x$ and $\sum y_n \to y$.

♠ If $\sum z_n$ converges then $\lim_{n\to\infty} z_n = 0$. The converse also holds.

Proof. Let $\epsilon > 0$ be given. Then that $\sum z_n$ converges, $\{S_N\}$ also converges. So, $\{S_N\}$ is Cauchy, so $\exists M \in \mathcal{N}$ such that

$$|S_n - S_m| < \epsilon \tag{117}$$

whenever $n, m \geq M$. Setting $n = m + 1$ we have

$$|z_n| = |S_{n+1} - S_n| < \epsilon. \tag{118}$$

□

♠ A series $\sum z_n$ is said to be absolutely convergent if $\sum |z_n|$ is convergent as a series of real, non-negative numbers.

♠ If $\sum z_n$ is absolute convergent than $\sum z_n$ is convergent.

Proof. Here is a sketch of the proof:

$$|S_N - S_M| = \left|\sum_{k=N+1}^{M} z_k\right| \leq \sum_{k=N+1}^{M} |z_k| \tag{119}$$

due to the triangle inequality. With this inequality, the Cauchyness of $\sum |z_k|$ implies the Cauchyness of $\sum z_k$. □

♠ The series $\sum_{n=0}^{\infty} z_n$ converges to $S \iff \lim_{N\to\infty} \rho_N = 0$ where $\rho_N = S - S_N = S - \sum_{n=0}^{N} z_n$ and S is some number that is to be the sum of the series.
♠ "Geometric series":

$$S_N = \frac{1 - z^{N+1}}{1 - z} = \sum_{n=0}^{N} z^n. \tag{120}$$

♠ For any $z \in \mathbb{C}$ such that $|z| < 1$, $\sum_{n=0}^{\infty}$ converges and its sum is $1/(1 - z)$.

Proof. For each $N \in \mathcal{N}$,

$$\rho_N = \frac{1}{1 - z} - \sum_{n=0}^{N} z^n = \frac{1}{1 - z} - \frac{1 - z^{N+1}}{1 - z} = \frac{z^{N+1}}{1 - z}. \tag{121}$$

Since $|z| < 1$, $\lim_{N\to\infty} z^{N+1} = 0$. So, $\lim_{N\to\infty} \rho_N = 0$. So, by one of the previous theorems, we have

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}. \tag{122}$$

□

57 Taylor's Theorem

Let $f(z)$ be analytic on a disk $\mathcal{B}_{R_0}(z_0)$, then for any $z \in \mathcal{B}_{R_0}(z_0)$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n. \tag{123}$$

Remarks:

1. In particular, the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ converges.
2. The sum is f.
3. For real functions $h : \mathbb{R} \to \mathbb{R}$. If h is differentiable on an open set containing x_0, it might not be twice differentiable.
4. For infinitely differentiable functions, now the series makes sense, but we might have h being representable by a Taylor series that is infinitely differentiable, but not equal to its Maclaurin series. For example:

$$h(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}. \tag{124}$$

Proof. Without loss of generality, assume that $z_0 = 0$ and consider $\mathcal{B}_{R_0}(z_0)$ on which f is analytic. Let $z \in \mathcal{B}_{R_0}(z_0)$. Let $|z_0| < |z| < R_0$, and define a s.c.c.(+) C centered at $z_0 = 0$ of radius R_0. Since z lives in the interior of C, Cauchy integral formula says

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z}\, dw. \tag{125}$$

Since $w \neq 0$, we write

$$\frac{1}{w - z} = \frac{1}{w} \frac{1}{1 - z/w} = \sum_{n=0}^{N} \frac{z^n}{w^{n+1}} + \frac{1}{w - z} \left(\frac{z}{w}\right)^{N+1}, \tag{126}$$

which is made possible by the fact that

$$\frac{1}{1 - a} = \frac{1 - a^{N+1}}{1 - a} + \frac{a^{N+1}}{1 - a} = \sum_{n=0}^{N} a^n + \frac{a^{N+1}}{1 - a}. \tag{127}$$

