PDE's & Calculus of Variations

Huan Q. Bui

MA411: PDE

Professor Evan Randles

May 6, 2019

Presentation layout

Shortest arc joining two points on a plane

Shortest arc joining two points on a plane

Idea: Let the "correct" path be $\bar{y}(x)$, then any path is

$$y(x) = \bar{y}(x) + \epsilon\eta(x)$$

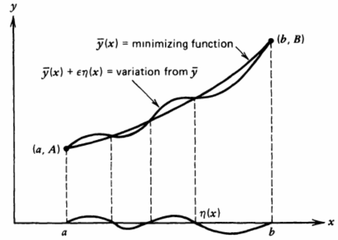
where $\eta(x_1) = \eta(x_2) = 0$, and ϵ is some constant parameter.

Shortest arc joining two points on a plane

<u>Idea</u>: Let the "correct" path be $\bar{y}(x)$, then any path is

$$y(x) = \bar{y}(x) + \epsilon\eta(x)$$

where $\eta(x_1) = \eta(x_2) = 0$, and ϵ is some constant parameter.

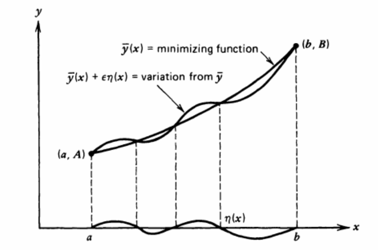


Shortest arc joining two points on a plane

Idea: Let the "correct" path be $\bar{y}(x)$, then any path is

$$y(x) = \bar{y}(x) + \epsilon\eta(x)$$

where $\eta(x_1) = \eta(x_2) = 0$, and ϵ is some constant parameter.



Distance, for any given variation $\eta(x)$:

$$L(\alpha) = \int ds = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx$$

The general picture...

The general picture...

$$S(\epsilon) = \int_a^b f[y', y, x]\, dx = \int_a^b f[\bar{y}' + \epsilon\eta', \bar{y} + \epsilon\eta, x]\, dx.$$

The general picture...

$$S(\epsilon) = \int_a^b f[y', y, x]\, dx = \int_a^b f[\bar{y}' + \epsilon\eta', \bar{y} + \epsilon\eta, x]\, dx.$$

Necessary condition: if $\bar{y}$ minimizes $S(\epsilon)$, then

The general picture...

$$S(\epsilon) = \int_a^b f[y', y, x]\, dx = \int_a^b f[\bar{y}' + \epsilon\eta', \bar{y} + \epsilon\eta, x]\, dx.$$

Necessary condition: if $\bar{y}$ minimizes $S(\epsilon)$, then

$$\frac{\partial S}{\partial \epsilon} = 0 \implies 0 = \frac{\partial f}{\partial \epsilon} = \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}.$$

The general picture...

$$S(\epsilon) = \int_a^b f[y', y, x]\, dx = \int_a^b f[\bar{y}' + \epsilon\eta', \bar{y} + \epsilon\eta, x]\, dx.$$

Necessary condition: if $\bar{y}$ minimizes $S(\epsilon)$, then

$$\frac{\partial S}{\partial \epsilon} = 0 \implies 0 = \frac{\partial f}{\partial \epsilon} = \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}.$$

Can show:

The general picture...

$$S(\epsilon) = \int_a^b f[y', y, x]\, dx = \int_a^b f[\bar{y}' + \epsilon\eta', \bar{y} + \epsilon\eta, x]\, dx.$$

Necessary condition: if $\bar{y}$ minimizes $S(\epsilon)$, then

$$\frac{\partial S}{\partial \epsilon} = 0 \implies 0 = \frac{\partial f}{\partial \epsilon} = \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}.$$

Can show:

$$\frac{\partial S}{\partial \epsilon} = \int_a^b \eta(x) \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) dx = 0.$$

The general picture...

$$S(\epsilon) = \int_a^b f[y', y, x]\, dx = \int_a^b f[\bar{y}' + \epsilon\eta', \bar{y} + \epsilon\eta, x]\, dx.$$