Next, by Cauchy's derivative formula,

$$f^{(n)}(0) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w - 0)^{n+1}}\, dw. \tag{128}$$

So we have

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - 0)^{n+1}}\, dw. \tag{129}$$

Next, let the error be

$$\begin{aligned}
\rho_N &= f(z) - \sum_{n=0}^{N} a_n z^n \\
&= \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z}\, dw - \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - 0)^{n+1}} z^n\, dw \\
&= \frac{1}{2\pi i} \oint_C f(w) \left[\frac{1}{w - z} - \sum_{n=0}^{N} \frac{z^n}{w^{n+1}} \right] dw \\
&= \frac{1}{2\pi i} \oint_C f(w) \frac{(z/w)^{N+1}}{w - z}\, dw.
\end{aligned} \tag{130}$$

Set

$$d = \min |w - z| \quad z \in C \tag{131}$$

and

$$M = \max |f(z)| \quad z \in \mathcal{B}_{R_0}(z_0 = 0) \tag{132}$$

then

$$\begin{aligned}|\rho_N| &= \frac{1}{2\pi}\left|\oint_C f(w)\frac{(z/w)^{N+1}}{w-z}\,dw\right| \\ &\le \frac{1}{2\pi}\frac{|z/w|^{N+1}}{d}M\mathcal{L}(C) \\ &= \frac{M|z/w|^{N+1}}{d}r_0 \end{aligned} \tag{133}$$

So, we have shown that given $z \in \mathcal{B}_{R_0}(0)$, $\exists |z| < r_0 < R_0$ for which

$$|\rho_N| \le M\frac{|z|^{N+1}}{d \cdot r_0^N} = \left(\frac{M|z|}{d}\right)\left(\frac{|z|}{r_0}\right)^N \; \forall N \in \mathbb{N}. \tag{134}$$

Since we've chosen $|z| < r_0 < R_0$, $|z|/r_0 < 1$. Given $\epsilon > 0$, $\exists N_0 \in \mathbb{N}$ for which $\forall N \ge N_0$,

$$\left(\frac{|z|}{r_0}\right)^N < \frac{\epsilon d}{M|z|}. \tag{135}$$

So, for all $N \ge N_0$,

$$|\rho_N| \le \frac{M|z|}{d}\left(\frac{|z|}{r_0}\right)^N < \epsilon. \tag{136}$$

Thus,

$$f(z) = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\sum_{n=0}^{N} a_n z^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n. \tag{137}$$

□

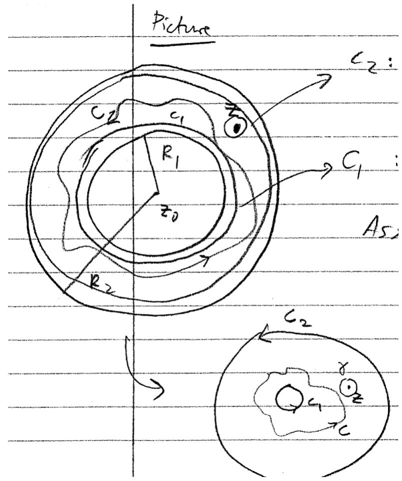
58 Laurent's Theorem

Let f be analytic on a region $\mathcal{D}$ defined by $R_1 < |z - z_0| < R_2$, and let a simple closed contour C endowed with a positive orientation in this annulus be given. Then, for each $z \in \mathcal{D}$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^{-n+1}} \tag{138}$$

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \quad b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{-n+1}}\,dz. \tag{139}$$



Proof. Without loss of generality, assume $z_0 = 0$. Let C_1, C_2, s.c.c.(+) be given such that C_2 encloses C_1, z, C; C encloses C_1, and the exterior of C_1 contains z, C. Also, let γ be a s.c.c.(+) around z, exterior to C_1 but interior to C_2. An appeal to Cauchy-Goursat for multiply-connected domain shows that

$$\oint_{C_2} \frac{f(s)}{s-z}\,ds - \oint_{C_1} \frac{f(s)}{s-z}\,ds - \oint_{C_\gamma} \frac{f(s)}{s-z}\,ds = 0. \tag{140}$$