Necessary condition: if $\bar{y}$ minimizes $S(\epsilon)$, then

$$\frac{\partial S}{\partial \epsilon} = 0 \implies 0 = \frac{\partial f}{\partial \epsilon} = \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}.$$

Can show:

$$\frac{\partial S}{\partial \epsilon} = \int_a^b \eta(x) \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) dx = 0.$$

This is true for any $\eta(x)$, so

The general picture...

$$S(\epsilon) = \int_a^b f[y', y, x]\, dx = \int_a^b f[\bar{y}' + \epsilon\eta', \bar{y} + \epsilon\eta, x]\, dx.$$

Necessary condition: if $\bar{y}$ minimizes $S(\epsilon)$, then

$$\frac{\partial S}{\partial \epsilon} = 0 \implies 0 = \frac{\partial f}{\partial \epsilon} = \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}.$$

Can show:

$$\frac{\partial S}{\partial \epsilon} = \int_a^b \eta(x) \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) dx = 0.$$

This is true for any $\eta(x)$, so

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0}$$

$\longrightarrow$ **Euler-Lagrange equation**.

Back to silly example

Back to silly example

$$f[y', y, x] = \sqrt{1 + (y')^2}$$

Back to silly example

$$f[y', y, x] = \sqrt{1 + (y')^2}$$

Applying the Euler-Lagrange equations:

Back to silly example

$$f[y', y, x] = \sqrt{1 + (y')^2}$$

Applying the Euler-Lagrange equations:

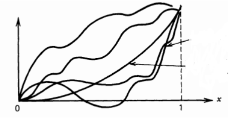
$$\begin{aligned}\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} &= 0\\ \implies y' &= \text{Constant}\\ \implies y &= ax + b.\end{aligned}$$

$\longrightarrow$ a straight line as expected.

A powerful method: Answer isn't always obvious

A powerful method: Answer isn't always obvious

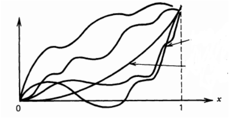
Ex: The Brachistochrone problem by Bernoulli, 1696.



Goal: minimize sliding time on a frictionless path.

A powerful method: Answer isn't always obvious

Ex: The Brachistochrone problem by Bernoulli, 1696.

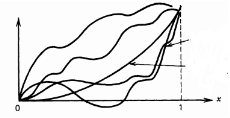


Goal: minimize sliding time on a frictionless path.

$$Time = \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} \sqrt{\frac{1+(x')^2}{y}}\, dy.$$

A powerful method: Answer isn't always obvious

Ex: The Brachistochrone problem by Bernoulli, 1696.



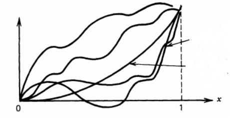
Goal: minimize sliding time on a frictionless path.

$$Time = \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} \sqrt{\frac{1+(x')^2}{y}}\, dy.$$

Applying Euler-Lagrange equation to $f[x', x, y]$, get

A powerful method: Answer isn't always obvious

Ex: The Brachistochrone problem by Bernoulli, 1696.



Goal: minimize sliding time on a frictionless path.

$$Time = \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} \sqrt{\frac{1+(x')^2}{y}}\, dy.$$

Applying Euler-Lagrange equation to $f[x', x, y]$, get

$$\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \end{cases} \longrightarrow \text{Cycloid}$$

Two-way street

1. PDE's can be formulated as minimization problems

Two-way street

1. PDE's can be formulated as minimization problems
2. Euler-Lagrange Equations as PDE's

Ex: Laplace's equation with Dirichlet BC:

$$(*) \begin{cases} \nabla^2 u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Ex: Laplace's equation with Dirichlet BC:

$$(*)\begin{cases}\nabla^2 u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega\end{cases}$$

Claim: Of all admissible w satisfying $w = g$, u solves $(*)$ $\iff$ u minimizes

$$S[w] = \frac{1}{2}\int_\Omega |\nabla w|^2\,dx.$$

Proof $\implies$

<u>Show</u>: u solves $(*) \implies S[u] \leq S[w] \, \forall$ admissible w.