Next, by Cauchy integral formula,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{C_\gamma} \frac{f(s)}{s-z}\,ds \\ &= \oint_{C_2} \frac{f(s)}{s-z}\,ds - \oint_{C_1} \frac{f(s)}{s-z}\,ds \\ &= \oint_{C_2} \frac{f(s)}{s-z}\,ds + \oint_{C_1} \frac{f(s)}{z-s}\,ds. \end{aligned} \tag{141}$$

For the first integral, we can make the following replacement

$$\begin{aligned} \frac{1}{s-z} &= \frac{1}{s}\left(\frac{1}{1-z/s}\right) \\ &= \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + \frac{1}{s-z}\left(\frac{z}{s}\right)^N. \end{aligned} \tag{142}$$

For the second integral, we can make the following replacement (interchanging the role of s and z)

$$\begin{aligned}\frac{1}{z-s} &= \frac{1}{z}\left(\frac{1}{1-s/z}\right)\\ &= \sum_{n=0}^{N-1}\frac{s^n}{z^{n+1}} + \frac{1}{z-s}\left(\frac{s}{z}\right)^N\\ &= \sum_{n=1}^{N}\frac{s^{n-1}}{z^n} + \frac{1}{z-s}\left(\frac{s}{z}\right)^N\\ &= \sum_{n=1}^{N}\frac{z^{-n}}{s^{-n+1}} + \frac{1}{z-s}\left(\frac{s}{z}\right)^N.\end{aligned} \tag{143}$$

And so we have

$$\begin{aligned}f(z) = \frac{1}{2\pi i}\oint_{C_2} f(s)\left[\sum_{n=0}^{N-1}\frac{z^n}{s^{n+1}} + \frac{1}{s-z}\left(\frac{z}{s}\right)^N\right] z^n\,dz\\ + \frac{1}{2\pi i}\oint_{C_1} f(s)\left[\sum_{n=1}^{N}\frac{z^{-n}}{s^{-n+1}} + \frac{1}{z-s}\left(\frac{s}{z}\right)^N\right] z^{-n}\,dz\\ = \sum_{n=0}^{N-1}\underbrace{\left[\frac{1}{2\pi i}\oint_{C_2}\frac{f(s)}{s^{n+1}}\,ds\right]}_{\alpha_n} z^n + \sum_{n=1}^{N}\underbrace{\left[\frac{1}{2\pi i}\oint_{C_1}\frac{f(s)}{s^{-n+1}}\,ds\right]}_{\beta_n} z^{-n} + \rho_N + \sigma_N\end{aligned} \tag{144}$$

where

$$\rho_N = \frac{1}{2\pi i}\oint_{C_2}\frac{f(s)}{s-z}\left(\frac{z}{s}\right)^N\,ds \tag{145}$$

$$\sigma_N = \frac{1}{2\pi i}\oint_{C_1}\frac{f(s)}{z-s}\left(\frac{s}{z}\right)^N\,ds. \tag{146}$$

Now, on C_2,

$$\frac{1}{|s-z|} \leq \frac{1}{R_2 - R}, \tag{147}$$

and on C_1,

$$\frac{1}{|z-s|} \leq \frac{1}{R - R_1}, \tag{148}$$

where $R = |z|$, $R_1 < R < R_2$. Setting $M = \max|f(s)|$ where $s \in C_1 \cap C_2$, by triangle inequality, we have that

$$|\rho_N| = \frac{1}{2\pi}\left|\oint_{C_2}\frac{f(s)}{s-z}\left(\frac{z}{s}\right)^N\,ds\right| \leq \frac{1}{2\pi}\frac{M}{R_2 - R}\left(\frac{R}{R_2}\right)^N 2\pi R_2 = \frac{M}{1 - R/R_2}\left(\frac{R}{R_2}\right)^N. \tag{149}$$

Similarly,

$$|\sigma_N| \leq \frac{M}{1 - R_1/R} \left(\frac{R_1}{R}\right)^N. \tag{150}$$

We see that $\rho_N \to 0$, $\sigma \to 0$ as $N \to \infty$. It follows (with ϵ's and N's similar to those in the proof of Taylor's theorem) that