Proof $\Longrightarrow$

Show: u solves $(*) \implies S[u] \leq S[w] \, \forall$ admissible w.

$$\text{Observe: } 0 = \int w\nabla^2 u = -\int \nabla u \cdot \nabla w.$$

Proof $\Longrightarrow$

Show: u solves $(*) \implies S[u] \leq S[w] \, \forall$ admissible w.

$$\text{Observe: } 0 = \int w \nabla^2 u = -\int \nabla u \cdot \nabla w.$$

So,

$$S[u'] = \frac{1}{2}\int |\nabla(u+v)|^2 = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{2}\int |\nabla w|^2 \geq \frac{1}{2}\int |\nabla u|^2.$$

Proof $\implies$

<u>Show</u>: u solves $(*) \implies S[u] \leq S[w] \, \forall$ admissible w.

$$\text{Observe: } 0 = \int w \nabla^2 u = -\int \nabla u \cdot \nabla w.$$

So,

$$S[u'] = \frac{1}{2}\int |\nabla(u+v)|^2 = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{2}\int |\nabla w|^2 \geq \frac{1}{2}\int |\nabla u|^2.$$

$\implies$ u minimizes $S[w]$.

Proof $\Longleftarrow$

Show: u minimizes $S[w] \implies u$ solves $(*)$.

Proof $\Longleftarrow$

Show: u minimizes $S[w] \implies u$ solves $(*)$.

Idea: Consider variation in u, so let $u \to u + \epsilon v$, v satisfies BC.

Proof $\Longleftarrow$

Show: u minimizes $S[w] \implies u$ solves $(*)$.

Idea: Consider variation in u, so let $u \to u + \epsilon v$, v satisfies BC.

$$S[u+\epsilon v] = \frac{1}{2}\int |\nabla(u+\epsilon v)|^2 = \frac{1}{2}\int |\nabla u|^2 + 2\epsilon \nabla u \cdot \nabla v + \epsilon^2 |\nabla v|^2.$$

Proof $\Longleftarrow$

Show: u minimizes $S[w] \implies u$ solves $(*)$.

Idea: Consider variation in u, so let $u \to u + \epsilon v$, v satisfies BC.

$$S[u+\epsilon v] = \frac{1}{2}\int |\nabla(u+\epsilon v)|^2 = \frac{1}{2}\int |\nabla u|^2 + 2\epsilon \nabla u \cdot \nabla v + \epsilon^2 |\nabla v|^2.$$

u minimizes S, so $\partial S / \partial \epsilon = 0$ at $\epsilon = 0$, so after a lot of simplification

$$\left.\frac{\partial S}{\partial \epsilon}\right|_{\epsilon=0} = -\int v \nabla^2 u = 0.$$

Proof $\Longleftarrow$

Show: u minimizes $S[w] \implies u$ solves $(*)$.

Idea: Consider variation in u, so let $u \to u + \epsilon v$, v satisfies BC.

$$S[u+\epsilon v] = \frac{1}{2}\int |\nabla(u+\epsilon v)|^2 = \frac{1}{2}\int |\nabla u|^2 + 2\epsilon \nabla u \cdot \nabla v + \epsilon^2 |\nabla v|^2.$$

u minimizes S, so $\partial S/\partial \epsilon = 0$ at $\epsilon = 0$, so after a lot of simplification

$$\left.\frac{\partial S}{\partial \epsilon}\right|_{\epsilon=0} = -\int v\nabla^2 u = 0.$$

This is true for any v, so $\boxed{\nabla^2 u = 0}$. So u solves $(*)$ as claimed.