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n + \sum_{n=1}^{\infty} \beta_n z^{-n}. \tag{151}$$

And by corollary to Cauchy-Goursat for multiply-connected regions,

$$\begin{aligned} \alpha_n &= \frac{1}{2\pi i} \int_C (\ )\, ds = a_n \\ \beta_n &= \frac{1}{2\pi i} \int_C (\ )\, ds = b_n \end{aligned} \tag{152}$$

for all n. □

59 More results about series

Consider a power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{153}$$

1. If $S(z)$ converges at some $z_1 \neq z_0$ the $S(z)$ converges on $\mathcal{B}_R(z_0)$ where $|z_0 - z_1| \leq R$.
2. The series converges uniformly and absolutely on every ball $\mathcal{B}$ properly contained in $\mathcal{B}_R(z_0)$.
3. On $\mathcal{B}_R(z_0)$, $S(z)$ is analytic, $S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$.
4. If C is a s.c.c.(+) and g is continuous on C and $C \subset \mathcal{B}_R(z_0)$ then

$$\oint_C fg\, dz = \sum_{n=0}^{\infty} \oint_C a_n g(z)(z - z_0)^n\, dz \tag{154}$$

5. Uniqueness of Laurent series: If $S(z) = \sum_{n\in\mathbb{Z}} c_n (z - z_0)^n$ converges on an annulus $R_1 \leq |z - z_0| \leq R_2$ then this is precisely the Laurent series of S at z_0.

60 Residues

For C a s.c.c.(+), let f have singularities at $z_1, z_2, \ldots, z_n$ enclosed by C. Then all the z_k's are isolated singularities, and there exist punctured disks $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ inside C which are on-overlapping whose centers contains z_k's, respectively.

Next, suppose that f has an isolated singularity at z_0. Then f has a Laurent series expansion on an annulus $0 < |z - z_0| < R$ with

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}. \tag{155}$$

Further, for any s.c.c.(+) C_k,

$$b_n = \frac{1}{2\pi i} \oint_{C_k} \frac{f(z)}{(z - z_0)^{-n+1}}\, dz \forall n = 1, 2, 3, \ldots \tag{156}$$

In particular,

$$b_1 = \frac{1}{2\pi i} \oint_{C_k} f(z)\, dz. \tag{157}$$

We shall call this coefficient of $1/(z - z_0)$ in the Laurent series expansion the residue of f at z_0, denoted

$$b_1 := \operatorname{Res}_{z=z_0} f(z). \tag{158}$$

This gives us a way to compute integrals by finding Laurent series expansions.

61 The Residue Theorem

Let C be a s.c.c.(+) and suppose that f is analytic on C and the interior to C except at a finite number of points $z_1, z_2, \ldots, z_n$, all enclosed by C. Then

$$\oint_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z). \tag{159}$$

Proof. Take $C_1, C_2, \ldots, C_n$ to be non-intersecting s.c.c.(+) inside C where each enclosed only the singular point z_k, respectively. Then f is analytic on $\operatorname{Int}(C) \setminus \cup^n \operatorname{Int} C_k$. By Cauchy-Goursat for multiply-connected region,

$$\oint_C f(z)\, dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\, dz. \tag{160}$$

But for each k, we also have

$$\oint_{C_k} f(z)\, dz = 2\pi i \operatorname{Res}_{z=z_k} f(z). \tag{161}$$

So,

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}_{z=z_k} f(z). \tag{162}$$

□

62 Classification of Singularities

If the principal part of the Laurent series expansion of f is identically zero then z_0 is said to be a removable singularity.

If z_0 is an isolated removable singularity for f for $z \neq z_0$ but $0 < |z - z_0| < R$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + 0. \tag{163}$$

At $z = z_0$, the left-hand side is a_0. So if we define

$$f_{ext}(z) = \begin{cases} f(z) & 0 < |z - z_0| < R \\ a_0 & z = z_0 \end{cases} \tag{164}$$

then

$$f_{ext}(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{165}$$

for all z such that $|z - z_0| < R$. This is called an extension of f. We note that $f_{ext}(z)$ is analytic on $\mathcal{B}_R(z_0)$. We have just removed the removable singularity.

When the principal part of f is nonzero and contains a finite number of summands

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} = \frac{b_1}{(z - z_0)} + \dots \frac{b_m}{(z - z_0)^m} \tag{166}$$

and $b_k \neq 0 \forall k \geq m + 1$ then z_0 is a pole of order m for f. When $m = 1$, z_0 is called a simple pole.

If the principal part of f is identically zero, then z_0 is a removable singularity for f, because f can be extended via its valid Taylor-Laurent series expansion to an analytic function on $\mathcal{B}_R(z_0)$.

z_0 is said to be an essential singularity of f it it is not removable or a pole, i.e., the principle part of the Laurent series of f contains an infinite number of non-zero terms.

63 Residues with Φ theorem

Let z_0 be an isolated singularity of f. Then z_0 is a pole or order m if and only if $\exists$ a function $\phi(z)$ which is non zero at z_0, analytic at z_0 and for which

$$f(z) = \frac{\phi(z)}{(z - z_0)^m} \tag{167}$$

for $z \in$ a nbh of z_0. In this case,

$$\operatorname{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}. \tag{168}$$

Proof. (→) Suppose that

$$f(z) = \frac{\phi(z)}{(z - z_0)^m} \tag{169}$$

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$. Then we have that $\phi(z)$ has a valid Taylor series expansion in $\mathcal{B}_R(z_0)$:

$$\phi(z) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z - z_0)^n. \tag{170}$$

With this, we can write $f(z)$ as

$$\begin{aligned} f(z) &= \frac{1}{(z - z_0)^m} \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z - z_0)^n \\ &= \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z - z_0)^{n-m} \\ &= \sum_{n=0}^{m-1} \frac{\phi^{(n)}(z_0)}{n!}(z - z_0)^{n-m} + \text{(Taylor)} \\ &= \sum_{k=1}^{m} \frac{\phi^{(n-k)}(z_0)}{(m-k)!}(z - z_0)^k + \text{(Taylor)}, \quad (k = m - n). \end{aligned} \tag{171}$$

And so z_0 is a pole of order m, since $\phi^{(0)}(z_0) \neq 0$. And of course, we get for free

$$\operatorname{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}. \tag{172}$$

(←) Conversely, assume that f has a pole at z_0 or order m. Then

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + 0 \ldots \\ &= \frac{1}{(z - z_0)^m} \left[\sum_{n=0}^{\infty} a_n (z - z_0)^{n+m} + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^{n-m}} \right] \\ &:= \frac{\phi(z)}{(z - z_0)^m} \end{aligned} \tag{173}$$

where $\phi(z)$ is defined to be the expression in the square brackets. With this, we see that $\phi(z)$ is analytic at z_0 and $\phi(z_0) = 0 + b_m \neq 0$ by hypothesis. □

64 Residues with p-q theorem

Let p, q be analytic at z_0. If $p(z_0) \neq 0, q'(z_0) \neq 0$, and $p'(z_0) = 0$ then

$$f(z) = \frac{p(z)}{q(z)} \tag{174}$$

has a simple pole of z_0 and

$$\mathrm{Res}_{z=z_0} f(z) = \mathrm{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}. \tag{175}$$

Proof. Since $q'(z_0) \neq 0$, q has a simple zero at z_0. So $1/q$ has a simple pole at z_0 and

$$\mathrm{Res}_{z=z_0} \frac{1}{q} = \frac{1}{q'(z_0)}. \tag{176}$$

Since $p(z_0) \neq 0$, we know that

$$\mathrm{Res}_{z=z_0} \frac{p}{q} = p(z_0)\, \mathrm{Res}_{z=z_0} \frac{1}{q} = \frac{p(z_0)}{q'(z_0)}. \tag{177}$$

□

Proof. This proof should be more elaborate than the previous proof: □

65 What happens near singularities?

If z_0 is a pole of order m for f, then

$$\lim_{z \to z_0} f(z) = \infty. \tag{178}$$

66 Removable singularity - Boundedness - Analyticity (RBA)

If z_0 is a removable singularity for f then f is bounded and analytic on a punctured nbh of z_0.