Euler-Lagrange Equations as PDE's

Recall: If y minimizes

$$S[y] = \int f[y', y, x]\, dx$$

Euler-Lagrange Equations as PDE's

Recall: If y minimizes

$$S[y] = \int f[y', y, x]\, dx$$

we have an associated Euler-Lagrange Equation:

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}.$$

Euler-Lagrange Equations as PDE's

<u>Recall</u>: If y minimizes

$$S[y] = \int f[y', y, x]\, dx$$

we have an associated Euler-Lagrange Equation:

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}.$$

$\longrightarrow$ Looks complicated, but if $\mathcal{L}$ is known then things often become simple :)

Principle of Least Action

$$S[y] = \int f[y', y, x]\, dx$$

Principle of Least Action

$$S[y] = \int f[y', y, x]\, dx$$

Associated Euler-Lagrange Equation:

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}.$$

Principle of Least Action

$$S[y] = \int f[y', y, x]\, dx$$

Associated Euler-Lagrange Equation:

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}.$$

In physics, S is called the **action**. f is called the **Lagrangian**, denoted $\mathcal{L}$.

Principle of Least Action

$$S[y] = \int f[y', y, x]\, dx$$

Associated Euler-Lagrange Equation:

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}.$$

In physics, S is called the **action**. f is called the **Lagrangian**, denoted $\mathcal{L}$.

$\longrightarrow$ **Principle of Least Action**: Systems tend to be such that $\boxed{\delta S = 0}$

Principle of Least Action

$$S[y] = \int f[y', y, x]\, dx$$

Associated Euler-Lagrange Equation:

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}.$$

In physics, S is called the **action**. f is called the **Lagrangian**, denoted $\mathcal{L}$.

$\longrightarrow$ **Principle of Least Action**: Systems tend to be such that $\boxed{\delta S = 0}$

$$\boxed{\mathcal{L} = \text{Kinetic energy} - \text{Potential Energy}}$$

$\longrightarrow$ Equations of motion are found as solutions to E-L PDE's.

Newton's Second Law from E-L

Consider

Newton's Second Law from E-L

Consider

$$\mathcal{L} = \frac{1}{2}mv^2 - U(x) = \mathcal{L}[x', x, t] = \mathcal{L}[v, x, t].$$

Newton's Second Law from E-L

Consider

$$\mathcal{L} = \frac{1}{2}mv^2 - U(x) = \mathcal{L}[x', x, t] = \mathcal{L}[v, x, t].$$

Then Euler-Lagrange equation tells us

Newton's Second Law from E-L

Consider

$$\mathcal{L} = \frac{1}{2}mv^2 - U(x) = \mathcal{L}[x', x, t] = \mathcal{L}[v, x, t].$$

Then Euler-Lagrange equation tells us

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x'}$$

Newton's Second Law from E-L

Consider

$$\mathcal{L} = \frac{1}{2}mv^2 - U(x) = \mathcal{L}[x', x, t] = \mathcal{L}[v, x, t].$$

Then Euler-Lagrange equation tells us

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x'} \\ -\frac{dU}{dx} &= \frac{d}{dt}(mv) \end{aligned}$$

Newton's Second Law from E-L

Consider

$$\mathcal{L} = \frac{1}{2}mv^2 - U(x) = \mathcal{L}[x', x, t] = \mathcal{L}[v, x, t].$$

Then Euler-Lagrange equation tells us

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x'} \\ -\frac{dU}{dx} &= \frac{d}{dt}(mv) \\ F &= ma \end{aligned}$$

Newton's Second Law from E-L

Consider

$$\mathcal{L} = \frac{1}{2}mv^2 - U(x) = \mathcal{L}[x', x, t] = \mathcal{L}[v, x, t].$$

Then Euler-Lagrange equation tells us

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x'} \\ -\frac{dU}{dx} &= \frac{d}{dt}(mv) \\ F &= ma \end{aligned}$$

$\longrightarrow$ In fact, Laplace's, Poisson's, wave eqns,... can be found this way!

THE END

Appendix

1. Since $w = 0$ at end points,

$$\int w\nabla^2 u = (w\nabla v)\Big|_a^b - \int \nabla u \cdot \nabla w = -\int \nabla u \cdot \nabla w$$

2.

$$|\nabla(u+v)|^2 = |\nabla u|^2 + |\nabla w|^2 + 2\nabla u \cdot \nabla w$$