67 The converse of RBA

Let f be analytic on $0 < |z - z_0| < \delta$ for some $\delta > 0$. If f is also bounded on $0 < |z - z_0| < \delta$, then if z_0 is a singularity for f, it must be removable.

Proof. By assumption, f has a Laurent series representation of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \tag{179}$$

where b_n in particular is given by

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{-n+1}} \, dz \tag{180}$$

where C is a s.c.c.(+) in the annulus of the analyticity. In particular, if $0 < \rho < \delta$, and $C_\rho := \{z, |z - z_0| = \rho\}$, (+) then

$$|b_n| = \left| \frac{1}{2\pi i} \oint_{C_\rho} \frac{f(z)}{(z - z_0)^{-n+1}} \, dz \right| \tag{181}$$

and if M is such that $f(z) \leq M \forall 0 < |z - z_0| < \delta$ then

$$|b_n| \leq \frac{1}{2\pi} \frac{M}{\rho^{-n+1}} 2\pi\rho = M\rho^n. \tag{182}$$

Since this is valid $\forall \rho < \delta$, we must have that $b_n = 0 \forall n$. □

68 Casorati-Weierstrass Theorem

Let f have an essential singularity at z_0. Then $\forall w_0 \in \mathbb{C}$ and $\epsilon > 0$,

$$|f(z) - w_0| < \epsilon \tag{183}$$

for some $z \in \mathcal{B}_\delta(z_0) \forall \delta 0$.
$\iff$ f is arbitrarily close to every complex number on every nbh of z_0.
$\iff$ $\forall \delta > 0, f(\mathcal{B}_\delta(z_0) \setminus \{z_0\})$ is dense on $\mathbb{C}$.
$\iff$ f gets close to every single point in a ball for any ball.
$\iff$ If z_0 is an essential singularity for f then f attains, except for at most one value, every complex number an infinite number of time on every nbh of z_0.

Proof. Assume to reach a contradiction that $\exists w_0 \in \mathbb{C}, \epsilon, \delta > 0$ s.t.

$$|f(z) - w_0| \geq \epsilon \forall 0 < |z - z_0| < \delta, \tag{184}$$

i.e., f does not get close to some value w_0 in some nbh of z_0 of radius δ. Then, consider

$$g(z) = \frac{1}{f(z) - w_0} \tag{185}$$

which is bounded and analytic on the punctured disk $0 < |z - z_0| < \delta$. At worst, z_0 is a removable singularity for g. Also note that $g(z)$ is not identically zero since f is not constant (as f has a singularity). With this,

$$g(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \tag{186}$$

which allows us to extend g to z_0. Let $m = \min(k = 0, 1, 2, \dots)$ such that $a_k \neq 0$, which exists because $g \neq 0$. Then

$$g(z) = (z - z_0)^m \sum_{k=0}^{\infty} a_k (z - z_0)^{k-m} = (z - z_0)^m \sum_{k=0}^{\infty} a_{k+m} (z - z_0)^k. \tag{187}$$

Call the sum $h(z)$, which $h(z_0) = a_m \neq 0$. So, in $\mathcal{B}_\delta(z_0) \setminus \{z_0\}$, we have

$$f(z) = w_0 + \frac{1}{g(z)}. \tag{188}$$

If $g(z_0) \neq 0 \iff m = 0$, then this formula allows s to extend f to z_0, which is then analytic, which makes z_0 a removable singularity. This is a contradiction. If $g(z_0) = 0$, then because $m \geq 1$ (by definition) and

$$f(z) = w_0 + \frac{1}{g(z)} = \frac{w_0 g(z) + 1}{(z - z_0)^m h(z)} := \frac{\phi(z)}{(z - z_0)^m}. \tag{189}$$

We see that $\phi(z_0) \neq 0$, and $\phi(z)$ is analytic. So, z_0 is a pole of order m of f. This is also a contradiction. □